Radiation in the Near Zone of a Hertzian Dipole

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1 Problem

Calculate the Poynting vector of the fields of a Hertzian oscillating dipole at all points in space. Show that the time-averaged Poynting vector has the same form in the near zone as it does in the far zone, which confirms that radiation exists both close to and far from the source.[1]

2 Solution

2.1 Hertzian Electric Dipole

The electric and magnetic fields of an ideal, point Hertzian electric dipole $\mathbf{p}$ can be written (in Gaussian units) as,

$$\mathbf{E} = k^2 p(\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \times \hat{\mathbf{r}} \frac{e^{i(kr-\omega t)}}{r} + p[3(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}] \left(\frac{1}{r^3} - \frac{ik}{r^2} \right) e^{i(kr-\omega t)}, \tag{1}$$

$$\mathbf{B} = k^2 p(\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \left(\frac{1}{r} - \frac{1}{ikr^2} \right) e^{i(kr-\omega t)}, \tag{2}$$

whose real parts are,

$$\mathbf{E} = k^2 p(\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \times \hat{\mathbf{r}} \frac{\cos(kr-\omega t)}{r} + p[3(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}] \left[\frac{\cos(kr-\omega t)}{r^3} + \frac{k\sin(kr-\omega t)}{r^2} \right], \tag{3}$$

$$\mathbf{B} = k^2 p(\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \left[\frac{\cos(kr-\omega t)}{r} - \frac{\sin(kr-\omega t)}{kr^2} \right], \tag{4}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector from the center of the dipole to the observer, $\mathbf{p} = p\cos\omega t\,\hat{\mathbf{p}}$ is the electric dipole moment vector, $\omega = 2\pi f$ is the angular frequency, $k = \omega/c = 2\pi/\lambda$ is the wave number and c is the speed of light [1, 2].

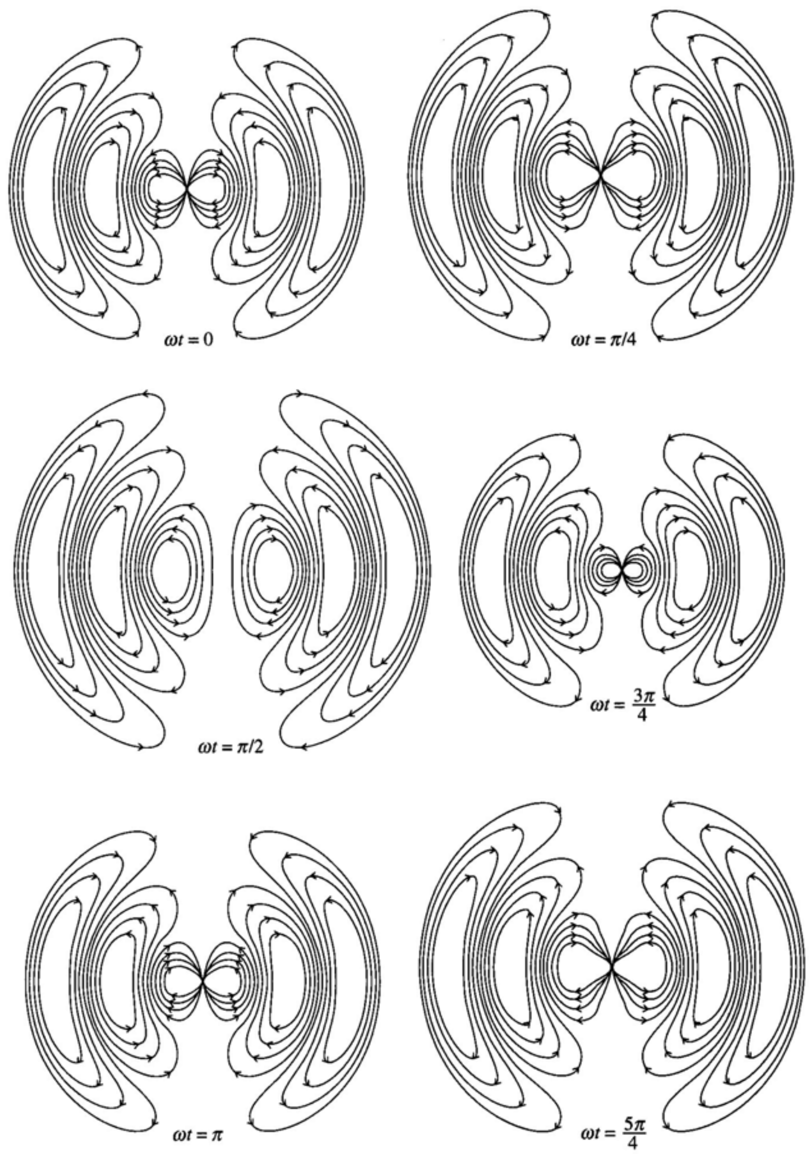
We say that the radiation part of these fields are the terms that vary as $1/r$,

$$\mathbf{E}_{\rm rad} = k^2 p(\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \times \hat{\mathbf{r}} \frac{\cos(kr-\omega t)}{r}, \tag{5}$$

[1] Some additional details, including discussion of the scalar and vector potentials of a Hertzian dipole in both the Coulomb and Lorenz gauges, are given in prob. 2 of
http://kirkmcd.princeton.edu/examples/ph501set8.pdf
Some consideration of Hertz vectors and scalars is given in the Appendix of
http://kirkmcd.princeton.edu/examples/smallloop.pdf

$$\mathbf{B}_{\rm rad} = k^2 p (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \frac{\cos(kr - \omega t)}{r} \, . \tag{6}$$

In the near zone of the dipole, where $kr \lesssim 1$, the radiation fields are smaller that the other components of $\mathbf{E}$ and $\mathbf{B}$. The most prominent feature of the fields in the near zone is that the electric field looks a lot like that of an electrostatic dipole, as shown in the figure below. Because field patterns that look like radiation are discernable only for $r \gtrsim \lambda$, there may be an impression that the radiation is created at some distance from an antenna, rather than at the antenna itself.



Since the radiated power comes from the antenna (from the power supply that drives the antenna), there must be a flow of energy out from the surface of the antenna into the surrounding space. The usual electrodynamic measure of energy flow is Poynting's vector [3] (in a medium with unit relative permeability),

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}. \tag{7}$$

When we use the fields (3)-(4) to calculate the Poynting vector we find six terms, some of which do not point along the radial vector $\hat{\mathbf{r}}$,

$$\begin{aligned}\mathbf{S} &= \frac{c}{4\pi}\left\{k^4p^2[(\hat{\mathbf{r}}\times\hat{\mathbf{p}})\times\hat{\mathbf{r}}]\times(\hat{\mathbf{r}}\times\hat{\mathbf{p}})\left[\frac{\cos^2(kr-\omega t)}{r^2}-\frac{\cos(kr-\omega t)\sin(kr-\omega t)}{kr^3}\right]\right.\\ &\quad +k^2p^2[3(\hat{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\hat{\mathbf{p}}]\times(\hat{\mathbf{r}}\times\hat{\mathbf{p}})\left[\frac{\cos^2(kr-\omega t)-\sin^2(kr-\omega t)}{r^4}\right.\\ &\quad\quad \left.\left.+\cos(kr-\omega t)\sin(kr-\omega t)\left(\frac{k}{r^3}-\frac{1}{kr^5}\right)\right]\right\}\\ &= \frac{c}{4\pi}\left\{k^4p^2\sin^2\theta\,\hat{\mathbf{r}}\left[\frac{\cos^2(kr-\omega t)}{r^2}-\frac{\cos(kr-\omega t)\sin(kr-\omega t)}{kr^3}\right]\right.\\ &\quad +k^2p^2[(3\cos^2\theta-1)\,\hat{\mathbf{r}}-2\cos\theta\,\hat{\mathbf{p}}]\left[\frac{\cos^2(kr-\omega t)-\sin^2(kr-\omega t)}{r^4}\right.\\ &\quad\quad \left.\left.+\cos(kr-\omega t)\sin(kr-\omega t)\left(\frac{k}{r^3}-\frac{1}{kr^5}\right)\right]\right\}, \end{aligned} \tag{8}$$

where θ is the angle between vectors $\mathbf{r}$ and $\mathbf{p}$. As well as the expected radial flow of energy, there is a flow in the direction of the dipole moment $\mathbf{p}$. Since the product $\cos(kr-\omega t)\sin(kr-\omega t)$ can be both positive and negative, part of the energy flow is inwards at times, rather than outwards as expected for pure radiation.

However, we obtain a simple result if we consider only the time-averaged Poynting vector, $\langle\mathbf{S}\rangle$. Noting that $\langle\cos^2(kr-\omega t)\rangle = \langle\sin^2(kr-\omega t)\rangle = 1/2$ and $\langle\cos(kr-\omega t)\sin(kr-\omega t)\rangle = (1/2)\langle\sin 2(kr-\omega t)\rangle = 0$, eq (8) leads to,

$$\langle\mathbf{S}\rangle = \frac{ck^4p^2\sin^2\theta}{8\pi r^2}\,\hat{\mathbf{r}}. \tag{9}$$

The time-average Poynting vector is purely radially outwards, and falls off as $1/r^2$ at all radii, as expected for a flow of energy that originates in the oscillating point dipole. The time-average angular distribution $d\langle P\rangle/d\Omega$ of the radiated power is related to the Poynting vector by,

$$\frac{d\langle P\rangle}{d\Omega} = r^2\,\hat{\mathbf{r}}\cdot\langle\mathbf{S}\rangle = \frac{ck^4p^2\sin^2\theta}{8\pi} = \frac{p^2\omega^4\sin^2\theta}{8\pi c^3}, \tag{10}$$

which is the expression usually derived for dipole radiation in the far zone. Here we see that this expression holds in the near zone as well.

We conclude that radiation, as measured by the time-averaged Poynting vector, exists in the near zone of an antenna as well as in the far zone.

The question sometimes arises as to whether the fields of an antenna could be pure radiation, with no nonpropagating near fields (that take energy to maintain). It is clear that a small oscillating dipole will generate nearby, quasistatic dipole fields. See sec. 6 of [4] for a kind of inverse argument, that any dipole radiation fields in the far zone imply, via a diffraction integral, that nonradiation fields are present in the near zone.

2.2 Hertzian Magnetic Dipole

If $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are solutions to Maxwell's equations in free space (*i.e.*, where the charge density ρ and current density $\mathbf{J}$ are zero), then the dual fields,

$$\mathbf{E}'(\mathbf{r},t) = -\mathbf{B}(\mathbf{r},t), \qquad \mathbf{B}'(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t), \tag{11}$$

are solutions there also. The Poynting vector is the same for the dual fields as for the original fields,[2,3]

$$\mathbf{S}' = \frac{c}{4\pi}\mathbf{E}' \times \mathbf{B}' = -\frac{c}{4\pi}\mathbf{B} \times \mathbf{E} = \mathbf{S}. \tag{12}$$

Taking the dual of fields (3)-(4) with the change of notation $\mathbf{p} \to \mathbf{m}$, we find the fields,

$$\mathbf{E}' = -k^2 m(\hat{\mathbf{r}} \times \hat{\mathbf{m}}) \left(\frac{1}{r} - \frac{1}{ikr^2}\right) e^{i(kr-\omega t)}, \tag{13}$$

$$\mathbf{B}' = k^2 m(\hat{\mathbf{r}} \times \hat{\mathbf{m}}) \times \hat{\mathbf{r}} \, \frac{e^{i(kr-\omega t)}}{r} + m[3(\hat{\mathbf{m}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{m}}] \left(\frac{1}{r^3} - \frac{ik}{r^2}\right) e^{i(kr-\omega t)}, \tag{14}$$

whose real parts are,

$$\mathbf{E}' = -k^2 m(\hat{\mathbf{r}} \times \hat{\mathbf{m}}) \left[\frac{\cos(kr-\omega t)}{r} - \frac{\sin(kr-\omega t)}{kr^2}\right], \tag{15}$$

$$\mathbf{B}' = k^2 m(\hat{\mathbf{r}} \times \hat{\mathbf{m}}) \times \hat{\mathbf{r}} \frac{\cos(kr-\omega t)}{r} + m[3(\hat{\mathbf{m}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{m}}] \left[\frac{\cos(kr-\omega t)}{r^3} + \frac{k\sin(kr-\omega t)}{r^2}\right], \tag{16}$$

which are also solutions to Maxwell's equations. These are the fields of an oscillating point magnetic dipole, whose peak magnetic moment is $\mathbf{m}$. In the near zone, the magnetic field (16) looks like that of a (magnetic) dipole.

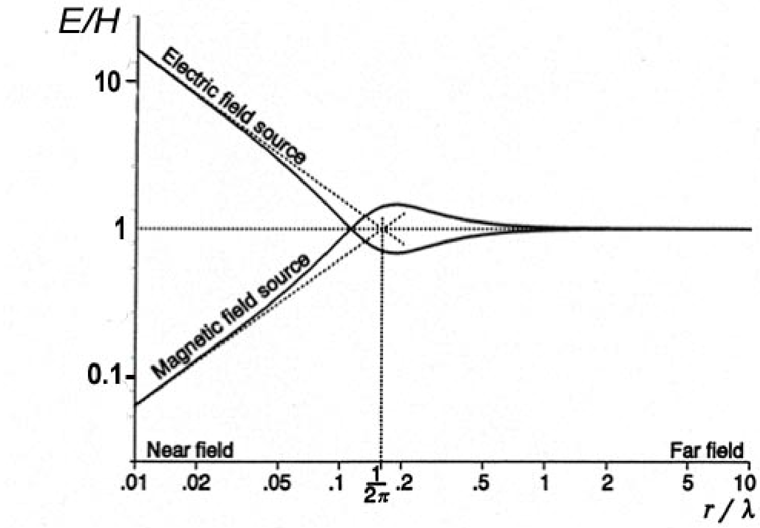
While the fields of eqs. (3)-(4) are not identical to those of eqs. (15)-(16), the Poynting vectors are the same in the two cases. Hence, the time-average Poynting vector, and also the angular distribution of the time-averaged radiated power are the same in the two cases. The radiation of a point electric dipole is the same as that of a point magnetic dipole (assuming that $\mathbf{m} = \mathbf{p}$), both in the near and in the far zones. Measurements of only the intensity of the radiation could not distinguish the two cases.

However, if measurements were made of both the electric and magnetic fields, then the near zone fields of an oscillating electric dipole, eqs. (3)-(4), would be found to be quite different from those of a magnetic dipole, eqs. (15)-(16). This is illustrated in the figure on p. 5, which plots the ratio $E/H = E/B$ of the magnitudes of the electric and magnetic fields as a function of the distance r from the center of the dipoles.

To distinguish between the cases of electric and magnetic dipole radiation, it suffices to measure only the polarization (*i.e.*, the direction, but not the magnitude) of either the electric of the magnetic field vectors.

[2]The energy radiated by a Hertzian magnetic dipole was first calculated by Fitzgerald in 1883 [5], prior to Hertz' calculations for electric dipoles. Apparently Fitzgerald considered atoms to involve charges oscillating in circles, and hence he thought that atoms would emit magnetic dipole radiation. He seems not to have noticed that such classical atoms would be unstable due to this radiation. See, for example, [7].

[3]An application of the concept of dual field to macroscopic, planar antennas is given in [6].



A Appendix: Self-Dual Antennas

Heaviside introduced the concept of dual electromagnetic fields in the 1880's while also considering magnetic charges and currents, ρ_e and $\mathbf{J}_m$, that are conceptually the "dual" of electric charges and currents, ρ_e and $\mathbf{J}_e$. In general, if fields $\mathbf{E}$ and $\mathbf{B}$ are generated by electric charges and currents ρ_e and $\mathbf{J}_e$, then the dual fields $\mathbf{E}'$ and $\mathbf{B}'$ of eq. (11) are generated by the dual magnetic charges $\rho_m' = \rho_e$ and $\mathbf{J}_m' = \mathbf{J}_e$. Since it appears that there are no magnetic charges and currents in Nature, it will not be possible in general to generate the dual fields $\mathbf{E}'$ and $\mathbf{B}'$. However, if only magnetic dipoles (or higher multipoles) are needed to generate the dual fields, it may be possible to do so with appropriate configurations of electric currents.

According to the definition (11) the dual fields cannot equal the original fields. However, it is possible that the dual fields are equal to the original fields to within a phase factor $\pm i$, in which case the fields are said to be self dual. That is,

$$\mathbf{E}' = -\mathbf{B} = \alpha \mathbf{E}, \qquad \mathbf{B}' = \mathbf{E} = \alpha \mathbf{B}, \qquad \text{requires that} \qquad \alpha^2 = -1, \tag{17}$$

such that,

$$\mathbf{E} = \pm i \mathbf{B}, \qquad \mathbf{B} = \mp i \mathbf{E}, \qquad \text{(self dual)}. \tag{18}$$

Perhaps the most familiar examples of self-dual electromagnetic fields are circularly polarized plane waves, such as,

$$\mathbf{E} = E_0(\hat{\mathbf{x}} \pm i\,\hat{\mathbf{y}})\, e^{i(kz-\omega t)}, \qquad \mathbf{B} = \hat{\mathbf{x}} \times \mathbf{E} = E_0(\hat{\mathbf{y}} \mp i\,\hat{\mathbf{x}})\, e^{i(kz-\omega t)} = \mp i \mathbf{E}. \tag{19}$$

A set of self-dual fields $(\mathbf{E}_{\rm sd}, \mathbf{B}_{\rm sd})$ can be formally generated from any solution $(\mathbf{E}, \mathbf{B})$ to Maxwell's equations according to,

$$\mathbf{E}_{\rm sd} = \mathbf{E} \mp i\mathbf{E}' = \mathbf{E} \pm i\mathbf{B}, \qquad \mathbf{B}_{\rm sd} = \mathbf{B} \mp i\mathbf{B}' = \mathbf{B} \mp i\mathbf{E} = \mp i\mathbf{E}_{\rm sd}. \tag{20}$$

Of course, in general it will not be possible physically generate these fields with only electric charges and currents. However, we can make a self-dual antenna by combining a Hertzian electric dipole antenna of moment $\mathbf{p}$ with a Hertzian magnetic dipole antenna of moment $\mathbf{m} = \pm i\mathbf{p}$. In this case, the far-zone electric field of waves emitted in direction has $\hat{\mathbf{r}}$ has direction $\mathbf{E}_{\rm sd} \propto \hat{\mathbf{r}} \times [(\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \pm i\,\hat{\mathbf{p}}] = (\hat{\mathbf{r}} \cdot \hat{\mathbf{p}})\hat{\mathbf{r}} - \hat{\mathbf{p}} \pm i\,\hat{\mathbf{r}} \times \hat{\mathbf{p}}$. Thus, when $\hat{\mathbf{r}}$ is perpendicular to $\hat{\mathbf{p}}$, the electric field has direction $-\hat{\mathbf{p}} \pm i\,\hat{\mathbf{r}} \times \hat{\mathbf{p}}$, so that the radiation emitted in this direction is circularly polarized (and therefore self dual as noted in eq. (19)).

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