Doubly Negative Metamaterials

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1 Problem

In 1967 Veselago considered the possibility of materials with (relative) permittivity ϵ and (relative) permeability μ both negative [1]. Building on the long history of artifical dielectrics for high-frequency applications (see, for example, [2, 3, 4]), in 1999 Pendry *et al.* proposed a technique for fabrication of metamaterials with both negative ϵ and negative μ within a certain frequency range [5]. Such materials were first realized in the laboratory in 2000 [6].

An idealized model of the frequency dependence of the permittivity and the permeability (both assumed to be linear and isotropic) is that,

$$\epsilon(\omega) = 1 \pm \frac{\omega_{\epsilon,p}^2}{\omega^2 - \omega_{\epsilon,0}^2 + i\Gamma_\epsilon \omega} \approx \frac{\omega^2 - \omega_{\epsilon,1}^2}{\omega^2 - \omega_{\epsilon,0}^2}, \qquad (1)$$

and,

$$\mu(\omega) = 1 \pm \frac{\omega_{\mu,p}^2}{\omega^2 - \omega_{\mu,0}^2 + i\Gamma_\mu \omega} \approx \frac{\omega^2 - \omega_{\mu,1}^2}{\omega^2 - \omega_{\mu,0}^2}, \qquad (2)$$

where the time dependence is assumed to have the form $e^{-i\omega t}$,

$$\omega_1 = \sqrt{\omega_0^2 \pm \omega_p^2}, \qquad (3)$$

ω_p is equivalent to the plasma frequency of the medium, and ω_0 is a resonant frequency of the medium that is associated with a small damping coefficient Γ which we neglect by restricting the analysis to frequencies not too close to ω_0 or ω_1. In passive metamaterials the $\pm$ sign is negative, while for an active metamaterial [7] with an inverted population the sign can be positive. *In general, there is no relation between the frequencies $\omega_{\epsilon,j}$ and $\omega_{\mu,j}$, but it suffices to suppose that either $\omega_{\epsilon,0} = \omega_{\mu,0}$, $\omega_{\epsilon,1} = \omega_{\mu,1}$ or $\omega_{\epsilon,0} = \omega_{\mu,1}$, $\omega_{\epsilon,1} = \omega_{\mu,0}$.*

(a) Discuss the relation between the phase and group velocities, $\mathbf{v}_p$ and $\mathbf{v}_g$, in a metamaterial for waves of the form $e^{i(kx-\omega t)}$, where for negligible damping we can write (see chap. 6, sec. 2 of [8] or sec. 85 of [9]),

$$\mathbf{v}_g = \frac{d\omega}{dk}\hat{\mathbf{x}} = \frac{\hat{\mathbf{x}}}{dk/d\omega} = \pm\frac{c\,\hat{\mathbf{x}}}{d[\omega n]/d\omega} = \pm\frac{c\,\hat{\mathbf{x}}}{n + \omega\ dn/d\omega}. \qquad (4)$$

with $n = \pm ck/\omega$ being the index of refraction, and c is the speed of light in vacuum. For what range of material parameters, if any, can the phase and group velocity be in opposite directions and/or the index n be considered as negative?

(b) By consideration of the phases of planes waves at the interface between an ordinary transparent medium and a metamaterial, deduce the form of Snell's law for waves incident on the interface at angle θ_i from within the ordinary medium, and refracted into the metamaterial at angle θ_t to the normal.

2 Solution

2.1 Phase and Group Velocity

In a medium free of external charge and current, that can be characterized by linear, isotropic (relative) permittivity ϵ and permeability μ, Maxwell's equations are (in Gaussian units),

$$\boldsymbol{\nabla} \cdot \mathbf{D} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \tag{5}$$

and the constitutive relations are,

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H}. \tag{6}$$

These can be combined to give a wave equation for, say, the magnetic field $\mathbf{H}$,

$$\nabla^2 \mathbf{H} = \frac{\epsilon\mu}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}. \tag{7}$$

In general we consider plane waves of the form,

$$\mathbf{H} = \mathbf{H}_0\, e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \tag{8}$$

where $\mathbf{H}_0$ and $\mathbf{k}$ can be complex functions of the angular frequency ω (but not of time). The approximations of the models (1)-(2) for metamaterials permit imply that the wave vector $\mathbf{k}$ is purely real. In this case, eqs. (7) and (8) combine to give the dispersion relation,

$$\mathbf{k}^2 \equiv k^2 = \frac{\epsilon\mu\omega^2}{c^2} \equiv \frac{n^2\omega^2}{c^2} \tag{9}$$

where the scalar wave number k is defined to be positive, and n is the index of refraction. For wave propagation, both ϵ and μ must be positive, or both negative.[1] Then,

$$k = \frac{\sqrt{\epsilon\mu}\,\omega}{c} = \pm\frac{n\omega}{c}, \tag{10}$$

where the index n could be either positive or negative,

$$n = \pm\sqrt{\epsilon\mu}. \tag{11}$$

We write the wave vector as $\mathbf{k} = k\,\hat{\mathbf{k}}$ with k positive, so the waveform (8) becomes,

$$\mathbf{H} = \mathbf{H}_0\, e^{i(k\,\hat{\mathbf{k}}\cdot\mathbf{x} - \omega t)}, \tag{12}$$

such that we identify the phase velocity $\mathbf{v}_p$ as,

$$\mathbf{v}_p = \frac{\omega}{k}\,\hat{\mathbf{k}} = \frac{c\,\hat{\mathbf{x}}}{\sqrt{\epsilon\mu}} = \pm\frac{c}{n}\,\hat{\mathbf{k}}. \tag{13}$$

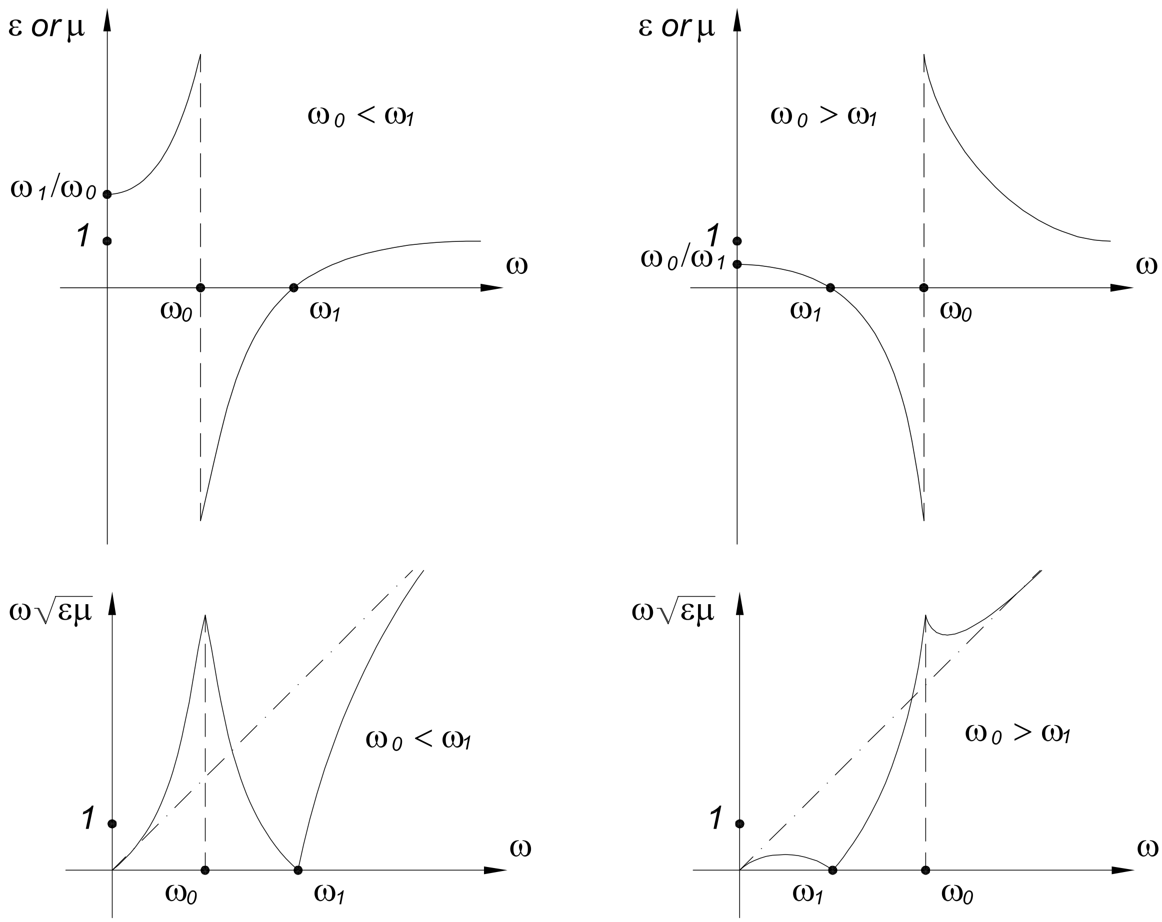
[1] For the special case that ϵ or μ (or both) vanish, see [10].

The phase-velocity vector $\mathbf{v}_p$ is in the direction of the wave vector $\mathbf{k} = k\,\hat{\mathbf{k}}$.

Turning to the group velocity, we combining eqs. (4) and (10) to write the group velocity of a packet of waves of the form (8) as,[2]

$$\mathbf{v}_g = \frac{1}{dk/d\omega}\,\hat{\mathbf{x}} = \frac{c\,\hat{\mathbf{x}}}{d[\omega\sqrt{\epsilon\mu}]/d\omega} = \pm\frac{c\,\hat{\mathbf{x}}}{d[\omega n]/d\omega}\,. \tag{14}$$

For passive metamaterials the model forms (1)-(2) hold for $\omega_0 < \omega_1$, as sketched on the left in the two figures below. In this case $d(\omega\sqrt{\epsilon\mu})/d\omega$ is negative for $\omega_0 < \omega < \omega_1$ where ϵ and μ are both negative. Thus, the phase and group velocities, eqs. (13) and (14), have the opposite signs in a doubly negative, passive metamaterial [11].



We can use the notation $n = -\sqrt{\epsilon\mu}$ together with $k = -n\omega/c$ to write the waveform (8) as,

$$\mathbf{H} = \mathbf{H}_0 e^{i(n\omega x/c - \omega t)}, \tag{15}$$

for the case of a wave with group velocity in the $+x$-direction and phase velocity in the $-x$-direction. This convention is often summarized by saying that a passive, doubly negative

[2]Care must be taken to avoid the mathematical oddity that one might write $df/d\omega = d\sqrt{(-f)^2}/d\omega = 2(1/2)\sqrt{(-f)/(-f)}\ d(-f)/d\omega = -df/d\omega$, which is a variant on the theme that while $1 = e^{2i\pi}$ we cannot say that $1 = \sqrt{1} = \sqrt{e^{2i\pi}} = e^{i\pi} = -1$.

metamaterial has a negative index of refraction.[3]

On the other hand, if active metamaterials can be constructed with behavior as shown in the right figures above, then for $\omega_1 < \omega < \omega_0$, where both ϵ and μ are negative, $d(\omega\sqrt{\epsilon\mu})/d\omega$ is positive, and the group and phase velocities are in the same direction. In this case it would be best to write simply $n = \sqrt{\epsilon\mu}$.

2.1.1 Energy Density

If a medium could have negative ϵ and negative μ at zero frequency, then the static energy density,

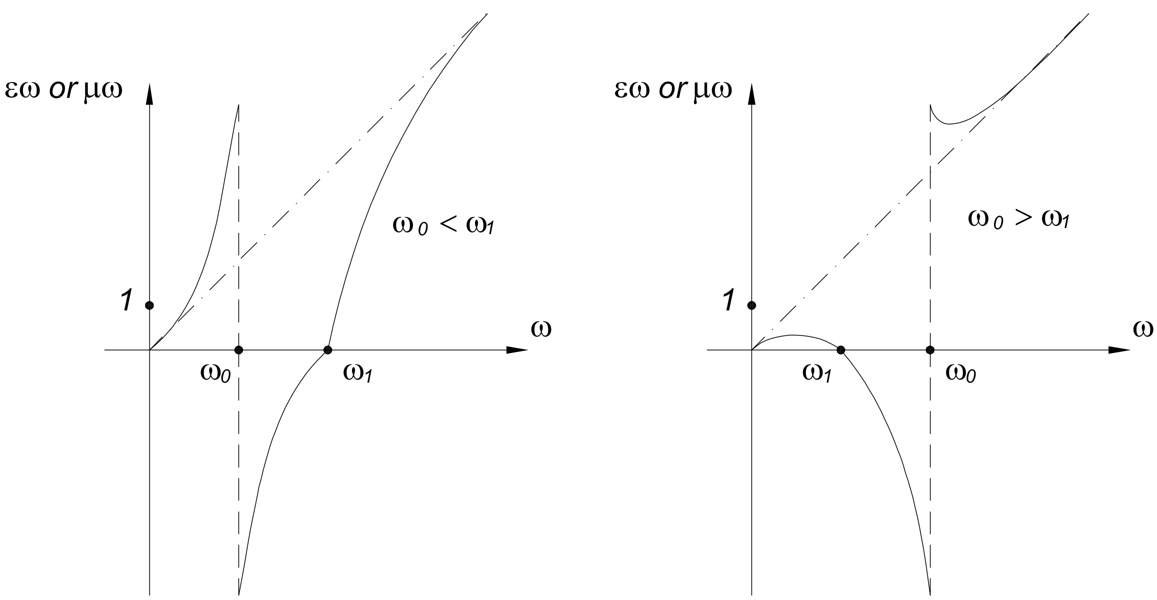
$$u = \frac{\epsilon E^2 + \mu H^2}{8\pi}\,, \tag{16}$$

would be negative, which is unreasonable for a passive material.

When the permittivity and permeability have frequency dependence, the energy density takes on a more complicated form (due to Brillouin (1932), reviewed in chap. IV of [8]; see also sec. 80 of [9]),

$$\langle u_\omega \rangle = \frac{1}{8\pi}\left(\frac{d(\epsilon\omega)}{d\omega}\left\langle E^2\right\rangle + \frac{d(\mu\omega)}{d\omega}\left\langle H^2\right\rangle\right). \tag{17}$$

The products $\epsilon\omega$ and $\mu\omega$ are illustrated for the models (1)-(2) in the figure below. We see that for the passive metamaterial ($\omega_0 < \omega_1$) the energy density is positive (as was anticipated by Veselago [1]). But for an active metamaterial the energy density is formally negative in the region $\omega_1 < \omega < \omega_0$ of doubly negative ϵ and μ. Such behavior is nonclassical, but occurs in gain media [12].[4]



[3]Of course, it is also consistent to write $n = \sqrt{\epsilon\mu}$ and $k = n\omega/c$. Conversely, any medium could be defined to have a negative index of refraction so long as we also write $k = -n\omega/c$.

[4]In the realm of electromagnetic antennas there exists lore called the "Chu limit" [13]-[17], which tacitly assumes that the density of electromagnetic energy in the "near zone" of the antenna is positive. However, if the near zone were an active medium, perhaps a metamaterial, the Chu limit would not necessarily hold [18]. It remains that to date no clear evasion of the Chu limit has been demonstrated.

2.1.2 "Left-Handed" Metamaterials

The fourth Maxwell equation (5) for a plane wave of form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ indicates that

$$\mathbf{k}\times\mathbf{E} = \frac{\omega}{c}\mathbf{B} = \frac{\mu\omega}{c}\mathbf{H} = \pm\sqrt{\frac{\mu}{\epsilon}}k\mathbf{H}, \qquad \text{or} \qquad \mathbf{H} = \pm\sqrt{\frac{\epsilon}{\mu}}\,\hat{\mathbf{k}}\times\mathbf{E}, \qquad \mathbf{B} = \sqrt{\epsilon\mu}\,\hat{\mathbf{k}}\times\mathbf{E}, \tag{18}$$

using eq. (10), where the upper sign holds for ordinary media and active metamaterials, while the lower sign holds for passive metamaterials. This result has led to the characterization of passive metamaterials as left-handed media, in contrast to ordinary media, and active metamaterials, for which $\hat{\mathbf{k}}$, $\mathbf{E}$ and $\mathbf{H}$ form a right-handed triad.[5] According to the discussion in sec. 2.1, only left-handed media should be characterized as having a negative index of refraction.

2.1.3 Momentum Density

In 1903 Max Abraham noted [19] that the Poynting vector [20], which describes the flow of energy in the electromagnetic field,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}, \tag{19}$$

when divided by c^2 has the additional significance of being the density of momentum stored in the electromagnetic field,[6]

$$\mathbf{p}_{\rm EM}^{\rm (A)} = \frac{\mathbf{E}\times\mathbf{H}}{4\pi c} \qquad \text{(Abraham)}. \tag{20}$$

Of course, $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ and $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, where $\mathbf{P}$ and $\mathbf{M}$ are the densities of electric and magnetic polarization, respectively.

In 1908 Hermann Minkowski gave an alternative derivation [23] that the electromagnetic-momentum density is,[7]

$$\mathbf{p}_{\rm EM}^{\rm (M)} = \frac{\mathbf{D}\times\mathbf{B}}{4\pi c} \qquad \text{(Minkowksi)}, \tag{21}$$

and the debate over the merits of these two expressions continues to this day. Minkowski died before adding to the debate, while Abraham published several times on it [25]. For recent reviews, see [26]-[29].

As both fields $\mathbf{D}$ and $\mathbf{H}$ include aspects of the material medium, if any, we consider the field-only momentum density to be simply,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c} \qquad \text{(field only)}. \tag{22}$$

This momentum density is directed along the wave vector $\mathbf{k}$, which is opposite to the group velocity, in doubly negative metamaterials.

[5]Note that $\hat{\mathbf{k}}$, $\mathbf{E}$ and $\mathbf{B}$ form a right-handed triad in all materials.

[6]This was discussed earlier by J.J. Thomson [21] and by Poincaré [22].

[7]See also, for example, sec. 2.1 of [24].

In general,

$$\mathbf{p}_{\mathrm{EM}}^{(\mathrm{A})} = \mu \mathbf{p}_{\mathrm{EM}}, \qquad \mathbf{p}_{\mathrm{EM}}^{(\mathrm{M})} = \epsilon \mathbf{p}_{\mathrm{EM}}, \tag{23}$$

such that in doubly negative metamaterials both $\mathbf{p}_{\mathrm{EM}}^{(\mathrm{A})}$ and $\mathbf{p}_{\mathrm{EM}}^{(\mathrm{M})}$ point in the direction of the group velocity, but are equal only if $\epsilon = \mu$.

In a dispersive medium with index $n(\omega)$ it is useful to introduce the quantity,

$$n_g = \frac{c}{v_g} = c\frac{dk}{d\omega} = \frac{d(\omega n)}{d\omega} = n + \omega \frac{dn}{d\omega}\,, \tag{24}$$

which is sometimes called the group-velocity index. This velocity is usually positive in a passive medium, but is negative near optical resonances [12]. The emerging consensus [24, 26, 27, 28] is that the Abraham momentum density corresponds to the momentum of a photon of angular frequency ω in a medium of group-velocity index n_g being $\hbar\omega \mathbf{v}_g/c^2 = \hbar\omega\,\hat{\mathbf{v}}_g/n_g c$, and is sometimes call the kinetic momentum density. The Minkowski momentum density corresponds to the momentum $n^2\hbar\omega \mathbf{v}_g/c^2 = n^2\hbar\omega\,\hat{\mathbf{v}}_g/n_g c$ of a photon of angular frequency ω, and is sometimes called the pseudomomentum or the quasimomentum.[8] The Abraham and Minkowski momenta are always in the same direction for ordinary media and for doubly negative metamaterials.

The momentum of a photon most often used in quantum theory is $\hbar\mathbf{k} = n\hbar\omega\,\hat{\mathbf{k}}/c$, which is often called the canonical momentum. The canonical momentum is in the direction of the wave vector $\mathbf{k}$, and so has the same direction as the field-only momentum density (22). In most media the Minkowski momentum is the same as the canonical momentum, but they are opposite in a passive, doubly negative, nondispersive metamaterial [30].[9]

Which of these momenta is the most physically relevant depends on the details of the experiment to measure the momentum.

2.2 Snell's Law

We consider the case of an ordinary medium with $\epsilon_1 > 0$, $\mu_1 > 0$ at $x < 0$, and a doubly negative metamaterial with $\epsilon_2 < 0$, $\mu_2 < 0$ at $x > 0$, as sketched in the figure on the next page.

We also consider plane waves of the form $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. One condition on the incident, reflected and transmitted waves is that the phase of all three waves must be the same at any point $(0, y, z)$ on the interface $x = 0$. This implies that,

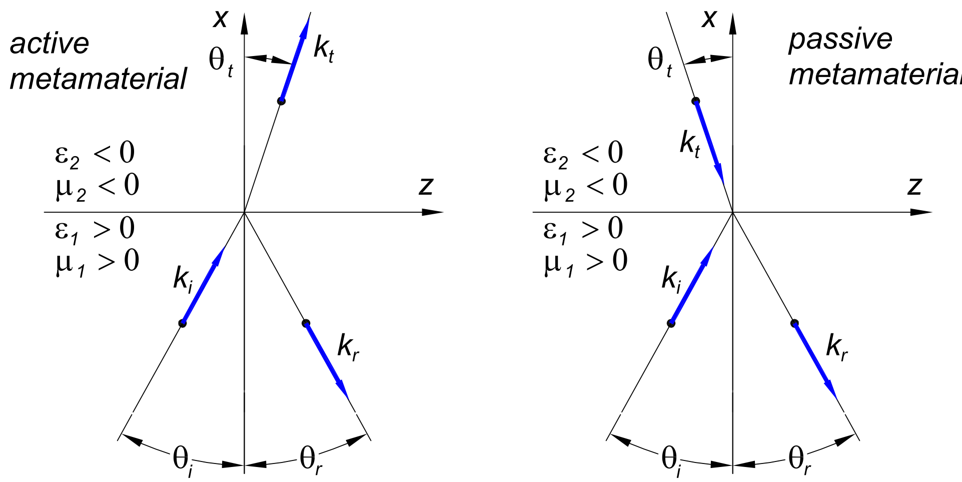
$$k_i \sin\theta_i = k_r \sin\theta_r = \pm k_t \sin\theta_t, \tag{25}$$

where for the moment we suppose that the transmitted wave is "ordinary" in the sense that the transmitted ray which passes through the origin lies in the first quadrant of the x-z

[8] Some people prefer to define the Minkowksi momentum as $\hbar\mathbf{k}$, in which case it is identical to the canonical momentum [26]. This view comes from consideration of the wave as a polariton, which is not strictly a photon, as a polariton includes matter, charged or neutral, in its wave function. The momentum of the polariton is $n^2\hbar\omega\mathbf{v}_g/c^2$, of which it is considered that the photon part is $\hbar\mathbf{k}$. I am not presently persuaded of the merits of this decomposition of a quantum state.

[9] So-called electrostatic waves can exist in media where $\mathbf{D}$, $\mathbf{B}$ and $\mathbf{H}$ are zero , while the wave is associated with a momentum density in the medium that is opposite to the direction of the wave vector [31].

plane, as shown in the left figure above for an active metamaterial. Then, the upper (lower) sign in eq. (25) holds for an active (passive) metamaterial.



Since the incident and reflected waves are in the same medium the magnitudes of the wave numbers k_i and k_r are the same (for waves of a given frequency ω),

$$k_i = k_r = \frac{\sqrt{\epsilon_1 \mu_1}\,\omega}{c} = \frac{n_1 \omega}{c}\,, \tag{26}$$

and hence,

$$\theta_r = \theta_i. \tag{27}$$

If we designate k_t to be a positive quantity,

$$k_t = \frac{\sqrt{\epsilon_2 \mu_2}\,\omega}{c} = \pm \frac{n_2 \omega}{c}\,, \tag{28}$$

where the index of refraction,

$$n_2 = \pm\sqrt{\epsilon_2 \mu_2}, \tag{29}$$

can be considered as negative for a passive metamaterial, as discussed in sec. 2.1. Then, eq. (25) also indicates that,

$$n_1 \sin\theta_i = n_2 \sin\theta_t, \tag{30}$$

where n_2, and hence θ_t as well, are negative for passive metamaterials. In the latter case, the transmitted ray which passes through the origin lies in the second quadrant of the x-z plane, as shown in the right figure above. This behavior is often called negative refraction.[10,11]

[10]Thus, we have the option of retaining Snell's law in the form $n_1 \sin\theta_i = n_2 \sin\theta_t$ for medium 2 being a passive metamaterial by considering n_2 and θ_t as both negative, with θ_t defined to be positive in the first quadrant of the x-z plane. Alternatively, we could consider n_2 to be positive, and modify Snell's law to be $n_1 \sin\theta_i = -n_2 \sin\theta_t$, where again θ_t is defined to be positive in the first quadrant of the x-z plane. And yet another alternative is to write Snell's law as $n_1 \sin\theta_i = n_2 \sin\theta_t$ with n_2 positive, but to define θ_t to be positive in the second quadrant of the x-z plane. Among these three alternatives the first is perhaps the most pleasing.

[11]Negative refraction in the sense considered here can also be occur for materials with positive $Re(n)$ if the losses are sufficiently high ($Im(n)$ large) [32, 33].

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