Mertz' Paramagnetic Fluid Pump

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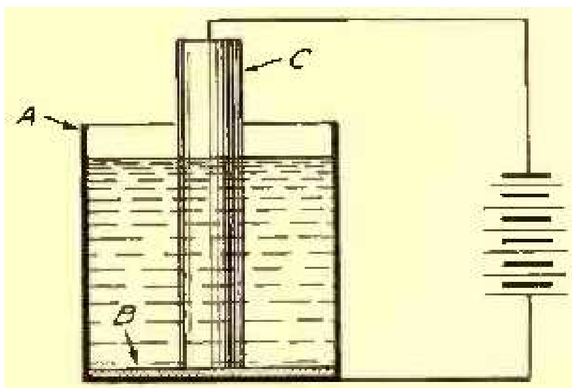
1 Problem

Discuss the principle of operation of an electromagnetic "pump" described by Mertz (1915) in a popular electrical magazine [1].

As sketched below, an annular volume of copper sulfate and water, a conducting, paramagnetic fluid,[1] surrounds a strong, cylindrical, conducting, permanent magnet C that rests on an insulating base B, and the liquid is surrounded by an outer conducting cylinder A. When a strong DC current flows between the outer cylinder and the cylindrical magnet, the copper-sulfate solution rotates azimuthally (and the current flows on spiraling paths in the liquid). The direction of rotation depends on the sign of the electric current.



2 Solution

2.1 $\mathbf{J} \times \mathbf{B}$ Force

The most common type of electromagnetic pump is based on the $\mathbf{J} \times \mathbf{B}$ force density in a current-carrying liquid,[2] which latter is typically nonmagnetic (*i.e.*, diamagnetic).

In Mertz' apparatus, if we approximate the current in the liquid as flowing in horizontal planes, the azimuthal component of the $\mathbf{J} \times \mathbf{B}$ force density would be $-J_r B_z$, in a cylindrical coordinate system (r, ϕ, z) with the z-axis along that of the cylinder. If the conducting cylinder C were nonmagnetic, there would be no z-component to the magnetic field (due only to the electric current driven by the battery). Hence, it is essential that cylinder C be a permanent magnet for there to be an azimuthal force on the liquid.

[1]Mertz' paper is one of the earliest to mention magnetic fluids. In recent years, so-called ferrofluids have found considerable application. See, for example, [2].

[2]This effect was perhaps first demonstrated by Hering [3], and analyzed by Northrup [4] (1907).

A version of Mertz' experiment using (table)salt water as the conducting liquid is reported at https://www.physicscentral.com/experiment/physicsathome/electricwhirlpool.cfm. That is, use of the paramagnetic copper-sulfate solution is not essential.

2.2 Force on Paramagnetic Ions

Here, we show that the magnetic force on the paramagnetic copper ions in the liquid adds to the $\mathbf{J} \times \mathbf{B}$ force discussed above.

To deduce this force, we model the magnetic moment $\mathbf{m}$ of an ion as two equal an opposite (effective) magnetic charges p separated by small distance $\mathbf{d}$, *i.e.*, $\mathbf{m} = p\,\mathbf{d}$. The force on an effective magnetic charge p at position $\mathbf{x}$ in a magnetic field $\mathbf{B}$ is $\mathbf{F} = p\,\mathbf{B}(\mathbf{x})$.[3] Then, the force on a magnetic dipole $\mathbf{m}$ with $-p$ at $\mathbf{x}$ and p at $\mathbf{x} + \mathbf{d}$ is,

$$\mathbf{F} = \lim_{\mathbf{d}\to 0,\ p\,\mathbf{d}=\mathbf{m}} [p\,\mathbf{B}(\mathbf{x}+\mathbf{d}) - p\,\mathbf{B}(\mathbf{x})] = p(\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{B}(\mathbf{x}) = (\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}. \tag{1}$$

Further, we suppose that the paramagnetic ions in the liquid are aligned (on average) with the magnetic field, such that,

$$\mathbf{m} = k\mathbf{B}. \tag{2}$$

for a (temperature-dependent) constant k. Hence, the force on a magnetic dipole in the liquid is,[4,5]

$$\mathbf{F} = k(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}. \tag{3}$$

In a cylindrical coordinate system (r, ϕ, z) with the z-axis along that of the cylinder, the permanent magnetic field has r- and z-components, while that due to the electrical current in the conducting cylinder C has only a ϕ-component. Both of these fields are azimuthally symmetric.[6]

In general, the magnetic force on the liquid has all three of r-, ϕ- and z-components, but we emphasize only the ϕ component here,

$$F_\phi = k\left(B_r(r,z)\frac{\partial}{\partial r} + B_z(r,z)\frac{\partial}{\partial z}\right)B_\phi(r,z). \tag{4}$$

[3]See Appendix A of [5] for discussion of the force on effective magnetic charges associated with magnetic materials that actually are based on Ampèrian "molecular currents".

[4]In general, $\boldsymbol{\nabla} B^2 = 2(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B} + 2\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})$. If there were no current in the liquid (and it were in a steady state), $\boldsymbol{\nabla}\times\mathbf{B} = 0$, and we could write the force as $\mathbf{F} = k\boldsymbol{\nabla} B^2/2$. Then, with $\mathbf{B}$ independent of ϕ, as in the present example, there would be no azimuthal force.

[5]In a linear magnetic medium, where $\mathbf{B} = \mu_r\mu_0\mathbf{H}$ and μ_r is the relative permeability, the magnetization density is $\mathbf{M} = n\mathbf{m}$ with n being the number density of magnetic dipoles $\mathbf{m}$. Also, $\mathbf{M} = \chi_{\mathrm{M}}\mathbf{H}$, where $\chi_{\mathrm{M}} = \mu_r - 1$ (see, for example, sec. 2.4 of [6]). Hence, $\mathbf{m} = (\mu_r - 1)\mathbf{B}/n\mu_r\mu_0$, and $k = (\mu_r - 1)/n\mu_r\mu_0$.

Paramagnetic materials have $\mu_r > 1$, $k > 0$, while diamagnetic materials have $1 > \mu_r > 0$, $k < 0$. Hence, the direction of the force (3) on a magnetic liquid in Mertz' example is opposite for paramagnetic and diamagnetic liquids.

[6]The spiraling current in the liquid contributes to all three of B_r, B_ϕ and B_z. This effect must be included in a detailed calculation; otherwise $F_\phi = 0$ as remarked in the preceding footnote.

The azimuthal force (4) is nonzero only if there exists both the field $(B_r, 0, B_z)$ due to the permanent magnet C and the field $(0, B_\phi, 0)$ due to the electrical current through the magnet.

Note that if the magnet C were very long, and the liquid only near its midplane in z, then B_r would be negligible, B_ϕ would depend only on r, and the azimuthal force would be negligible.

A Appendix: The Ampère Force Law (Sept. 7 2021)

Mertz' experiment provided a tabletop demonstration (in his view) of the (Biot-Savart) $\mathbf{J} \times \mathbf{B}$ force law. He did not mention Ampère's force law, which also predicts the observed azimuthal rotation of the conducting liquid, as discussed below.

A.1 Historical Background

In 1820-1822, Ampère examined the force between two circuits, carrying steady currents I_1 and I_2, and inferred that this could be written (here in vector notation, which Ampère did not use) as (pp. 21-24 of [7]),[7]

$$\mathbf{F}_{\text{on }1} = \oint_1 \oint_2 d^2\mathbf{F}_{\text{on }1}, \quad d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2 [3(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2) - 2\, d\mathbf{l}_1 \cdot d\mathbf{l}_2] \frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\text{on }2}, \quad (5)$$

where $\mathbf{r} = \mathbf{l}_1 - \mathbf{l}_2$ is the distance from a current element $I_2\, d\mathbf{l}_2$ at $\mathbf{r}_2 = \mathbf{l}_2$ to element $I_1\, d\mathbf{l}_1$ at $\mathbf{r}_1 = \mathbf{l}_1$.[8,9] The integrand $d^2\mathbf{F}_{\text{on }1}$ of eq. (5) has the appeal that it changes sign if elements 1 and 2 are interchanged, and so suggests a force law for current elements that obeys Newton's third law.

In 1825, Ampère noted, p. 214 of [10], p. 29 of [7], p. 366 of the English translation in

[7]A historical survey of the development of electrodynamics in the 1800's is given in the Appendix to [9]. A thoughtful online site about Ampère is http://www.ampere.cnrs.fr/parcourspedagogique/index-en.php.

[8]Ampère sometimes used the notation that the angles between $d\mathbf{l}_i$ and $\mathbf{r}$ are θ_i, and the angle between the plane of $d\mathbf{l}_1$ and $\mathbf{r}$ and that of $d\mathbf{l}_2$ and $\mathbf{r}$ is ω. Then, $d\mathbf{l}_1 \cdot d\mathbf{l}_2 = dl_1\, dl_2 (\sin\theta_1 \sin\theta_2 \cos\omega + \cos\theta_1 \cos\theta_2)$, and the force element of eq. (5) can be written as,

$$d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2\, dl_1\, dl_2 (\cos\theta_1 \cos\theta_2 - 2\sin\theta_1 \sin\theta_2 \cos\omega) \frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\text{on }2}. \quad (6)$$

[9]Ampère also noted the equivalents to,

$$d\mathbf{l}_1 = \frac{\partial \mathbf{r}}{\partial l_1} dl_1, \qquad \mathbf{r} \cdot d\mathbf{l}_1 = \mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial l_1} dl_1 = \frac{1}{2} \frac{\partial r^2}{\partial l_1} dl_1 = r \frac{\partial r}{\partial l_1} dl_1, \qquad d\mathbf{l}_2 = -\frac{\partial \mathbf{r}}{\partial l_2} dl_2, \qquad \mathbf{r} \cdot d\mathbf{l}_2 = -r \frac{\partial r}{\partial l_2} dl_2, \quad (7)$$

where l_1 and l_2 measure distance along the corresponding circuits in the directions of their currents. Then,

$$d\mathbf{l}_1 \cdot d\mathbf{l}_2 = -d\mathbf{l}_1 \cdot \frac{\partial \mathbf{r}}{\partial l_2} dl_2 = -\frac{\partial}{\partial l_2}(\mathbf{r} \cdot d\mathbf{l}_1)\, dl_2 = -\frac{\partial}{\partial l_2}\left(r \frac{\partial r}{\partial l_1}\right) dl_1\, dl_2 = -\left(\frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2} + r \frac{\partial^2 r}{\partial l_1 \partial l_2}\right) dl_1\, dl_2, \quad (8)$$

and eq. (5) can also be written in forms closer to those used by Ampère,

$$d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2\, dl_1\, dl_2 \left[2r \frac{\partial^2 r}{\partial l_1 \partial l_2} - \frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2}\right] \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} 2 I_1 I_2\, dl_1\, dl_2 \frac{\partial^2 \sqrt{r}}{\partial l_1 \partial l_2} \frac{\hat{\mathbf{r}}}{\sqrt{r}} = -d^2\mathbf{F}_{\text{on }2}. \quad (9)$$

[8], that for a closed circuit, eq. (5) can be rewritten as,[10]

$$\mathbf{F}_{\text{on } 1} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})\, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}}}{r^2} = \oint_1 I_1 d\mathbf{l}_1 \times \frac{\mu_0}{4\pi} \oint_2 \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}\,, \tag{11}$$

in vector notation (which, of course, he did not use).[11] Ampère made very little comment on this result.[12] However, in retrospect, we see that the form (11) lends itself to the interpretation that the force between closed circuits with steady currents can be written in terms of a magnetic field **B** as,

$$\mathbf{F} = \oint I\, d\mathbf{l} \times \mathbf{B}, \qquad \text{where} \qquad \mathbf{B} = \frac{\mu_0}{4\pi} \oint \frac{I\, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}\,, \tag{12}$$

both equations of which are often called the Biot-Savart law.[13]

Already in 1820 Ampère came to the vision that all magnetic effects are due to electrical currents.[14,15]

A.2 Application to Mertz' Experiment

In Ampère's view, the axially magnetized rod should be considered as containing "molecular currents", and uniform axial magnetization **M**corresponds to a uniform, azimuthal surface current, $\mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward unit vector from the cylindrical surface of the rod.

If a current element $I\, d\mathbf{l}$ in the conducting liquid interacted with a current element $I'\, d\mathbf{l}'$ on the surface of magnetized rod according to eq. (5), the force be would be along the line the line of centers of the two current elements. For $d\mathbf{l}$ pointing to the axis of the rod, an element

[10]Note that for a fixed point 2, $d\mathbf{l}_1 = d\mathbf{r}$, and $dr = d\mathbf{r} \cdot \hat{\mathbf{r}} = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. Then, for any function $f(r)$, $df = (df/dr)\, dr = (df/dr)\, d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. In particular, for $f = -1/r$, $df = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}/r^2$, so the first term of the first form of eq. (10) is a perfect differential with respect to $\mathbf{l}_1$. Hence, when integrating around a closed loop 1, the first term does not contribute, and it is sufficient to write (as first argued by Neumann, p. 67 of [11]),

$$\mathbf{F}_{\text{on } 1} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r^2}\, \hat{\mathbf{r}} = -\mathbf{F}_{\text{on } 2}. \tag{10}$$

[11]Ampère's force law for closed circuits with steady currents can be written in many other ways as well. Maxwell gave an early survey of this in Arts. 510-526 of [12].

[12]As a consequence, the form (11) is generally attributed to Grassmann [13], as in [14], for example.

[13]Biot and Savart [15, 18] actually studied on the force due to an electric current I in a wire on one pole, p, of a long, thin magnet. Their initial interpretation of the results was somewhat incorrect, which was remedied by Biot in 1821 and 1824 [19, 20] with a form that can be written in vector notation (and in SI units) as,

$$\mathbf{F} = \frac{\mu_0\, p}{4\pi} \oint \frac{I\, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2}\,, \tag{13}$$

where $\mathbf{r}$ is the distance from a current element $I\, d\mathbf{l}$ to the magnetic pole. There was no immediate interpretation of eq. (13) in terms of a magnetic field, $\mathbf{B} = \mathbf{F}/p$.

[14]The brief report of a lecture by Ampère on Sept. 18, 1820 [21] ends with *je réduisis tous les phénomènes magnétiques à des effets purement électriques*. See also [22].

[15]The confirmation that permanent magnetism, due to the magnetic moments of electrons, is Ampèrian (rather than Gilbertian = due to pairs of opposite magnetic charges) came only after detailed studies of positronium (e^+e^- "atoms") in the 1940's [23, 24].

$I'\,d\mathbf{l}'$ on the surface of the rod, but not in the plane containing the element $I\,d\mathbf{l}$ and the axis of the rod, implies a force component in the direction of $\mathbf{M}\times d\mathbf{l}$ according to Ampère's prescription (5). This is consistent with the observed behavior in Mertz' experiment, so his experiment does not distinguish between Ampère's force law and the Biot-Savant force law.

A.2.1 Model Calculation for an Infinte Solenoid

For ease of analytic computation, we consider the case where the "external" magnetic field in Mertz' experiment is due to an infinite solenoid, about the z-axis, of radius a, with axial magnetic field $\mathbf{B} = \mu_0 I\,\hat{\mathbf{z}}$ inside the solenoid (and "zero" outside it), where I is the azimuthal current density per unit length aroung the solenoid.

Then, for a radial current element in the conducting liquid, $\mathbf{J} = -J\,\hat{\mathbf{r}}$ in a cylindrical coordinate system (r,ϕ,z), the Biot-Savart force is,

$$d\mathbf{F} = \mathbf{J}\,d\mathrm{Vol}\times\mathbf{B} = \mu_0 I J\,d\mathrm{Vol}\,\hat{\boldsymbol{\phi}}, \tag{14}$$

if the current element is inside the solenoid, while the force if zero otherwise.

According to Ampère's prescription (5), the force on this current element, at $(r,\phi,z) = (b,0,0)$, is,

$$d\mathbf{F} = \frac{\mu_0}{4\pi} I J\,d\mathrm{Vol}\int_{-\infty}^{\infty} dz \int_{\phi=0}^{2\pi}$$
$$\left[3\left(\hat{\mathbf{R}}\cdot -\hat{\mathbf{r}}(\phi=0)\right)\left(\hat{\mathbf{R}}\cdot a\,d\phi\,\hat{\boldsymbol{\phi}}(\phi)\right) - 2\left(-\hat{\mathbf{r}}(\phi=0)\right)\cdot a\,d\phi\,\hat{\boldsymbol{\phi}}(\phi)\right]\frac{\hat{\mathbf{R}}}{R^2}\,, \tag{15}$$

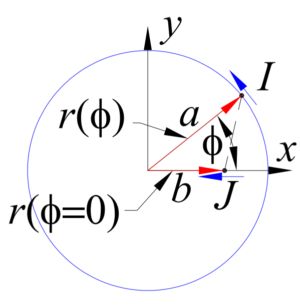
where,

$$\mathbf{R} = b\,\hat{\mathbf{r}}(\phi=0) - a\,\hat{\mathbf{r}}(\phi) - z\,\hat{\mathbf{z}} = (b - a\cos\phi)\,\hat{\mathbf{x}} - a\sin\phi\,\hat{\mathbf{y}} - z\,\hat{\mathbf{z}}, \tag{16}$$

$$R^2 = a^2 + b^2 - 2ab\cos\phi + z^2, \tag{17}$$

$$\hat{\mathbf{R}}\cdot -\hat{\mathbf{r}}(\phi=0) = -\frac{b - a\cos\phi}{R}\,, \qquad \hat{\mathbf{R}}\cdot a\,d\phi\,\hat{\boldsymbol{\phi}}(\phi) = -\frac{ab\sin\phi\,d\phi}{R} \tag{18}$$

$$-\hat{\mathbf{r}}(\phi=0))\cdot a\,d\phi\,\hat{\boldsymbol{\phi}}(\phi) = a\sin\phi\,d\phi. \tag{19}$$



Thus,

$$d\mathbf{F} = \frac{\mu_0}{4\pi} I J\,d\mathrm{Vol}\int_{-\infty}^{\infty} dz \int_{0}^{2\pi} d\phi$$
$$\left[\frac{3ab\sin\phi(b - a\cos\phi)}{R^2} - 2a\sin\phi\right]\frac{(b - a\cos\phi)\,\hat{\mathbf{x}} - a\sin\phi\,\hat{\mathbf{y}} - z\,\hat{\mathbf{z}}}{R^3}\,. \tag{20}$$

The $\hat{\mathbf{x}}$-component integral is odd in $\sin\phi$, and the $\hat{\mathbf{z}}$-integral is odd in z, so both of these integrals vanish. The remaining $\hat{\mathbf{y}}$-integral is,

$$
\begin{aligned}
dF_y &= \frac{\mu_0}{4\pi} a^2 I J\, d\mathrm{Vol} \int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi \left[-\frac{3b\sin^2\phi(b - a\cos\phi)}{R^5} + \frac{2\sin^2\phi}{R^3} \right] \\
&= \frac{\mu_0}{4\pi} a^2 I J\, d\mathrm{Vol} \int_0^{2\pi} d\phi \left[-\frac{4b\sin^2\phi(b - a\cos\phi)}{(a^2+b^2-2ab\cos\phi)^2} + \frac{4\sin^2\phi}{a^2+b^2-2ab\cos\phi} \right] \\
&= \frac{\mu_0}{\pi} I J\, d\mathrm{Vol} \int_0^{2\pi} d\phi \frac{(1-(b/a)\cos\phi)\sin^2\phi}{[1-2(b/a)\cos\phi+(b/a)^2]^2} \\
&= \frac{\mu_0}{2\pi} I J\, d\mathrm{Vol} \int_0^{2\pi} d\phi \frac{(1-(b/a)\cos\phi)(1-\cos 2\phi)}{[1-2(b/a)\cos\phi+(b/a)^2]^2} \\
&= \frac{\mu_0}{2\pi} I J\, d\mathrm{Vol} \int_0^{2\pi} d\phi \frac{1-(b/a)\cos\phi-\cos 2\phi+(b/a)\cos\phi\cos 2\phi}{[1-2(b/a)\cos\phi+(b/a)^2]^2} \\
&= \frac{\mu_0}{2\pi} I J\, d\mathrm{Vol} \int_0^{2\pi} d\phi \frac{1-(b/2a)\cos\phi-\cos 2\phi+(b/2a)\cos 3\phi}{[1-2(b/a)\cos\phi+(b/a)^2]^2}\,,
\end{aligned}
\tag{21}
$$

using Dwight [25] 200.03 and 200.05. The remaining integrals in eq. (21) are given in Gradshteyn and Ryzhik [26] 3.616.7, and must be evaluated separately for $b < a$ and $b > a$,

$$
\begin{aligned}
dF_y(b<a) &= \frac{\mu_0}{2\pi} I J\, d\mathrm{Vol} \frac{2\pi}{(1-(b/a)^2)^3} \left[b^2/a^2 \left(2 + \frac{1-(b/a)^2}{b^2/a^2} \right) - \frac{b^4/a^4}{2} \left(2 + 2\frac{1-b^2/a^2}{(b/a)^2} \right) \right. \\
&\qquad \left. - b^4/a^4 \left(2 + 3\frac{1-b^2/a^2}{b^2/a^2} \right) + \frac{b^6/a^6}{2} \left(2 + 4\frac{1-b^2/a^2}{b^2/a^2} \right) \right] \\
&= \frac{\mu_0 I J\, d\mathrm{Vol}}{(1-(b/a)^2)^3} \left(1 + \frac{b^2}{a^2} - \frac{b^2}{a^2} - \frac{3b^2}{a^2} + \frac{b^4}{a^4} + \frac{2b^4}{a^4} - \frac{b^6}{a^6} \right) \\
&= \frac{\mu_0 I J\, d\mathrm{Vol}}{(1-(b/a)^2)^3} \left(1 - \frac{3b^2}{a^2} + \frac{3b^4}{a^4} - \frac{b^6}{a^6} \right) = \mu_0 I J\, d\mathrm{Vol},
\end{aligned}
\tag{22}
$$

in agreement with the (much simpler to derive) Biot-Savart result, noting that at the current element $\mathbf{J}\, d\mathrm{Vol}$ the $\hat{\mathbf{y}}$-direction is also the $\hat{\boldsymbol{\phi}}$-direction.

For completeness, if the current element $\mathbf{J}\, d\mathrm{Vol}$ is outside the infinite solenoid,

$$
\begin{aligned}
dF_y(b>a) &= \frac{\mu_0}{2\pi} I J\, d\mathrm{Vol} \frac{2\pi}{(b/a)^2-1)^3} \left[\left(2 + b^2/a^2 - 1 \right) - \frac{1}{2} \left(2 + 2(b^2/a^2-1) \right) \right. \\
&\qquad \left. - \frac{1}{b^2/a^2} \left(2 + 3(b^2/a^2-1) \right) + \frac{1}{2b^2/a^2} \left(2 + 4(b^2/a^2-1) \right) \right] \\
&= \frac{\mu_0 I J\, d\mathrm{Vol}}{(b/a)^2-1)^3} \left(1 + b^2/a^2 - b^2/a^2 + \frac{1}{b^2/a^2} - 3 - \frac{1}{b^2/a^2} + 2 \right) = 0.
\end{aligned}
\tag{23}
$$

A.2.2 Comment

The result that the force on the current element considered above is the same according to both the Ampère and Biot-Savart force laws is an example of a general result for steady currents flowing in closed circuits, as discussed in [27, 28].

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