Fields and Moments of a Moving Electric Dipole

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1 Problem

Discuss the fields of a moving point electric dipole, and characterize these fields in terms of multipole moments. It suffices to consider uniform motion with velocity v small compared to the speed c of light in vacuum.

2 Solution

This note is an elaboration of [1].

The relativistic transformations of densities $\mathbf{P}$ and $\mathbf{M}$ of electric and magnetic dipole moments (also called polarization and magnetization densities, respectively) were first discussed by Lorentz [2], who noted that they follow the same transformations as do the magnetic and electric fields $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ and $\mathbf{E} = \mathbf{D} - 4\pi\mathbf{P}$, respectively,

$$\mathbf{P} = \gamma\left(\mathbf{P}_0 + \frac{\mathbf{v}}{c} \times \mathbf{M}_0\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{P}_0)\,\hat{\mathbf{v}}, \quad \mathbf{M} = \gamma\left(\mathbf{M}_0 - \frac{\mathbf{v}}{c} \times \mathbf{P}_0\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{M}_0)\,\hat{\mathbf{v}}, \tag{1}$$

where the inertial rest frame moves with velocity $\mathbf{v}$ with respect to the (inertial) lab frame of the polarization densities, and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

Considerations of the fields moving electric dipoles perhaps first arose in the context of Čerenkov radiation when the dipole velocity v exceeds the speed of light c/n in a medium of index of refraction n [3].[2]

Later discussions by Frank of his pioneering work are given in [4, 5].

As noted by Frank [5], specialization of eq. (1) to point electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$, and to low velocities, leads to the forms,

$$\mathbf{p} \approx \mathbf{p}_0 + \frac{\mathbf{v}}{c} \times \mathbf{m}_0, \qquad \mathbf{m} \approx \mathbf{m}_0 - \frac{\mathbf{v}}{c} \times \mathbf{p}_0, \qquad \mathbf{p}_0 \approx \mathbf{p} - \frac{\mathbf{v}}{c} \times \mathbf{m}, \qquad \mathbf{m}_0 \approx \mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p}, \tag{2}$$

where $\mathbf{p}_0$ and $\mathbf{m}_0$ and the moments in the rest frame of the point particle, while $\mathbf{p}$ and $\mathbf{m}$

[1]This note was written by the author in his private capacity. No official support or endorsement by the Center for Disease Control and Prevention is intended or should be inferred.

[2]That a moving current leads to an apparent charge separation was noted in [6] (1880). Considerations of the "motional" electric dipole, $\mathbf{p} = \mathbf{v}/c \times \mathbf{m}_0$ when $\mathbf{p}_0 = 0$, due the motion of a magnetic dipole appears, in eq. (17) of [7]. See also eq. (13) of [8], which was inspired by Art. 600 of [9]. This effect was also discussed by Frenkel [10, 11] following the prediction by Uhlenbeck and Goudsmit [12] that an electron has an intrinsic magnetic moment.

are the moments when the particle has velocity $\mathbf{v}$ in the lab frame.[3,4,5] However, the fields associated with an electric and/or magnetic dipole moving at low velocity are not simply the instantaneous fields of the moments $\mathbf{p}$ and $\mathbf{m}$, which leads to ambiguities in interpretations of the fields as due to moments.

Before deducing the fields, we note a past misunderstanding regarding eq. (2).

2.1 Fisher's Claim

In sec. III of [14] it is claimed that the magnetic moment of a point electric dipole $\mathbf{p}_0$ which moves with velocity $\mathbf{v}$ can be calculated according to,

$$\mathbf{m} = \frac{1}{2c}\int \mathbf{r}\times\mathbf{J}\,d\mathrm{Vol} = \frac{1}{2c}\int \mathbf{r}\times\rho_0\mathbf{v}\,d\mathrm{Vol} = -\frac{\mathbf{v}}{2c}\times\int\rho_0\mathbf{r}\,d\mathrm{Vol} = -\frac{\mathbf{v}}{2c}\times\mathbf{p}_0, \tag{4}$$

which differs from eq. (2) by a factor of 2.

Furthermore, support for this result appears to be given in probs. 6.21, 6.22 and 11.27 of [15].

However, we should recall that the origin of the first equality in (4) is a multipole expansion of the (quasistatic) vector potential of a current distribution. See, for example, sec. 5.6 of [15]. That form depends on the current density $\mathbf{J}$ having zero divergence. In general,

$$\boldsymbol{\nabla}\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}\,, \tag{5}$$

where ρ is the electric charge density.[6] The charge density of a moving electric dipole is time dependent, such that $\boldsymbol{\nabla}\cdot\mathbf{J}\neq 0$, and we cannot expect the analysis of eq. (4) to be valid.[7]

2.2 The Fields of a Moving Electric Dipole

The literature on calculations of the fields of a moving, point electric dipole is very extensive, including [16]-[33]. Most of these works emphasize the radiation fields of an ultrarelativistic dipole. Here, we consider only the low-velocity limit of an electric dipole, of strength $\mathbf{p}_0$ in its rest frame, that is at the origin at time $t = 0$ with velocity $\mathbf{v}$. We work in the quasistatic approximation, in which the potentials are the same in the Coulomb and Lorenz gauges.

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[3] Ref. [5] recounts a past controversy that these transformations might involve the index n if the magnetic dipole were not equivalent to an Ampèrian current loop.

[4] The moments $\mathbf{p}$ and $\mathbf{m}$ associated with the densities $\mathbf{P}$ and $\mathbf{M}$ in a volume $V = V_0/\gamma$ transform for arbitrary v/c according to,

$$\mathbf{p} = \mathbf{p}_0 + \frac{\mathbf{v}}{c}\times\mathbf{m}_0 - (1-1/\gamma)(\hat{\mathbf{v}}\cdot\mathbf{p}_0)\,\hat{\mathbf{v}}, \quad \mathbf{m} = \mathbf{m}_0 - \frac{\mathbf{v}}{c}\times\mathbf{p}_0 - (1-1/\gamma)(\hat{\mathbf{v}}\cdot\mathbf{m}_0)\,\hat{\mathbf{v}}. \tag{3}$$

[5] In experimental searches for electric dipole moments of electrons, neutrons, *etc.*, which have intrinsic magnetic moments, the discussion (see, for example, [13]) emphasizes the rest frame of the particle, where the magnetic interaction energy due to lab-frame fields $\mathbf{B}$ and $\mathbf{E}$ is $U \approx -\mathbf{m}_0\cdot(\mathbf{B}-\mathbf{v}/c\times\mathbf{E}) - \mathbf{p}_0\cdot(\mathbf{E}+\mathbf{v}/c\times\mathbf{B})$, where $\mathbf{v}$ is the lab-frame velocity of the particle. This could be rewritten entirely in terms of lab-frame quantities as $U \approx -\mathbf{m}\cdot\mathbf{B} - \mathbf{p}\cdot\mathbf{E}$, but this in not done in the literature.

[6] For moving media, $\boldsymbol{\nabla}\cdot\mathbf{J} = -\partial\rho/\partial t - (\mathbf{v}\cdot\boldsymbol{\nabla})\rho = -d\rho/dt$.

[7] Thanks to Grigory Vekstein for pointing this out.

2.2.1 Multipole Expansions of the Potentials

Following the spirit of [14], and secs. 4.1 and 5.6 of [15], we consider multipole expansions of the scalar potential V and the vector potential $\mathbf{A}$ at time $t = 0$, when the moving electric dipole is that the origin. Then, the (quasistatic) scalar potential is,

$$V(\mathbf{r}, t = 0) = \frac{\mathbf{p}_0 \cdot \mathbf{r}}{r^3}\,, \tag{6}$$

as the charge distribution has only a dipole moment at this time. The (quasistatic) vector potential has the expansion,

$$\begin{aligned} A_i(\mathbf{r}, t = 0) &= \frac{1}{cr} \int J_i(\mathbf{r}', t = 0)\, d\mathrm{Vol}' + \frac{\mathbf{r}}{cr^3} \cdot \int \mathbf{r}' J_i(\mathbf{r}', t = 0)\, d\mathrm{Vol}' + \cdots \\ &= \frac{\mathbf{v}}{cr} \int \rho_0(\mathbf{r}', t = 0)\, d\mathrm{Vol}' + \frac{v_i \mathbf{r}}{cr^3} \cdot \int \mathbf{r}' \rho_0(\mathbf{r}', t = 0)\, d\mathrm{Vol}' + \cdots \\ &= \frac{v_i \mathbf{r} \cdot \mathbf{p}_0}{cr^3}, \end{aligned} \tag{7}$$

where ρ_0 is the electric charge distribution of the electric dipole in its rest frame, $\int \rho_0(\mathbf{r})\, d\mathrm{Vol} = 0$, $\int \rho_0(\mathbf{r})\, \mathbf{r}\, d\mathrm{Vol} = \mathbf{p}_0$, and $\mathbf{J} = \rho_0 \mathbf{v}$ is the current density of the moving electric dipole in the lab frame. It is perhaps not evident that all higher-order terms vanish in eq. (7), but this will be confirmed in sec. 2.2.2. Then,

$$\mathbf{A}(\mathbf{r}, t = 0) = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v}}{cr^3} = \left(-\frac{\mathbf{v}}{c} \times \mathbf{p}_0\right) \times \frac{\mathbf{r}}{r^3} + \frac{(\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{cr^3} \equiv \mathbf{A}_m + \mathbf{A}_p, \tag{8}$$

where,

$$\mathbf{A}_m = \frac{\mathbf{m} \times \mathbf{r}}{r^3} = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v} - (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{cr^3} \qquad \text{with} \qquad \mathbf{m} = -\frac{\mathbf{v}}{c} \times \mathbf{p}_0, \tag{9}$$

which is the vector potential expected from eq. (2) for a moving electric dipole moment (with no intrinsic magnetic moment), and,

$$\mathbf{A}_p = \frac{(\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{cr^3}\,. \tag{10}$$

It is something of a convention to say that $\mathbf{A}_m$ is the vector potential of the magnetic moment of the moving electric dipole in that the term $\mathbf{A}_p$ is also a vector potential with $1/r^2$ dependence as associated with dipole potentials. The decomposition $\mathbf{A} = \mathbf{A}_m + \mathbf{A}_p$ is of possible mathematical convenience but has no well-defined physical significance. For example, we could also write,

$$\mathbf{A} = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v}}{cr^3} = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v} - (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{2cr^3} + \frac{(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v} + (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{2cr^3} = \mathbf{A}_a + \mathbf{A}_s, \tag{11}$$

where,

$$\mathbf{A}_a = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v} - (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{2cr^3} = \frac{1}{2}\mathbf{A}_m, \qquad \mathbf{A}_s = \frac{(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v} + (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{2cr^3}\,, \tag{12}$$

are the antisymmetric and symmetric combinations of the terms $(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v}/2cr^3$ and $(\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}/2cr^3$. The decomposition (11) is advocated in probs. 6.21 and 6.22 of [15], but should not be construed as implying that the magnetic moment of the moving electric dipole is $\mathbf{A}_m/2$, just as the decomposition (8) does not imply that the magnetic moment of the moving electric dipole is $\mathbf{A}_m$.[8]

It would seem most correct simply to say that $1/r^2$ term in the vector potential of the moving electric dipole is $(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v}/cr^3$ without supposing this to be the sum of two effects of different physical character. Nonetheless, we can be led to the decompositions (8) and (11) by other arguments, each compelling in its way, as considered in secs. 2.2.3 and 2.2.4.

2.2.2 Fields and Potentials via Lorentz Transformations

Before going further, it is useful to note the most direct method of obtaining the potentials and fields of a moving electric dipole is via a Lorentz transformation from its rest frame, which has velocity $\mathbf{v}$ with $v \ll c$ with respect to the lab frame.

The potentials in the rest frame of the electric dipole, assumed to be at the origin, are,

$$V^\star = \frac{\mathbf{p}_0 \cdot \mathbf{r}^\star}{r^{\star 3}}, \qquad \mathbf{A}^\star = 0, \tag{13}$$

where quantities in the rest frame are denoted with the superscript *. The Lorentz transformation of the 4-vectors $(ct, \mathbf{r})$ and $(V, \mathbf{A})$ to the lab frame at time $t = 0$ yield, for $v \ll c$, where $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1$,

$$\mathbf{r}_\parallel^\star = \gamma(\mathbf{r}_\parallel - \mathbf{v}t) \approx \mathbf{r}_\parallel, \qquad \mathbf{r}_\perp^\star = \mathbf{r}_\perp, \qquad i.e., \qquad \mathbf{r}^\star \approx \mathbf{r}, \tag{14}$$

$$V = \gamma(V^\star + \mathbf{A}^\star \cdot \mathbf{v}/c) \approx V^\star = \frac{\mathbf{p}_0 \cdot \mathbf{r}^\star}{r^{\star 3}} \approx \frac{\mathbf{p}_0 \cdot \mathbf{r}}{r^3}, \tag{15}$$

$$\mathbf{A} = \gamma(\mathbf{A}^\star + V^\star \mathbf{v}/c) \approx \frac{(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v}}{cr^3}. \tag{16}$$

Hence, the multipole expansion (8) of the vector potential $\mathbf{A}$ at order $1/r^2$ is actually the "exact" result in the low-velocity approximation.

While the fields $\mathbf{E}$ and $\mathbf{B}$ can now be calculated from the potentials according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{17}$$

it is more straightforward to deduce the fields $\mathbf{E}$ and $\mathbf{B}$ as the Lorentz transforms of the fields in the rest frame of the electric dipole,

$$\mathbf{E}^\star = \frac{3(\mathbf{p}_0 \cdot \mathbf{r}^\star)\mathbf{r}^\star}{r^{\star 5}} - \frac{\mathbf{p}_0}{r^{\star 3}} - \frac{4\pi}{3}\mathbf{p}_0 \delta^3 \mathbf{r}^\star, \qquad \mathbf{B}^\star = 0. \tag{18}$$

Then,[9] at time $t = 0$ when $\mathbf{r}^\star \approx \mathbf{r}$,

$$\mathbf{E} = \gamma\left(\mathbf{E}^\star - \frac{\mathbf{v}}{c} \times \mathbf{B}^\star\right) - (\gamma - 1)(\mathbf{E}^\star \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} \approx \mathbf{E}^\star = \frac{3(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}_0}{r^3} - \frac{4\pi}{3}\mathbf{p}_0 \delta^3 \mathbf{r}, \tag{19}$$

[8]That Jackson was aware of both decompositions (8) and (11) is indicated in a letter of Mar. 20, 1990 on this theme [34], and somewhat indirectly in prob. 11.28 of [15]. *V.H. thanks J.D.J. for a copy of this letter.*

[9]See, for example, sec. 11.10 of [15].

$$
\begin{aligned}
\mathbf{B} &= \gamma\left(\mathbf{B}^\star + \frac{\mathbf{v}}{c}\times\mathbf{E}^\star\right) - (\gamma-1)(\mathbf{B}^\star\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} \approx \frac{\mathbf{v}}{c}\times\mathbf{E}^\star \approx \frac{\mathbf{v}}{c}\times\mathbf{E} \\
&= \frac{\mathbf{v}}{c}\times\left(\frac{3(\mathbf{p}_0\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}_0}{r^3} - \frac{4\pi}{3}\mathbf{p}_0\delta^3\mathbf{r}\right) \equiv \mathbf{B}_m + \mathbf{B}_p \equiv \mathbf{B}_a + \mathbf{B}_s,
\end{aligned} \tag{20}
$$

where the partial fields $\mathbf{B}_m$, $\mathbf{B}_p$, $\mathbf{B}_a$ and $\mathbf{B}_s$ will be displayed in secs. 2.2.3 and 2.2.4.

2.2.3 Potentials Deduced from Lab-Frame Charge and Current Densities

In its rest-frame the point electric dipole $\mathbf{p}_0$ (located at the origin) has polarization density,

$$
\mathbf{P}^\star = \mathbf{p}_0\,\delta^3\mathbf{r}^\star. \tag{21}
$$

The associated charge density is,

$$
\rho^\star = -\boldsymbol{\nabla}^\star\cdot\mathbf{P}^\star = -\mathbf{p}_0\cdot\boldsymbol{\nabla}^\star\delta^3\mathbf{r}^\star. \tag{22}
$$

In the lab frame the charge density is,

$$
\rho = \gamma\rho^\star \approx \rho^\star \approx -\mathbf{p}_0\cdot\boldsymbol{\nabla}\delta^3\mathbf{r} = -\boldsymbol{\nabla}\cdot\mathbf{P}, \tag{23}
$$

where the lab-frame polarization density is,

$$
\mathbf{P} = \mathbf{p}_0\,\delta^3\mathbf{r}, \tag{24}
$$

at the instant when the dipole is at the origin, noting that $\mathbf{r}\approx\mathbf{r}^\star$ and $\boldsymbol{\nabla}^\star = \boldsymbol{\nabla}$ in the low-velocity approximation.[10]

When the dipole is moving with a velocity $\mathbf{v}$ in the lab frame it has current density,

$$
\begin{aligned}
\mathbf{J} &= \gamma\rho^\star\mathbf{v} \approx \rho^\star\mathbf{v} = -\mathbf{v}(\mathbf{p}_0\cdot\boldsymbol{\nabla})\delta^3\mathbf{r} \\
&= -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{p}_0\,\delta^3(\mathbf{r}) - \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{p}_0\,\delta^3\mathbf{r}) + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{p}_0\,\delta^3\mathbf{r} \\
&= -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{p}_0\,\delta^3\mathbf{r} - \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{p}_0\,\delta^3\mathbf{r}) \\
&\equiv \mathbf{J}_p + \mathbf{J}_m,
\end{aligned} \tag{25}
$$

noting that the operator $\boldsymbol{\nabla}$ does not act on the constant vectors $\mathbf{p}_0$ and $\mathbf{v}$. The current density $\mathbf{J}_p$ can be thought of as an "electric-polarization current",

$$
\mathbf{J}_p = -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{p}_0\,\delta^3\mathbf{r} \equiv -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{P} = -\frac{d\mathbf{P}}{dt}\,. \tag{26}
$$

Likewise, the current density $\mathbf{J}_m$ can be thought of as a "magnetic-polarization current",

$$
\mathbf{J}_m = -\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{p}_0\,\delta^3\mathbf{r}) = c\boldsymbol{\nabla}\times\mathbf{m}\,\delta^3\mathbf{r} \equiv c\boldsymbol{\nabla}\times\mathbf{M}, \tag{27}
$$

using the convention that $\mathbf{m} = -\mathbf{v}/c\times\mathbf{p}_0$ of eq. (2), and

$$
\mathbf{M} = \mathbf{m}\,\delta^3\mathbf{r}. \tag{28}
$$

[10] The electric field $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ has zero divergence, $\boldsymbol{\nabla}\cdot\mathbf{D} = \boldsymbol{\nabla}\cdot\mathbf{E} + 4\pi\boldsymbol{\nabla}\cdot\mathbf{P} = 4\pi\rho + 4\pi\boldsymbol{\nabla}\cdot\mathbf{P} = 0$, in view of eq. (23).

Then the sum of the polarization currents (26) and (27) equals the “convection” current (25).

The (quasistatic) vector potentials associated with the polarization currents (26) and (27) when the electric dipole is at the origin in the lab frame are,

$$\begin{aligned}\mathbf{A}_p(\mathbf{r}) &= \frac{1}{c}\int \frac{\mathbf{J}_p(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' = -\frac{1}{c}\int \frac{(\mathbf{v}\cdot\boldsymbol{\nabla}')\mathbf{p}_0\,\delta^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' = -\frac{\mathbf{p}_0}{c}\int \frac{(\mathbf{v}\cdot\boldsymbol{\nabla}')\,\delta^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' \\ &= \frac{\mathbf{p}_0}{c}\int \frac{[\mathbf{v}\cdot(\mathbf{r}-\mathbf{r}')]\,\delta^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,d\mathrm{Vol}' = \frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{p}_0}{cr^3}\,, \end{aligned} \tag{29}$$

as previously found in eq. (10), and,

$$\begin{aligned}\mathbf{A}_m(\mathbf{r}) &= \frac{1}{c}\int \frac{\mathbf{J}_m(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' = \int \frac{\boldsymbol{\nabla}'\times\mathbf{m}\,\delta^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' = -\mathbf{m}\times\int \frac{\boldsymbol{\nabla}'\,\delta^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}' \\ &= \mathbf{m}\times\int \frac{(\mathbf{r}-\mathbf{r}')\,\delta^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,d\mathrm{Vol}' = \frac{\mathbf{m}\times\mathbf{r}}{r^3} = -\frac{(\mathbf{v}\times\mathbf{p}_0)\times\mathbf{r}}{cr^3} = \frac{(\mathbf{p}_0\cdot\mathbf{r})\mathbf{v}-(\mathbf{r}\cdot\mathbf{v})\mathbf{p}_0}{cr^3}\,, \end{aligned} \tag{30}$$

as previously found in eq. (9).

The magnetic field (20) can now be written as,

$$\mathbf{B} = \frac{\mathbf{v}}{c}\times\left(\frac{3(\mathbf{p}_0\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}_0}{r^3}\right) = \mathbf{B}_m + \mathbf{B}_p. \tag{31}$$

where,

$$\begin{aligned}\mathbf{B}_m = 2\mathbf{B}_a &= \boldsymbol{\nabla}\times\mathbf{A}_m = -\boldsymbol{\nabla}\times\frac{(\mathbf{v}\times\mathbf{p}_0)\times\mathbf{r}}{cr^3} \\ &= -\frac{\mathbf{v}\times\mathbf{p}_0}{cr^3}\,\boldsymbol{\nabla}\cdot\mathbf{r} + \mathbf{r}\left(\boldsymbol{\nabla}\cdot\frac{\mathbf{v}\times\mathbf{p}_0}{cr^3}\right) - (\mathbf{r}\cdot\boldsymbol{\nabla})\frac{\mathbf{v}\times\mathbf{p}_0}{cr^3} + \left(\frac{\mathbf{v}\times\mathbf{p}_0}{cr^3}\cdot\boldsymbol{\nabla}\right)\mathbf{r} \\ &= -\frac{3\mathbf{v}\times\mathbf{p}_0}{cr^3} - \frac{3[\mathbf{r}\cdot(\mathbf{v}\times\mathbf{p}_0)]\mathbf{r}}{cr^5} + \frac{3\mathbf{v}\times\mathbf{p}_0}{cr^3} + \frac{\mathbf{v}\times\mathbf{p}_0}{cr^3} \\ &= \frac{3[(-\mathbf{v}/c\times\mathbf{p}_0)\cdot\mathbf{r}]\mathbf{r}}{r^5} - \frac{-\mathbf{v}/c\times\mathbf{p}_0}{r^3} = \frac{3(\mathbf{m}\cdot\mathbf{r})]\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3} \\ &= \boldsymbol{\nabla}\times\frac{(\mathbf{p}_0\cdot\mathbf{r})\mathbf{v}-(\mathbf{r}\cdot\mathbf{v})\mathbf{p}_0}{cr^3} \\ &= (\mathbf{p}_0\cdot\mathbf{r})\boldsymbol{\nabla}\times\frac{\mathbf{v}}{cr^3} - \frac{\mathbf{v}}{cr^3}\times\boldsymbol{\nabla}(\mathbf{p}_0\cdot\mathbf{r}) - (\mathbf{v}\cdot\mathbf{r})\boldsymbol{\nabla}\times\frac{\mathbf{p}_0}{cr^3} + \frac{\mathbf{p}_0}{cr^3}\times\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{r}) \\ &= -\frac{3(\mathbf{p}_0\cdot\mathbf{r})\mathbf{r}\times\mathbf{v}}{cr^5} - \frac{\mathbf{v}\times\mathbf{p}_0}{cr^3} + \frac{3(\mathbf{v}\cdot\mathbf{r})\mathbf{r}\times\mathbf{p}_0}{cr^5} + \frac{\mathbf{p}_0\times\mathbf{v}}{cr^3} \\ &= \mathbf{v}\times\left(\frac{3(\mathbf{p}_0\cdot\mathbf{r})\mathbf{r}}{cr^5} - \frac{\mathbf{p}_0}{cr^3}\right) - \mathbf{p}_0\times\left(\frac{3(\mathbf{v}\cdot\mathbf{r})\mathbf{r}}{cr^5} - \frac{\mathbf{v}}{cr^3}\right) \\ &= \mathbf{v}\times\frac{3(\mathbf{p}_0\cdot\mathbf{r})\mathbf{r}}{cr^5} - \frac{2\mathbf{v}\times\mathbf{p}_0}{cr^3} - \mathbf{p}_0\times\frac{3(\mathbf{v}\cdot\mathbf{r})\mathbf{r}}{cr^5}\,, \end{aligned} \tag{32}$$

and,

$$\mathbf{B}_p = \boldsymbol{\nabla}\times\mathbf{A}_p = \boldsymbol{\nabla}\times\frac{(\mathbf{r}\cdot\mathbf{v})\mathbf{p}_0}{cr^3} = (\mathbf{r}\cdot\mathbf{v})\boldsymbol{\nabla}\times\frac{\mathbf{p}_0}{cr^3} + \boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{v})\times\frac{\mathbf{p}_0}{cr^3}$$

$$
\begin{aligned}
&= -3(\mathbf{r}\cdot\mathbf{v})\mathbf{r}\times\frac{\mathbf{p_0}}{cr^5} + \mathbf{v}\times\frac{\mathbf{p_0}}{cr^3} = \mathbf{p}_0\times\left(\frac{3(\mathbf{r}\cdot\mathbf{v})\mathbf{r}}{cr^5} - \frac{\mathbf{v}}{cr^3}\right) \\
&= \mathbf{v}\times\frac{\mathbf{p}_0}{cr^3} + \mathbf{p}_0\times\frac{3(\mathbf{r}\cdot\mathbf{v})\mathbf{r}}{cr^5}\,. \qquad (33)
\end{aligned}
$$

The consistency of this decomposition tends to reinforce the identification of eq. (27) as "the" magnetic moment of the moving electric dipole, even though the quantities $\mathbf{A}_p$ and $\mathbf{B}_p$ also have the $1/r^2$ and $1/r^3$ dependence, respectively, expected for dipole potentials and fields. However, alternative decompositions of the electric and magnetic lead to different interpretations. One such decomposition is considered in the following section.

2.2.4 Jackson's Argument

The preceding analyses have assumed for simplicity that the electric dipole is at the origin. Problems 6.21 and 6.22 of [15] encourage us to consider the situation when the electric dipole is not at the origin.

If the dipole is at position $\mathbf{r}_0$, most of the preceding results still hold with the substitutions $\mathbf{r}\to\mathbf{r}-\mathbf{r}_0$ and $r\to|\mathbf{r}-\mathbf{r}_0|$. However, the multipole moments of charge and current densities depend on the choice of origin.

The total electric charge, $Q=\int\rho\,d\mathrm{Vol}$ is, of course, independent of the choice of origin. The electric dipole moment $\mathbf{p}=\int\rho\,\mathbf{r}\,d\mathrm{Vol}$ is independent of the choice of origin if the total charge Q is zero (as for an electric dipole moment). Hence, the electric dipole moment of a dipole charge distribution is independent of the choice of origin. However, the electric quadrupole moment of a dipole charge distribution depends on the choice of origin.

The magnetic moment of a current density that obeys $\boldsymbol{\nabla}\cdot\mathbf{J}=0$ is independent of the choice of origin. In such cases $\int\mathbf{J}\,d\mathrm{vol}=0$, and we can write,

$$
\mathbf{m}=\frac{1}{c}\int\mathbf{r}\times\mathbf{J}\,d\mathrm{Vol}\to\frac{1}{c}\int(\mathbf{r}-\mathbf{r}_0)\times\mathbf{J}\,d\mathrm{Vol}=\mathbf{m}+\frac{\mathbf{r}_0}{c}\times\int\mathbf{J}\,d\mathrm{Vol}=\mathbf{m}. \qquad (34)
$$

In the present example of a moving electric dipole the current density has nonzero divergence, and the magnetic dipole moment (and higher magnetic moments) depend on the choice of origin. This reinforces the sense of previous sections that it is difficult to identify "the" magnetic moment of a moving electric dipole.

Turning to the argument implied in probs. 6.21 and 6.22 of [15], it is suggested that the vector potential (16) be rewritten in the form (11) as the sum of antisymmetric and symmetric terms. The motivation here is not obvious in quasistatic examples, but this technique is of use when considering a Taylor expansion of the retarded vector potential of a time-harmonic current distribution $\mathbf{J}(\mathbf{r},t)=\mathbf{J}(\mathbf{r})\,e^{-i\omega t}$, at distances far from these currents,

$$
\begin{aligned}
\mathbf{A}(\mathbf{r},t) &= \frac{1}{c}\int\frac{\mathbf{J}(\mathbf{r}',t'=t-R/c))}{R}\,d\mathrm{Vol}'\approx\frac{e^{-i\omega t}}{cr}\int\mathbf{J}(\mathbf{r}')\,e^{ikR}\,d\mathrm{Vol}' \\
&\approx \frac{e^{i(kr-\omega t)}}{cr}\int\mathbf{J}(\mathbf{r}')\,e^{-ik\,\hat{\mathbf{r}}\cdot\mathbf{r}'}\,d\mathrm{Vol}' \\
&\approx \frac{e^{i(kr-\omega t)}}{cr}\left(\int\mathbf{J}(\mathbf{r}')\,d\mathrm{Vol}'-ik\int\mathbf{J}(\mathbf{r}')(\hat{\mathbf{r}}\cdot\mathbf{r}')\,d\mathrm{Vol}'+\cdots\right), \qquad (35)
\end{aligned}
$$

where $R = |\mathbf{r} - \mathbf{r}'| \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}'$. As is well known (see, for example, secs. 9.2-3 of [15]), the first term corresponds to the vector potential of electric dipole radiation, and the second term is the sum of the vector potentials of magnetic dipole radiation and electric quadrupole radiation. The separation of the second term into the two types of radiation is accomplished by considering its symmetric and antisymmetric parts,

$$\begin{aligned}
\mathbf{J}(\mathbf{r}')(\hat{\mathbf{r}} \cdot \mathbf{r}') &= \frac{(\hat{\mathbf{r}} \cdot \mathbf{r}')\mathbf{J}(\mathbf{r}') - (\hat{\mathbf{r}} \cdot \mathbf{J}(\mathbf{r}'))\mathbf{r}'}{2} + \frac{(\hat{\mathbf{r}} \cdot \mathbf{r}')\mathbf{J}' + (\hat{\mathbf{r}} \cdot \mathbf{J}(\mathbf{r}'))\mathbf{r}'}{2} \\
&= \frac{(\mathbf{r}' \times \mathbf{J}(\mathbf{r}')) \times \hat{\mathbf{r}}}{2} + \frac{(\hat{\mathbf{r}} \cdot \mathbf{r}')\mathbf{J}(\mathbf{r}') + (\hat{\mathbf{r}} \cdot \mathbf{J}(\mathbf{r}'))\mathbf{r}'}{2} \\
&= c\,\mathbf{M}(\mathbf{r}') \times \hat{\mathbf{r}} + \frac{(\hat{\mathbf{r}} \cdot \mathbf{r}')\mathbf{J}(\mathbf{r}') + (\hat{\mathbf{r}} \cdot \mathbf{J}(\mathbf{r}'))\mathbf{r}'}{2},
\end{aligned} \tag{36}$$

where $\mathbf{M}(\mathbf{r}) = \mathbf{r} \times \mathbf{J}(\mathbf{r})/2c$ is the magnetization density associated with current density $\mathbf{J}$.

The identification of $\mathbf{r} \times \mathbf{J}/2c$ as a magnetization density seems justified if $\mathbf{J}$ has zero divergence, which is not the case, in general, in radiation problems. It is noteworthy that [35] (sec. 71) discusses both magnetic moments and radiation for a collection of point charges, rather than for a current density $\mathbf{J}$, which deftly avoids the present ambiguities.

For completeness, we record that the magnetic field associated with the symmetric part, $\mathbf{A}_s$, of the vector potential (11) is, at time $t = 0$ when the moving electric dipole is at the origin,

$$\begin{aligned}
\mathbf{B}_s &= \boldsymbol{\nabla} \times \mathbf{A}_s = \boldsymbol{\nabla} \times \frac{(\mathbf{p_0} \cdot \mathbf{r})\mathbf{v} + (\mathbf{r} \cdot \mathbf{v})\mathbf{p_0}}{2cr^3} \\
&= (\mathbf{p}_0 \cdot \mathbf{r})\boldsymbol{\nabla} \times \frac{\mathbf{v}}{2cr^3} - \frac{\mathbf{v}}{2cr^3} \times \boldsymbol{\nabla}(\mathbf{p}_0 \cdot \mathbf{r}) + (\mathbf{v} \cdot \mathbf{r})\boldsymbol{\nabla} \times \frac{\mathbf{p}_0}{2cr^3} - \frac{\mathbf{p}_0}{2cr^3} \times \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{r}) \\
&= -\frac{3(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{r} \times \mathbf{v}}{2cr^5} - \frac{\mathbf{v} \times \mathbf{p}_0}{2cr^3} - \frac{3(\mathbf{v} \cdot \mathbf{r})\mathbf{r} \times \mathbf{p}_0}{2cr^5} - \frac{\mathbf{p}_0 \times \mathbf{v}}{2r^3} \\
&= \mathbf{v} \times \frac{3(\mathbf{p}_0 \cdot \mathbf{r})\mathbf{r}}{2cr^5} + \mathbf{p}_0 \times \frac{3(\mathbf{v} \cdot \mathbf{r})\mathbf{r}}{2cr^5} = -3\,\mathbf{r} \times \frac{(\mathbf{v} \cdot \mathbf{r})\mathbf{p}_0 + (\mathbf{p}_0 \cdot \mathbf{r})\mathbf{v}}{2cr^5}\,.
\end{aligned} \tag{37}$$

When the moving dipole is at the origin the magnetic field varies purely as $1/r^3$, so the fourth Maxwell equation (for points away from the dipole itself),

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\,, \tag{38}$$

tells us that the electric field must include terms that vary as $1/r^4$, which are associated with the (time-dependent) electric quadrupole moment. However, eq. (38) should not be construed as implying that the changing electric quadrupole moment "creates" the magnetic field.

It is instructive to consider $\boldsymbol{\nabla} \times \mathbf{B}_s$, for which we note that,

$$\begin{aligned}
\boldsymbol{\nabla} \times \frac{\mathbf{r} \times (\mathbf{v} \cdot \mathbf{r})\mathbf{p}_0}{r^5} &= \boldsymbol{\nabla} \times \frac{(\mathbf{v} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} \times \mathbf{p}_0}{r^3} \\
= \boldsymbol{\nabla}\frac{1}{r^3} \times (\mathbf{v} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} \times \mathbf{p}_0 &+ \frac{\boldsymbol{\nabla} \times [(\mathbf{v} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} \times \mathbf{p}_0]}{r^3}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{3}{r^4}(\mathbf{v}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p}_0) + \frac{\boldsymbol{\nabla}(\mathbf{v}\cdot\hat{\mathbf{r}})\times(\hat{\mathbf{r}}\times\mathbf{p}_0)}{r^3} + (\mathbf{v}\cdot\hat{\mathbf{r}})\frac{\boldsymbol{\nabla}\times(\hat{\mathbf{r}}\times\mathbf{p}_0)}{r^3} \\
&= -\frac{3}{r^4}(\mathbf{v}\cdot\hat{\mathbf{r}})[(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{p}_0] + \frac{(\mathbf{v}\cdot\boldsymbol{\nabla})\,\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p}_0)}{r^3} + (\mathbf{v}\cdot\hat{\mathbf{r}})\frac{(\mathbf{p}_0\cdot\boldsymbol{\nabla})\,\hat{\mathbf{r}}-\mathbf{p}_0(\boldsymbol{\nabla}\cdot\hat{\mathbf{r}})}{r^3} \\
&= \frac{3(\mathbf{v}\cdot\hat{\mathbf{r}})\mathbf{p}_0-3(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{r^4} \\
&\quad + \frac{[\mathbf{v}-(\mathbf{v}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}]\times(\hat{\mathbf{r}}\times\mathbf{p}_0)}{r^4} + (\mathbf{v}\cdot\hat{\mathbf{r}})\frac{\mathbf{p}_0-(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}]-2\mathbf{p}_0}{r^4} \\
&= \frac{3(\mathbf{v}\cdot\hat{\mathbf{r}})\mathbf{p}_0-3(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{r^4} \\
&\quad + \frac{(\mathbf{v}\cdot\mathbf{p}_0)\,\hat{\mathbf{r}}-(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}}{r^4} - \frac{(\mathbf{v}\cdot\hat{\mathbf{r}})\mathbf{p}_0+(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}]}{r^4} \\
&= \frac{-5(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}+(\mathbf{v}\cdot\mathbf{p}_0)\,\hat{\mathbf{r}}+2(\mathbf{v}\cdot\hat{\mathbf{r}})\mathbf{p}_0}{r^4}.
\end{aligned} \tag{39}$$

Using eq. (39) with $\mathbf{p}_0$ and $\mathbf{v}$ interchanged, we obtain,

$$\boldsymbol{\nabla}\times\frac{\mathbf{r}\times[(\mathbf{v}\cdot\mathbf{r})\mathbf{p}_0+(\mathbf{p}_0\cdot\mathbf{r})\mathbf{v}]}{r^5} = \frac{-10(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}+2(\mathbf{v}\cdot\mathbf{p}_0)\,\hat{\mathbf{r}}+2(\mathbf{v}\cdot\hat{\mathbf{r}})\mathbf{p}_0+2(\mathbf{p}_0\cdot\hat{\mathbf{r}})\mathbf{v}}{r^4}. \tag{40}$$

Then, recalling eq. (37), we find that,

$$\boldsymbol{\nabla}\times\mathbf{B}_s = \frac{15(\mathbf{v}\cdot\hat{\mathbf{r}})(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-3(\mathbf{v}\cdot\mathbf{p}_0)\,\hat{\mathbf{r}}-3(\mathbf{v}\cdot\hat{\mathbf{r}})\mathbf{p}_0-3(\mathbf{p}_0\cdot\hat{\mathbf{r}})\mathbf{v}}{cr^4}. \tag{41}$$

As noted in prob. 6.21(c) of [15], when the moving dipole is at position $\mathbf{r}_0=\mathbf{v}t$ its electric field, eq. (19) with $\mathbf{r}\to\mathbf{r}-\mathbf{r}_0$, has the multipole expansion with respect to the origin,

$$\begin{aligned}
\mathbf{E} &= \frac{3[\mathbf{p}_0\cdot(\mathbf{r}-\mathbf{r}_0)](\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^5} - \frac{\mathbf{p}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \\
&= \frac{3(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-\mathbf{p}_0}{r^3} + \frac{15(\mathbf{r}_0\cdot\hat{\mathbf{r}})(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}-3(\mathbf{r}_0\cdot\mathbf{p}_0)\,\hat{\mathbf{r}}-3(\mathbf{r}_0\cdot\hat{\mathbf{r}})\mathbf{p}_0-3(\mathbf{p}_0\cdot\hat{\mathbf{r}})\mathbf{r}_0}{r^4} + \cdots \\
&= \mathbf{E}_{\rm dipole}+\mathbf{E}_{\rm quadrupole}+\cdots,
\end{aligned} \tag{42}$$

for $r_0 \ll r$. The electric dipole field $\mathbf{E}_{\rm dipole}$ (in contrast to the field $\mathbf{E}$ of the moving electric dipole) is constant in time (as is the electric dipole moment $\mathbf{p}\approx\mathbf{p}_0$). Thus,[11]

$$\boldsymbol{\nabla}\times\mathbf{B}_s = \frac{1}{c}\frac{\partial\mathbf{E}_{\rm quadrupole}}{\partial t} \approx \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}. \tag{43}$$

This establishes a relation between the partial magnetic field $\mathbf{B}_s$ and the electric quadrupole field $\mathbf{E}_{\rm quadrupole}$ of the moving electric dipole, but this should not be interpreted as a cause-and-effect relation.

It remains that the magnetic field of a moving electric dipole $\mathbf{p}_0$ at order $1/r^3$ is not exactly that of the "magnetic moment" $-\mathbf{v}/c\times\mathbf{p}_0$, so that it is somewhat delicate to use the terminology "magnetic moment" in this example.

[11] One can show that $\boldsymbol{\nabla}\times\mathbf{B}_a=0$, starting from the next to last form of eq. (32) and using the identities $(\mathbf{a}\cdot\boldsymbol{\nabla})\,\hat{\mathbf{r}} = [\mathbf{a}-(\mathbf{a}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}]/r$ and $(\mathbf{a}\cdot\boldsymbol{\nabla})(\mathbf{b}\cdot\hat{\mathbf{r}}) = [\mathbf{a}\cdot\mathbf{b}-(\mathbf{a}\cdot\hat{\mathbf{r}})(\mathbf{b}\cdot\hat{\mathbf{r}})]/r$, so that eq. (38) is satisfied for $\mathbf{B}=\mathbf{B}_a+\mathbf{B}_s$.

3 Moving Magnetic Dipole

For completeness, we include a discussion of the case of a magnetic dipole $\mathbf{m}_0$ that has velocity $\mathbf{v}$ in the lab frame. See also [29].

The potentials in the rest frame of magnetic dipole (where the magnetic moment is $\mathbf{m}_0$), assuming the dipole to be at the origin, are,

$$V^\star = 0, \qquad \mathbf{A}^\star = \frac{\mathbf{m}_0 \times \mathbf{r}^\star}{r^{\star 3}}, \tag{44}$$

where quantities in the rest frame are denoted with the superscript *. The Lorentz transformation of the 4-vector $(V, \mathbf{A})$ to the lab frame at time t when the dipole is at position $\mathbf{r}_0 = \mathbf{v}t$ yield, for $v \ll c$, where $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1$, and $\mathbf{r}^\star \approx \mathbf{r} - \mathbf{r}_0 \equiv \mathbf{R}$, is,

$$V = \gamma(V^\star + \mathbf{A}^\star \cdot \mathbf{v}/c) \approx \frac{\mathbf{m}_0 \times \mathbf{R} \cdot \mathbf{v}}{cR^3} = \frac{\mathbf{v}/c \times \mathbf{m}_0 \cdot \mathbf{R}}{R^3}, \tag{45}$$

$$\mathbf{A} = \gamma(\mathbf{A}^\star + V^\star \mathbf{v}/c) \approx \mathbf{A}^\star = \frac{\mathbf{m}_0 \times \mathbf{R}}{R^3}. \tag{46}$$

These potentials have a crisper interpretation than those of eqs. (6) and (8) for a moving electric dipole, in that we can say that the potentials of a magnetic dipole which moves with $v \ll c$ correspond to magnetic dipole moment $\mathbf{m} = \mathbf{m}_0$ and electric dipole moment,

$$\mathbf{p} = \frac{\mathbf{v}}{c} \times \mathbf{m}_0, \tag{47}$$

with respect to its instantaneous position.

The lab-frame fields $\mathbf{E}$ and $\mathbf{B}$ are the Lorentz transforms of the fields in the rest frame of the magnetic dipole,

$$\mathbf{E}^\star = 0, \qquad \mathbf{B}^\star = \frac{3(\mathbf{m}_0 \cdot \mathbf{r}^\star)\mathbf{r}^\star}{r^{\star 5}} - \frac{\mathbf{m}_0}{r^{\star 3}}. \tag{48}$$

Then,

$$\mathbf{E} \approx -\frac{\mathbf{v}}{c} \times \mathbf{B}^\star \approx -\frac{\mathbf{v}}{c} \times \mathbf{B} = -\frac{\mathbf{v}}{c} \times \left(\frac{3(\mathbf{m}_0 \cdot \mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{m}_0}{R^3} \right), \tag{49}$$

$$\mathbf{B} \approx \mathbf{B}^\star = \frac{3(\mathbf{m}_0 \cdot \mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{m}_0}{R^3}. \tag{50}$$

The electric field can also be written as,

$$\mathbf{E} = \mathbf{E}_p + \mathbf{E}_m, \tag{51}$$

with,

$$\mathbf{E}_p = -\boldsymbol{\nabla} V = \frac{3(\mathbf{p} \cdot \mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{p}}{R^3} = \frac{3(\mathbf{v}/c \times \mathbf{m}_0 \cdot \mathbf{R})\mathbf{R}}{R^5} - \frac{\mathbf{v}/c \times \mathbf{m}_0}{R^3}, \tag{52}$$

and[12]

$$\mathbf{E}_m = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}\frac{\mathbf{m}_0 \times \mathbf{R}}{R^3} = -\mathbf{m}_0 \times \left(\frac{3(\mathbf{v}\cdot\mathbf{R})\mathbf{R}}{cR^5} - \frac{\mathbf{v}}{cR^3}\right), \tag{53}$$

where $\mathbf{E}_p$ and $\mathbf{E}_m$ can also be interpreted as the electric fields associated with the polarization and magnetization densities of the moving magnetic dipole, respectively. The fields (49), (50), (52) and (53) are the duals of the fields (20), (19), (33) and (32), respectively, of a moving electric dipole, and suffer from the ambiguity that the electric (magnetic) field of the moving magnetic (electric) dipole is not simply that of the electric (magnetic) moment of the moving dipole.

If we decompose the electric field of the moving magnetic dipole into antisymmetric and symmetric pieces (with respect to interchange of symbols $\mathbf{m}_0$ and $\mathbf{v}$),

$$\mathbf{E} = \mathbf{E}_a + \mathbf{E}_s, \tag{54}$$

then,

$$\begin{aligned}
\mathbf{E}_a = \frac{\mathbf{E}_p}{2} &= \frac{3[(\mathbf{v}/c \times \mathbf{m}_0/2)\cdot\mathbf{R}]\mathbf{R}}{R^5} - \frac{\mathbf{v}/c \times \mathbf{m}_0/2}{R^3} \\
&= -\frac{\mathbf{v}}{2} \times \left(\frac{3(\mathbf{m}_0\cdot\mathbf{R})\mathbf{R}}{cR^5} - \frac{\mathbf{m}_0}{cR^3}\right) + \frac{\mathbf{m}_0}{2} \times \left(\frac{3(\mathbf{v}\cdot\mathbf{R})\mathbf{R}}{cR^5} - \frac{\mathbf{v}}{cR^3}\right), &(55)\\
\mathbf{E}_s &= \mathbf{E} - \mathbf{E}_a = 3\,\mathbf{R} \times \frac{(\mathbf{v}\cdot\mathbf{R})\mathbf{m}_0 + (\mathbf{m}_0\cdot\mathbf{R})\mathbf{v}}{2cR^5}, &(56)
\end{aligned}$$

and,

$$\nabla \times \mathbf{E}_a = 0, \qquad \nabla \times \mathbf{E}_s = -\frac{1}{c}\frac{\partial \mathbf{B}_{\rm quadrupole}}{\partial t}, \tag{57}$$

where the magnetic field (48) of the moving magnetic dipole at $\mathbf{r}_0$ can be expanded for $r_0 \ll r$ as, recaling that $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$,

$$\begin{aligned}
\mathbf{B} &= \frac{3[\mathbf{m}_0\cdot(\mathbf{r}-\mathbf{r}_0)](\mathbf{r}-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}_0|^5} - \frac{\mathbf{m}_0}{|\mathbf{r}-\mathbf{r}_0|^3} \\
&= \frac{3(\mathbf{m}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}_0}{r^3} + \frac{15(\mathbf{r}_0\cdot\hat{\mathbf{r}})(\mathbf{m}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - 3(\mathbf{r}_0\cdot\mathbf{m}_0)\,\hat{\mathbf{r}} - 3(\mathbf{r}_0\cdot\hat{\mathbf{r}})\mathbf{m}_0 - 3(\mathbf{m}_0\cdot\hat{\mathbf{r}})\mathbf{r}_0}{r^4} + \cdots \\
&= \mathbf{B}_{\rm dipole} + \mathbf{B}_{\rm quadrupole} + \cdots.
\end{aligned} \tag{58}$$

The form of eq. (55) permits us to consider that the electric dipole moment of the moving magnetic moment is $\mathbf{v}/c \times \mathbf{m}_0/2$, rather than that given in eq. (47), so again we must exercise care in using the term "moment" to characterize a moving intrinsic moment.

For discussion of a moving magnetic dipole in an external electric field, see [37].

[12]The form (53) was noted in prob. 12.4 of [36] and in prob. 11.29 of [15].

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