Do Neutrino Oscillations Conserve Energy?

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1 Problem

If neutrinos have mass, they can be at rest. If neutrinos can oscillate between types with different masses, then Einstein's relation $E = mc^2$ [1], where c is the speed of light in vacuum, implies that the (rest) energy of a neutrino changes during a mass oscillation. Does the phenomenon of neutrino-mass oscillation imply that energy is not conserved, or (if energy is always conserved) are claims of such oscillation a hoax?

2 Solution

The spirit of this solution is that energy and momentum are conserved in all interactions, but that the energies and momenta of quantum states do not necessarily have definite values until these quantities are measured (to some accuracy). For example, if a neutrino is produced in the process,

$$A \to B + \nu, \tag{1}$$

and all three energies E_A, E_B and E_ν are somehow measured, it will be found that $E_A = E_B + E_\nu$ (to the accuracy of the measurements). However, this does not mean that the neutrino had energy E_ν prior to the measurement. In general, the quantum state $\psi(A, B, \nu)$ prior to the measurements involves (entangled) substates ψ_i with a spread of energies, where the energies of substate i obey $E_{A,i} = E_{B,i} + E_{\nu,i}$.

If the "particles" A, B and ν have definite rest mass/energies m_A, m_B and m_ν, conservation of energy and momentum imply that in the rest frame of particle A,

$$E_\nu = \frac{(m_A^2 + m_\nu^2 - m_B^2)c^2}{2m_A} \approx \frac{m_A c^2}{2}\left(1 - \frac{m_B^2}{m_A^2}\right), \tag{2}$$

where the approximation holds when $m_\nu \ll m_B$ (as is typical for neutrino production). However, "particles" A and B have decay-time constants τ_A and τ_B (where τ_B is possibly infinite). If these particles are not observed/measured, we cannot say that they have definite mass/energy even in their rest frames, but rather we say that their rest mass/energy has a spread $\Delta mc^2 \approx \hbar/\tau$, where m is the central (most probable) value for the particle's rest mass. See, for example, [2]. Consequently, the neutrino energy has a spread given by,

$$\Delta E_\nu = \begin{cases} \Delta m_A c^2 \left(1 - \frac{E_\nu}{m_A c^2}\right) \approx \frac{\hbar}{\tau_A}\frac{m_B^2}{m_A^2} & (\tau_A \ll \tau_B,\ m_\nu \ll m_B), \\ \Delta m_B c^2 \frac{m_B}{m_A} \approx \frac{\hbar}{\tau_B}\frac{m_B}{m_A} & (\tau_A \gg \tau_B). \end{cases} \tag{3}$$

Example: For $\pi \to \mu + \nu$, $\tau_\mu \approx 2\ \mu$s $\gg \tau_\pi \approx 26$ ns, and $m_\mu^2/m_\pi^2 \approx 1/2$, so $\Delta E_\nu \approx \hbar/2\tau_\pi = \hbar c/2c\tau_\pi \approx 200$ MeV-fermi / $2 \cdot 7.8$ m $\approx 10^{-14}$ MeV, while in the pion rest frame, $E_\nu \approx m_\pi c^2/2 \approx 70$ MeV.

The most copious sources of (anti)neutrinos on Earth are the decays of heavy nuclei in nuclear reactors,

$$A \to B + e + \bar{\nu}, \tag{4}$$

where A and B are heavy nuclei with various lifetimes. In these three-bodies decays, the (anti)neutrino energy (in the rest frame of nucleus A) can be anywhere between zero and $(m_A - m_B - m_e)c^2$ with $E_{\bar{\nu},\max} \approx 10$ MeV. The (anti)neutrino energy distribution is peaked at $\langle E_{\bar{\nu}} \rangle \approx E_{\bar{\nu},\max}/2 \approx 5$ MeV, and the characteristic energy spread of the (anti)neutrinos is $\Delta E_{\bar{\nu}} / \langle E_{\bar{\nu}} \rangle \approx 1/4$. This energy spread is much larger than that due to effects of the lifetimes of nuclei A and B.

Similarly, the neutrinos from muon decay, $\mu \to \nu_\mu + e + \bar{\nu}_e$, have quantum states with large energy spreads.

If the neutrino oscillations are to be recognized as oscillations in the energy spectrum (at a fixed distance from the source), the energy resolution of the detector must be better than the period of the energy oscillations. In the quantum view, if the oscillations cannot be observed they cannot be said to exist. This notion is incorporated in the so-called coherence length for neutrino oscillations (which must be longer than the distance from source to detector if the detector can resolve the oscillations).

We now turn to a version of the standard analysis of neutrino oscillations in sec. 2.1, add the effect of detector resolution in sec. 2.2, consider an important effect of source size in sec. 2.3, and return to the theme of entangled states in sec. 2.4.

2.1 Two-Neutrino Oscillations

To place the present problem in a more specific context, we consider the standard analysis of neutrino oscillations supposing that there are only two types of neutrinos, both with mass. Production of these neutrinos in a weak interaction via a W-boson emphasizes the so-called flavor states, ν_a and ν_b, while the neutrino states with definite mass are ν_1 and ν_2. These two pairs of states are related by 2×2 unitary matrix with a single parameter, the mixing angle θ_{12} [3],[1]

$$\begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = \begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} \\ -\sin\theta_{12} & \cos\theta_{12} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_{12} & -\sin\theta_{12} \\ \sin\theta_{12} & \cos\theta_{12} \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix}, \tag{5}$$

which implies the states a and b can transform back and forth between each other.[2] The neutrino flavor states a and b do not have a well defined mass, according to eq. (5), if the

[1] The two-neutrino mixing angle θ_{12} was introduced prior to its relative, the Cabibbo angle [4], that describes the weak-interaction coupling of the u-quark to the d-s quark system, where u, d and s are flavor states of the strong interaction, which differ from the flavor states of the weak interaction. The formalism for the strong-weak three-quark mixing was introduced in [5], and the present notation in terms of quark-mixing angles first appeared in [6]. The latter notation is also commonly used for three-neutrino mixing.

[2] The possibility of such transitions in a two-state system of elementary particles was first noted by Gell-Mann and Pais in 1955 [7] for the K^0-$\bar{K}^0$ system, and first considered for neutrinos by Pontecorvo in 1957

neutrino states 1 and 2 have different masses (as occurs in Nature).[3]

When a neutrino is produced in a nuclear decay, or in the decay of a meson, it is produced in a flavor state rather than a mass state, and it typically accompanied by the associated flavor antilepton. Energy and momentum are conserved in this decay process, but the energy and momentum of the neutrino are different depending for the different neutrino mass state components of the neutrino flavor state.

We review the standard formalism for neutrino oscillations (perhaps first given in [12], and in somewhat more detail in [13]).

The usual procedure is to consider plane-wave states of neutrinos 1 and 2 that have well defined energies E_i and momenta P_i large compared to their (rest) masses m_i, which wave/particles propagate essentially at the speed c of light in vacuum.[4] Then, for propagation along the x-axis, the momenta P_i are,

$$c^2 P_i^2 = E_i^2 - m_i^2 c^4, \qquad P_i \approx \frac{E_i}{c}\left(1 - \frac{m_i^2 c^4}{2E_i^2}\right), \tag{6}$$

and,

$$\psi_i(x,t) = \psi_{i,0}\, e^{i(P_i x - E_i t)/\hbar} \approx \psi_{i,0}\, e^{iE_i(x/c - t)/\hbar}\, e^{-im_i^2 c^3 x/2E_i\hbar}. \tag{7}$$

Now, a neutrino created in a decay at, say, time $t = 0$ is not really in a plane-wave state (7), but rather has a wave packet with a spread of energies ΔE, which implies the time spread of the wave packet is $\Delta t \approx \hbar/\Delta E$ and a spatial width $\Delta x \approx \hbar c/\Delta E$. If wave packets of neutrino mass types 1 and 2 are created together (at the origin and at time $t = 0$), then these packets continue to overlap significantly, and interfere, until their centroids are separated by roughly the pulse width Δx. This occurs at the so-called coherence time $t_{\rm coh}$ related by,

$$\Delta x \approx \frac{\hbar c}{\Delta E} = |v_1 - v_2|\, t_{\rm coh} = \left|\frac{c^2 P_1}{E_1} - \frac{c^2 P_2}{E_2}\right| t_{\rm coh} = \left|\frac{m_1^2 c^4}{2E_1^2} - \frac{m_2^2 c^4}{2E_2^2}\right| c t_{\rm coh}, \tag{8}$$

where $E = (E_1 + E_2)/2$ is the average energy of the two neutrinos. We introduce the

[8]. In meson-antimeson systems such as K^0-$\bar K^0$, the meson and antimeson have the same mass (assuming CPT invariance is valid), and can decay to the same final states, such that transitions $K^0 \leftrightarrow \bar K^0$ are possible. The neutrino oscillations considered here are not between neutrinos and antineutrinos, but between different flavor states of neutrinos (or of antineutrinos). For discussion of possible $\nu \leftrightarrow \bar\nu$ oscillations, see [9].

[3]If a neutrino could be produced in either of the mass states 1 or 2, it would remain in that state until observed (provided it propagates in vacuum; propagation through matter involves interactions that depend on neutrino flavor which lead to oscillations between neutrino mass states [10, 11]). If there were a method of observation of mass states, the neutrino would always be observed in the same mass state in which it was created.

[4]We work in the lab frame. In contrast, discussion of K^0-$\bar K^0$ oscillations are typically given in the "rest frame" of the K, which is not strictly well defined since the eigenstates K_L^0 and K_S^0 has different masses. However, the neutral-Kaon mass difference is very small, $\Delta m_K/\overline{m}_K \approx 10^{-14}$, so little error is incurred by this procedure. However, for neutrinos it could be that $\Delta m_\nu/\overline{m}_\nu > 1$, so the notion of a single rest frame for oscillating neutrinos is doubtful.

coherence length $L_{\rm coh}$ according to (see, for example, [14, 15, 16]),[5,6]

$$L_{\rm coh} = ct_{\rm coh} = \frac{\hbar c}{\Delta E \left| \frac{m_1^2 c^4}{2E_1^2} - \frac{m_2^2 c^4}{2E_2^2} \right|} \approx \frac{2E^2 \hbar c}{\Delta E \, |m_1^2 - m_2^2| \, c^4} \,. \tag{9}$$

The usual analysis continues with the approximation (often not stated explicitly) that the first phase factor in eq. (7) can be ignored, and we write,[7]

$$\psi_i(x,t) \approx \psi_{i,0} \, e^{-im_i^2 c^3 x/2E_i\hbar}, \qquad \text{where} \qquad x \approx ct. \tag{10}$$

The exponential phase factor $e^{-im_i^2 c^3 x/2E_i\hbar}$ is slightly different for mass states 1 and 2, which leads to an oscillatory interference term in the spatial/time dependence of an initial single-flavor state.

For a neutrino created at the origin at time $t = 0$ in a pure flavor state a with $\psi_{a,0} = 1$, $\psi_{b,0} = 0$, the initial mass states are,

$$\psi_{1,0} = \cos\theta_{12}, \qquad \psi_{2,0} = \sin\theta_{12}, \tag{11}$$

according to eq. (5), so the evolution of the flavor states is, using eq. (10),

$$\psi_a(x) = \cos\theta_{12}\psi_1 + \sin\theta_{12}\psi_2 = \cos^2\theta_{12} \, e^{-im_1^2 c^3 x/2E_1\hbar} + \sin^2\theta_{12} \, e^{-im_2^2 c^3 x/2E_2\hbar}, \tag{12}$$

$$\psi_b(x) = -\sin\theta_{12}\psi_1 + \cos\theta_{12}\psi_2 = -\cos\theta_{12}\sin\theta_{12}\left(e^{-im_1^2 c^3 x/2E\hbar} - e^{-im_2^2 c^3 x/2E\hbar}\right). \tag{13}$$

The probability that the initial flavor state a is still a after the neutrino has traveled distance x is,

$$\begin{aligned} P_{a\to a}(x) &= |\psi_a(x)|^2 = \cos^4\theta_{12} + \sin^4\theta_{12} + 2\cos^2\theta_{12}\sin^2\theta_{12}\cos\left[\left(\frac{m_1^2}{E_1} - \frac{m_2^2}{E_2}\right)\frac{c^3 x}{2\hbar}\right] \\ &= \cos^4\theta_{12} + \sin^4\theta_{12} + 2\cos^2\theta_{12}\sin^2\theta_{12}\left(1 - 2\sin^2\frac{\Delta m_{12}^2 c^3 x}{4E\hbar}\right) \\ &= 1 - \sin^2 2\theta_{12}\sin^2\frac{\Delta m_{12}^2 c^3 x}{4E\hbar} = 1 - \sin^2 2\theta_{12}\sin^2\frac{x}{L_{\rm osc}}\,, \end{aligned} \tag{14}$$

where the squared mass difference Δm_{12}^2 and oscillation length L are given by,[8]

$$\Delta m_{12}^2 = \left(\frac{m_1^2}{E_1} - \frac{m_2^2}{E_2}\right) E \approx m_1^2 - m_2^2, \qquad L_{\rm osc} = \frac{4E\hbar}{\Delta m_{12}^2 c^3} \approx \frac{4E\hbar c}{|m_1^2 - m_2^2| \, c^4}\,. \tag{15}$$

[5]The neutrino coherence length was perhaps first discussed in [17].

[6]Calling the length defined by eq. (9) the coherence length is perhaps unfortunate in that the meaning here is significantly different from the usage in optics, where the optical coherence length is usually taken to be the spatial width $\hbar c/\Delta E$ of a wave packet (in vacuum) with energy spread ΔE.

[7]It is actually more common to write $\psi_i(x,t) \approx \psi_{i,0}\, e^{-im_i^2 c^4 t/2E_i\hbar}$ where $t = x/c$. Writing $\psi_i(x,t)$ as a function of x is closer to experimental practice, as emphasized in [15, 16].

[8]The oscillations of a neutral meson-antimeson system are usually expressed in the (nominal) rest frame of the meson as $\cos(|m_1 - m_2|\, c^2 t^\star/\hbar)$ (see, for example, [18]). In the lab frame the oscillation has, for $E \gg mc^2$, the approximate form $\cos[(|m_1 - m_2|\, c^2 x/\hbar c)(mc^2/E)] = \cos(\left|m_1^2 - m_2^2\right| c^4 x/2E\hbar c) = 1 - 2\sin^2(\left|m_1^2 - m_2^2\right| c^4 x/4E\hbar c)$, where $m = (m_1 + m_2)/2$ is the average mass of the states 1 and 2 of definite lifetime (sometimes called the "long" and "short" states as in K_L^0 and K_S^0). Again, $L_{\rm osc} = 4E\hbar c/\left|m_1^2 - m_2^2\right| c^4$.

and again $E = (E_1 + E_2)/2$ is the average neutrino energy. The period of the oscillation in x is $\lambda_x = \pi L_{\rm osc}$ for fixed energy E, and the period of the oscillation in E is approximately,

$$\lambda_E \approx \frac{\pi L_{\rm osc} E}{x} = \frac{E}{N_{\rm osc}} \qquad \left(N_{\rm osc} \equiv \frac{x}{\lambda_x} \gg 1 \right), \tag{16}$$

when the fixed distance $x = N_{\rm osc}\lambda_x \gg \lambda_x$, where $N_{\rm osc} \gg 1$ is the number of the oscillation being observed.

The probability that the initial flavor state a is has become flavor b after the neutrino has traveled distance x is,

$$\begin{aligned} P_{a\to b}(x) &= |\psi_b(x)|^2 = 2\cos^2\theta_{12}\sin^2\theta_{12}\left\{1 - \cos\left[\left(\frac{m_1^2}{E_1} - \frac{m_2^2}{E_2}\right)\frac{c^3 x}{2\hbar}\right]\right\} \\ &= \sin^2 2\theta_{12}\sin^2\frac{\Delta m_{12}^2 c^3 x}{4E\hbar} = \sin^2 2\theta_{12}\sin^2\frac{x}{L_{\rm osc}} = 1 - P_{a\to a}(x). \end{aligned} \tag{17}$$

Equations (14) and (17) are the standard representation of two-neutrino oscillations for neutrinos produce in a flavor state.

The coherence length (9) is related to the oscillation length (15) by,

$$L_{\rm coh} \approx \frac{E}{\Delta E} L_{\rm osc} \gtrsim L_{\rm osc}, \tag{18}$$

which indicates that in the extreme case that the neutrino wave packet is minimally narrow, with energy spread $\Delta E \approx E$ as for (anti)neutrinos from the decay of heavy nuclei, only a few oscillations might be observable (if the neutrino is detected without any measurement of its energy). However, as discussed on p. 2, neutrinos from the two-body decay of mesons have $\Delta E \ll E$ and very large numbers of oscillations could be observed in principle.

Under the assumption that the relevant energy spectrum is approximately Gaussian with variance σ_E, it has become conventional to write,[9]

$$L_{\rm coh} = \frac{E}{\sqrt{2}\pi\sigma_E} L_{\rm osc}. \tag{19}$$

2.2 Effect of Detector Resolution

If the neutrino is detected in a manner that determines its energy to some accuracy $\Delta E_{\rm det}$ which is smaller than the energy spread ΔE associated with the source, the energy spread that appears in eqs. (18)-(19) should be $\Delta E_{\rm det}$ rather than the source-related energy spread as considered above.[10] In practice, the relative energy resolution of neutrino detectors is a few percent, which has no effect on the coherence of oscillations of neutrinos from two-body decays, but will be the determining factor for the coherence length of neutrinos from three-body decays. That is, a detector with sensitivity to neutrino energy makes a selection

[9] The definition (19) may have first been given in eq. (24) of [20].

[10] Strictly, $1/\sigma_E^2 = 1/\sigma_{E_{\rm source}}^2 + 1/\sigma_{E_{\rm det}}^2$, as perhaps first discussed in eq. (30) of [21]. See also eq. (53) of [22] and eq. (15) of [20]. Earlier discussion of the role of the detector, as in [23], emphasized its size rather than its energy resolution.

among the full spectrum of energy of neutrinos incident upon it, which permits observation of oscillations of neutrinos from a three-body decay at greater distance from the source than would be possible if the detector merely identified the presence of a neutrino but with no information as to its energy (or momentum).

In particular, if oscillations are to be observed in the energy spectrum of neutrinos from a three-body decay at a single distance x from the source (rather than, say, as oscillations as a function of distance for neutrinos of any energy), the energy resolution of the detector must be better than 1/4 of a period λ_E of the oscillation in energy. Then, eq. (16) implies that $E/\Delta E_{\rm det} \gtrsim 4N_{\rm osc}$, and the coherence length is $L_{\rm coh} \gtrsim 4N_{\rm osc}L_{\rm osc} = 4x/\pi$ for observation of oscillation number $N_{\rm osc}$ at distance $x = N_{\rm osc}\lambda_x = N_{\rm osc}\pi L_{\rm osc}$. That is, the requirement that the detector resolution be good enough to resolve the energy oscillations insures that the coherence length for the oscillations is long enough that they can be observed.[11,12]

For example, in the context of a three-neutrino scenario, where $L_{12} \approx 30L_{13}$, it is possible to resolve the so-called mass hierarchy by observation of rapid 1-3 oscillations with $N_{\rm osc} \approx 30$ at the peak of the first (slower) 1-2 oscillation [25]. The relative detector energy resolution for neutrinos needs to be better than 1/120 to resolve the oscillations, whereas to avoid any effects of decoherence, the energy resolution should be somewhat better than this.

While only a single oscillation has been observed in neutrino experiments (and in the K^0-$\bar K^0$ system) to date, oscillations over nine periods have recently been observed in the B_s^0-$\bar B_s^0$ system [26], which indicates that $E_{B\bar B}/\Delta E \gtrsim 10$ for B_s^0 production at a hadron collider. Since heavy quark states such as the B_s^0 are produced in pp collisions via "fusion" of gluons whose initial energies are not well defined, but the relative detector energy resolution for the B_s^0 is less that 10%, the experimental results [26] are evidence that $\Delta E \approx \Delta E_{\rm det}$ in this case, where good detector resolution has extended the coherence length of the meson-antimeson oscillations.

2.2.1 Decoherence when the Neutrino Energy is Not Used in the Analysis

To illustrate further the notion of "decoherence", we consider the relative rate of electron antineutrinos that could be detected as a function of distance from a nuclear reactor, if the neutrino energy were not measured (or knowledge of the neutrino energy not used in the analysis).[13]

Then, as the energy of the detected neutrinos, roughly $2 < E < 8$ MeV, varies by a factor of ≈ 4, the oscillation length of these neutrinos varies by a factor of 4, and the oscillations become "smeared out" with distance from the reactor. At large distances, oscillations cannot be observed *vs.* distance, and the survival probability is constant at $P(L) \approx 1 - \sin^2(2\theta_{12}) \left\langle \sin^2(\Delta m_{12}^2 L/4E) \right\rangle \approx 1 - 0.5\sin^2(2\theta_{12}) \approx 0.6$ for oscillations where $\sin^2(2\theta_{12}) \approx 0.8$, as in [29] (KamLAND), from which the left figure below is taken.[14]

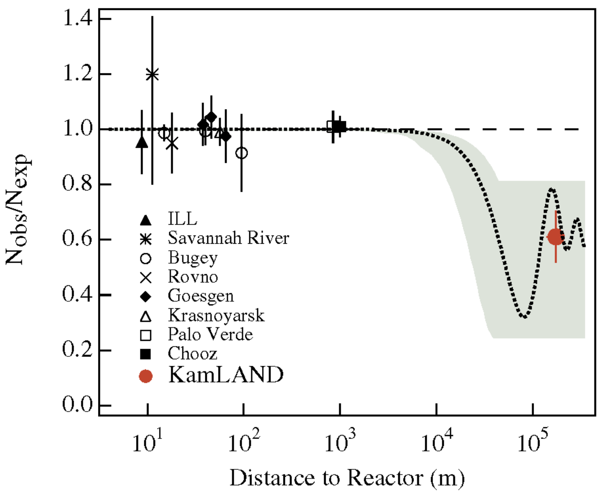
[11] If the energy resolution is barely sufficient to resolve the oscillations, the coherence length is only slightly larger than the source-detector distance, and there may be some loss of amplitude of the oscillations.

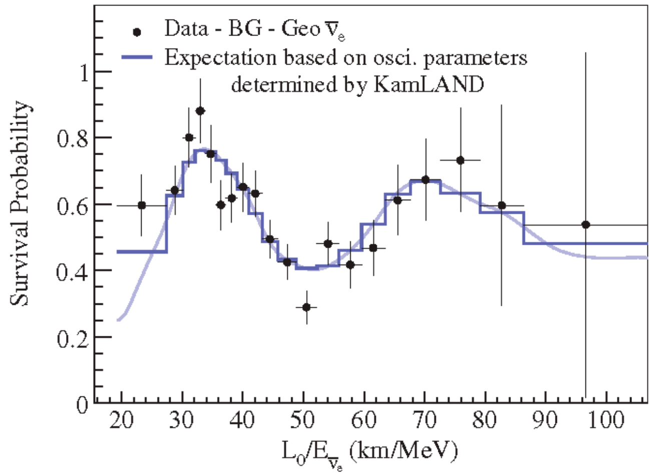
[12] For comments on decoherence in the Daya Bay reactor antineutrino experiment, see [24].

[13] The Daya Bay analyses reported in [27, 28] are not of this type, but use the observed neutrino energy in a fit of the data to a model of the oscillating-neutrino interaction rate *vs.* distance.

[14] Discussion of decoherence in the KamLAND data is given in [30].

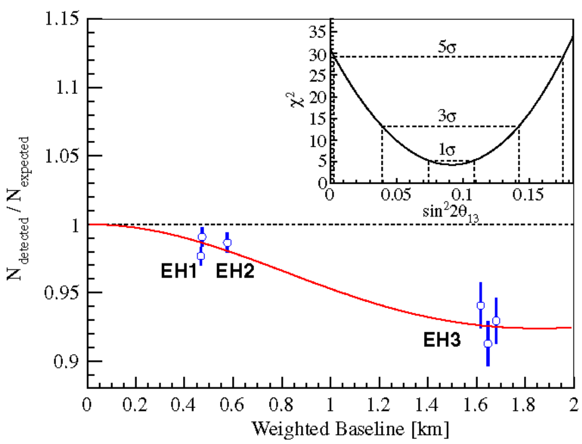
See also [31], where the damping of the oscillations to $1 - 0.5\sin^2(2\theta_{12})$ is called an effect of quantum

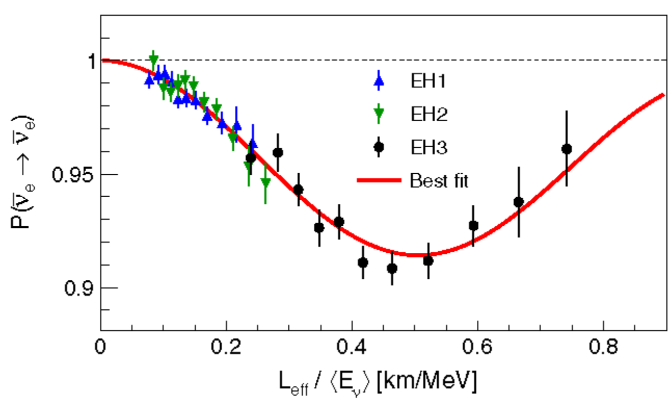




On the other hand, the reconstructed neutrino energy E can be used to plot the data *vs.* L/E, as in the right figure above (from [35]), in which can evidence for neutrino oscillations is more clearly seen.

To illustrate this effect for the Daya Bay experiment, where the relevant neutrino-mixing angle is θ_{13}, with $\sin^2(2\theta_{13}) \approx 0.09$, we recall the left figure below (from [27]), in which the neutrino energy is not used in making the plot, and the right figure below (from [28]), in which the energy is used. Again, better evidence for oscillations is obtained when the measured neutrino energy (with its uncertainty due to the detector energy resolution) is used.





We illustrate this point further with a calculation based on parameters for $1-3$ neutrino oscillations, assuming two different energy bands in the analysis shown in the left figure below.

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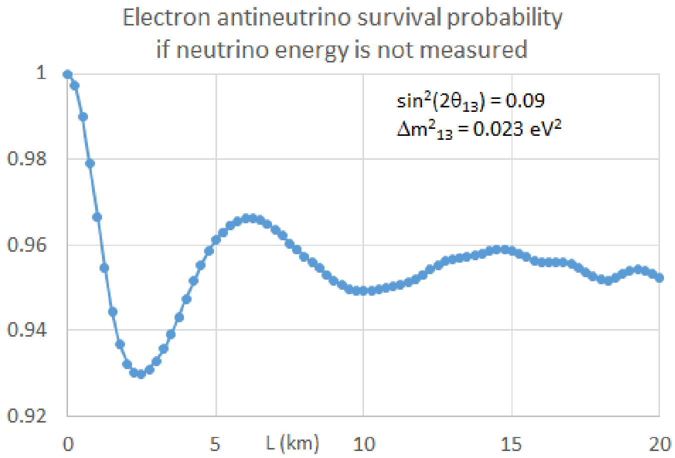
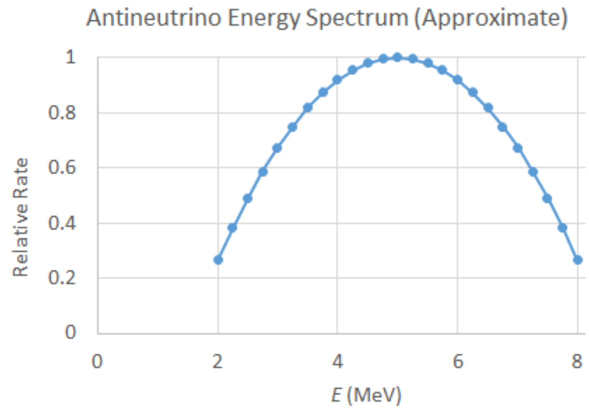
decoherence.

Discussion of decoherence in data from atmospheric and astrophysical neutrinos is given, for example, in [32, 33, 34].





The red curve is for an analysis that ignores the neutrino energy, such that the neutrino oscillation is damped/distorted beyond $\approx L_{\rm osc}(\langle E\rangle)/[\Delta E/E]$ ($\approx$ 2 km for this Daya Bay example)).[15] The blue curve in the left figure above is for an analysis that restricts the neutrino energy to $4.5 < E < 5$ MeV. The coherence length in this case is $\approx$ 15 km, about 6 times longer than for the analysis with $2 < E < 8$ MeV, with only slight degradation of the amplitude of the oscillation at 15 km $= L_{\rm coh}(4.5 < E < 5)$.

In these examples, neutrino oscillations occur, but the effect is not observable as an oscillation at large distances, which loss of information we call "decoherence."[16]

The energy range ΔE used in the data analysis can be changed/varied after the data are collected. This "delayed choice" affects the amount of "decoherence" in the analysis. However, even if the range of reconstructed energy E is made very narrow in the analysis, the restricted data sample corresponds to neutrinos of energy range $\approx \sqrt{2}\pi\sigma_E$, where σ_E is the detector energy resolution. Hence, the coherence length in a data analysis cannot be larger than $EL_{\rm osc}/\sqrt{2}\pi\sigma_E$, which could be called the "quantum coherence length," but it could be shorter if a choice is made after the data were collected to use $\Delta E > \sqrt{2}\pi\sigma_E$. In the latter case, we could speak of the "classical coherence length" $EL_{\rm osc}/\Delta_E$.

The amount of decoherence depends on the range ΔE of energies sampled in the detector/data analysis, as well as on the source-detector distance. Decoherence is often stated as an effect of the "environment" on a quantum system, and in the present examples, the "environment" includes the "empty space" between the source and the detector, as well as the detector itself.

These examples reinforce that the quantity $L_{\rm osc}(\langle E\rangle)/[\Delta E/E]$ should be regarded as the coherence length $L_{\rm coh}$ in an experiment where neutrinos within energy range ΔE are observed.

[15]If there were no "smearing"/"decoherence", the first minimum in the red curve in the above left plot would have value 1 - 0.09 = 0.91 rather than 0.93 (at $L \approx 2.5$ km $\approx L_{\rm osc}(\langle E\rangle) \approx L_{\rm coh}$). Hence, some effect of "decoherence" is already observable in the figure at $L_{\rm coh} \approx 2.5$ km.

[16]Some people (for example, [36]) consider that the "smearing" of the oscillations due to limited energy resolution in a neutrino detector is not an effect of "decoherence", although the "smearing" precludes observation of the oscillations at large distances. In this view, the "coherence length" is not the length over which an oscillatory signal can be well observed, but a more abstract concept of less relevance to experimental measurements.

In sum, **the coherence length depends on the detector/data analysis, as well as the neutrino-production process**.

2.2.2 Further Details

When a neutrino (or antineutrino) of flavor a is produced in the decay,

$$A \to B + \nu_a, \tag{20}$$

in the rest frame of particle A of mass m_A, and the neutrino flavor state ν_a is related to neutrino mass eigenstates ν_1 and ν_2 of masses m_1 and m_2 by

$$|\nu_a\rangle = \cos\theta_{12}|\nu_1\rangle + \sin\theta_{12}|\nu_2\rangle, \tag{21}$$

the final state wavefunction in entangled, and can be written as,

$$|B, \nu_a\rangle = \cos\theta_{12}|B_1\rangle|\nu_1\rangle + \sin\theta_{12}|B_2\rangle|\nu_2\rangle. \tag{22}$$

Energy and momentum conservation are that,

$$m_A = E_{B_1} + E_{\nu_1} = E_{B_2} + E_{\nu_2}, \qquad 0 = \mathbf{P}_{B_1} + \mathbf{P}_{\nu_1} = \mathbf{P}_{B_2} + \mathbf{P}_{\nu_2}, \tag{23}$$

where, of course, the energy and momentum for a state of mass m are related by $E^2 = m^2c^4 + P^2c^2$.

We are particularly interested in the case of a three-body β-decay, where B is a two-particle system, and the neutrino energies form a continuum over a range of several MeV.

In general, the neutrino is detected with an energy resolution smaller than the width of its β-decay spectrum, so that for detected neutrinos, we can speak of σ_E ($\approx 0.08\sqrt{E}$ for the Daya Bay experiment) as the detector energy resolution rather than as the width of the β-decay spectrum.[17]

The usual argument[18] is that the process of detection of a neutrino leaves it with a definite energy E, even if this value is not well known due to the uncertainty in the measurement of that energy, which is reported as $\bar{E}$. Then, the observed behavior of detected neutrinos is to be obtained by weighting their survival probability by an approximately Gaussian detector-resolution function.

From eq. (14), the probability that a neutrino of energy E and flavor a is still of that flavor after traveling distance x is,

$$P_{a\to a}(x, E) = 1 - \sin^2 2\theta_{12} \sin^2 \frac{\Delta m_{12}^2 c^3 x}{4E\hbar}\,. \tag{24}$$

The probability that a neutrino is detected as having energy $\bar{E}$ by a detector with rms energy resolution $\sigma_E(\bar{E})$ is,

$$P_{a\to a}(x, \bar{E}) \propto \int dE\, e^{-(E-\bar{E})/2\sigma_E^2} P_{a\to a}(x, E). \tag{25}$$

[17] In the unrealistic case of extremely fine detector resolution σ_E would not go to zero, but to a small value governed by other considerations, such as the size of the atom that contained the state A. That is, $\sigma_E^2 = \sigma_{E_{\rm det}}^2 + \sigma_{E_{\rm other}}^2$.

[18] See, for example, [37].

The Gaussian factor in eq. (24) can be rewritten as,[19]

$$\begin{aligned} e^{-(E-\bar{E})^2/2\sigma_E^2} &= e^{-x^2E^2\bar{E}^2(E-\bar{E})^2/2\sigma_E^2E^2\bar{E}^2x^2} = e^{-E^2\bar{E}^2(x/E-x/\bar{E})^2/2\sigma_E^2x^2} \approx e^{-\bar{E}^4(x/E-x/\bar{E})^2/2\sigma_E^2x^2} \\ &\equiv e^{-\bar{E}^2(w-\bar{w})^2/2\sigma_E^2\bar{w}^2}, \end{aligned} \tag{26}$$

where $w = x/E$ and $\bar{w} = x/\bar{E}$. The probability (25) can now be represented in the (normalized) form,

$$\begin{aligned} P_{a\to a}(\bar{w}) &= \int dw\, \frac{e^{-\bar{E}^2(w-\bar{w})^2/2\sigma_E^2\bar{w}^2}}{\sqrt{2\pi}\sigma_E\bar{w}/\bar{E}} \left(1 - \sin^2 2\theta_{12} \sin^2 \frac{\Delta m_{12}^2 c^3 w}{4\hbar}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_{12} \int dw\, \frac{e^{-\bar{E}^2(w-\bar{w})^2/2\sigma_E^2\bar{w}^2}}{\sqrt{2\pi}\sigma_E\bar{w}/\bar{E}} \left(1 - \cos \frac{\Delta m_{12}^2 c^3 w}{2\hbar}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_{12} \left(1 - \int dw'\, \frac{e^{-\bar{E}^2 w'^2/2\sigma_E^2\bar{w}^2}}{\sqrt{2\pi}\sigma_E\bar{w}/\bar{E}} \cos \frac{\Delta m_{12}^2 c^3 (w'+\bar{w})}{2\hbar}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_{12} \left(1 - \cos \frac{\Delta m_{12}^2 c^3 \bar{w}}{2\hbar}\, e^{-\sigma_E^2 \Delta m_{12}^4 c^6 \bar{w}^2/8\bar{E}^2\hbar^2}\right) \\ &= 1 - \frac{1}{2}\sin^2 2\theta_{12} \left(1 - \cos \frac{2x}{L_{\rm osc}(\bar{E})}\, e^{-2(\sigma_E^2/\bar{E})^2 x^2/L_{\rm osc}^2(\bar{E})}\right), \end{aligned} \tag{27}$$

using Gradshteyn and Ryzhik 3.896.4 [38]. For very fine detector energy resolution, $\sigma_E/\bar{E} \ll 1$, we recover the form (14). For coarse energy resolution the survival probability does not oscillate, but simply has the value $1 - (1/2)\sin^2 2\theta_{12}$ independent of x, and we say that the oscillations have decohered, as illustrated in the top left figures on pp. 3-4. The damping/coherence length in the last form of eq. (27) is,

$$L_{\rm coh}(\bar{E}) = \frac{L_{\rm osc}(\bar{E})}{\sqrt{2}\,\sigma_E/\bar{E}}\,. \tag{28}$$

2.3 Effect of Source Size

If the neutrino source is large compared to an oscillation length the evidence for neutrino oscillations in a detector will be "washed out". This is not strictly an effect of decoherence, in that neutrinos produced in different primary interactions do not interfere with one another.[20,21]

This effect is important in studies of oscillations of reactor neutrinos, where the distances between the detector and multiple reactors must be not too different. Also, since supernova neutrinos have energies and oscillations lengths similar to those for reactor neutrinos, but

[19] This type of transformation was used in [22], on which this section is based.

[20] Dirac has written [39] "Each photon then interferes only with itself. Interference between two different photons never occurs."

[21] The source of the neutrino could be determined by detection of its partner B in the production reaction (1) (without precise determination of the energy or momentum of B), so in the case of multiple source points the observed probability distributions (14) and (17) are the sum of those for the various possible production points.

the size of supernovas is large compared to the kilometer scale of the oscillation lengths, oscillations of supernova neutrinos cannot be observed.[22,23]

2.4 Entanglement in the Decay $A \to \nu_a \bar{l}_a$

To clarify how energy and momentum are conserved in neutrino oscillations of an initial flavor state ν_a, we suppose that the neutrino is produced in the decay of a parent meson A via the two-body mode,

$$A \to \nu_a \bar{l}_a, \tag{29}$$

where $\bar{l}_a$ is the charged antilepton of flavor a. We can suppose that the parent particle is at rest, and that $m_A - m_{l_a} \gg m_i$, so the total neutrino energy is large compared to the neutrino rest-mass/energy and the neutrino speed is close to c.[24] The initial neutrino state just after the decay is,

$$|\nu_a\rangle = \cos\theta_{12}\,|\nu_1\rangle + \sin\theta_{12}\,|\nu_2\rangle, \tag{30}$$

where the neutrino mass states have different energies E_i and momenta $\mathbf{P}_i$. The entire final state just after the decay is the entangled combination,

$$\cos\theta_{12}\,|\nu_1\rangle|\bar{l}_{a,1}\rangle + \sin\theta_{12}\,|\nu_2\rangle|\bar{l}_{a,2}\rangle, \tag{31}$$

where the two states $|\bar{l}_{a,1}\rangle$ and $|\bar{l}_{a,2}\rangle$ of the antilepton have different energies, momenta and velocities,

$$E'_i = m_A c^2 - E_i, \qquad \mathbf{P}'_i = -\mathbf{P}_i, \qquad E'^2_i - c^2 P'^2_i = m^2_{l_a} c^4, \qquad \mathbf{v}'_i = \frac{c^2 \mathbf{P}'_i}{E'_i}\,, \tag{32}$$

where primed quantities describe the antilepton.

In general, the antilepton $\bar{l}_a$ interacts with its environment long before the neutrino is detected. If this interaction resulted in a "measurement" of the energy and momentum of the antilepton that could distinguish between the two values of E'_i and $\mathbf{P}'_i$, thereby determining

[22] A separate issue is that for oscillations to be observed at distances from the source that are very large compared to an oscillation length, the period of the oscillations in the energy spectrum is very short, and extremely good detector energy resolution would be required to resolve the oscillations from a "point" source.

[23] Yet another issue for stellar and supernova neutrinos is that their coherence length is limited by electromagnetic collisions of the source particles just prior to production of a neutrino. See, for example, [17, 40, 41].

[24] If the neutrino produced in the decay (29) is later measured to have flavor b, we have in a sense observed the reaction $A \to \nu_b \bar{l}_a$. This is nominally forbidden by lepton-number conservation. What we learn is that lepton-number conservation applies only to the primary (weak) interaction (29), and not to the subsequent time evolution of the neutrino. That is, Nature does not require that if the neutrino is measured (some time after its production) to have flavor b then the antilepton also have that flavor.

For example, there is not sufficient energy available in nuclear decays to produce muon antineutrinos (via decays of the form $(A, Z \to (A, Z+2)\,\mu^- \bar{\nu}_\mu)$, but the allowed production of electron antineutrinos $\bar{\nu}_e$ does lead to observed oscillations in the survival probability $P_{\bar{\nu}_e \to \bar{\nu}_e}(t)$, with the implication that the final state at times after the neutrino production can be $(A, Z+2)\,e^- \bar{\nu}_\mu$.

the value of index i at the time of the interaction, then the neutrino would be observed (generally at a later time) as if it had the same value of index i.[25]

A complication is that the neutrino mass states are not directly observable; rather only the flavor states a and b are observable. Assuming the interaction of the antilepton occurs at distance from the decay point small compared to a neutrino oscillation length, the probabilities that the antilepton would be "measured" to have indices 1 and 2 are $\cos^2\theta_{12}$ and $\sin^2\theta_{12}$, respectively. Then, the probabilities that the neutrino is observed to have flavors a and b when the antilepton is "measured" to have index 1 are $\cos^4\theta_{12}$ and $\cos^2\theta_{12}\sin^2\theta_{12}$, respectively, while probabilities that the neutrino is observed to have flavors a and b when the antilepton is "measured" to have index 2 are $\sin^4\theta_{12}$ and $\cos^2\theta_{12}\sin^2\theta_{12}$, respectively. Hence, the total probability of observing neutrino flavor state a (at any time), if the antilepton is "measured" quickly after its production, is $\cos^4\theta_{12} + \sin^4\theta_{12} = 1 - 2\cos^2\theta_{12}\sin^2\theta_{12} = 1 - (1/2)\sin^2 2\theta_{12}$ and the probability of observing flavor state b is $2\cos^2\theta_{12}\sin^2\theta_{12} = (1/2)\sin^2 2\theta_{12}$. While the initial neutrino state was flavor a, a later observation of flavor state b is not strictly an oscillation, as the probability of this observation is independent of time.

Note that an experiment which observes flavor neutrino states at only a single time/distance from the production point cannot distinguish neutrino oscillations from the scenario in which the partner antilepton is "measured" by its environment.[26] That is, past claims of evidence for neutrino oscillations based on data collected only at a single distance from the neutrino source are something of a "hoax", in that such claims are valid only if neutrino oscillations actually do exist.[27]

The first compelling evidence for neutrino oscillations was the observation of ν_μ "disappearance" in atmospheric neutrinos (as detected under the Earth's surface) [44], which showed a dependence on neutrino energy of the disappearance probability. Similarly, the evidence for $\bar{\nu}_e$ disappearance in reactor neutrino experiments [45, 46] with detectors as multiple distances is also evidence for neutrino oscillation.

The evidence for neutrino oscillation in experiments, in which the partner antilepton (or partner lepton in case of antineutrino production, as at nuclear reactors) interacts with its environment long before the neutrino is detected, indicates that the interactions of the antilepton with its environment do not constitute a "measurement" that destroys the entanglement in the wavefunction of the neutrino.[28]

Shortly after the neutrino is produced, it is appropriate to consider that the entangled wavefunction is no longer given by eq. (31), but by something like,

$$\cos\theta_{12}\,|\nu_1\rangle|\mathrm{env}_1\rangle + \sin\theta_{12}\,|\nu_2\rangle|\mathrm{env}_2\rangle, \tag{33}$$

[25] A similar situation is realized is certain optical experiments with entangled pairs of photons. An appropriate measurement of one photon (with the choice of type of measurement being delayed until after the entangled pair has been created) can destroy interference effects that are otherwise observable for the second photon. See, for example, [42].

[26] The recent observation of ν_e events from a beam of ν_ν in the T2K experiment [19] is not by itself evidence for neutrino oscillations, but only for neutrino mixing.

[27] The observation of a non-electron-neutrino component of the flux of solar neutrinos [43] was carefully presented as evidence (by itself) only for neutrino mixing and not for neutrino oscillation.

[28] This contrasts with the view that interactions with the "box"/environment of the macroscopic cat in Schrödinger's famous cat paradox [47] result in decoherence of any entangled quantum state of the cat, rendering it to be "classical".

where the environmental states $|\text{env}_i\rangle$ are localized near the origin (production point) and have energies that are large compared to the neutrino energies E_i, and which differ by $E'_1 - E'_2 = E_2 - E_1$. then the difference in the energies of the two relevant environmental states is much smaller compared to their average energies than is the ratio $2(E_1 - E_2)/(E_1 + E_2)$ for the neutrino mass states. Hence, we can ignore the tiny phase differences in the environmental states in the wavefunction (33), such that the effective time dependence (up to an overall phase factor) of the system is the same as that previously assumed in eqs. (12)-(13). The standard neutrino-oscillation analysis of sec. 2.1 holds to a very good approximation for the entangled neutrino-production state, which state clarifies how energy and momentum are conserved in neutrino oscillations.[29]

A Appendix: Oscillation Length If No Antilepton Interactions

If the antilepton $\bar{l}_a$ of eq. (29) never interacts (is never measured), then the evolution of the state $|\nu_i(x,t)\rangle|\bar{l}_{a,i}(x',t')\rangle$ would be,

$$\begin{aligned} |\nu_i(x,t)\rangle|\bar{l}_{a,i}(x',t')\rangle &= |\nu_{i,0}\rangle\, e^{i(P_i x - E_i t)/\hbar}|\bar{l}_{a,i,0}\rangle\, e^{i(P'_i x' + E'_i t')/\hbar} \\ &\approx |\nu_{i,0}\rangle|\bar{l}_{a,i,0}\rangle\, e^{2iP_i x/\hbar}\, e^{-i[E_i - (m_A c^2 - E_i)]x/\hbar c} \\ &= |\nu_{i,0}\rangle|\bar{l}_{a,i,0}\rangle\, e^{2i(P_i - E_i/c)x/\hbar}\, e^{i m_A c^2 x/\hbar c} \\ &\approx |\nu_{i,0}\rangle|\bar{l}_{a,i,0}\rangle\, e^{-i m_i^2 c^3 x/E_i\hbar}\, e^{-i m_A c^2 x/\hbar c}, \end{aligned} \tag{34}$$

in the plane-wave approximation, noting that the wave packets are significant only for $x \approx ct$ and $x' = -ct' \approx -x$ in the simplifying assumption that the antilepton has $v' \approx c$, and recalling eqs. (6) and (32). To within the phase factor $e^{-i m_A c^2 x/\hbar c}$ which is common to both indices i, this form is the same as that of eq. (7), except for the absence of a factor of 2 in the factor $e^{-i m_i^2 c^3 x/E_i\hbar}$. Then, the oscillation analysis of sec. 2.1 follows as before, except that the oscillation length differs by a factor of 2,

$$L_{\text{osc}} = \frac{2E\hbar}{\Delta m_{12}^2 c^3} \approx \frac{2E\hbar c}{|m_1^2 - m_2^2|\, c^4} \qquad (\text{no interactions of } \bar{l}_a,\ E'_{\bar{l}_a} \gg m_{\bar{l}_a} c^2). \tag{35}$$

[29] A somewhat similar argument is given in sec. 10.2 of [48]. In contrast, interactions of the antilepton with the environment are ignored in [49], which leads to a factor of ≈ 2 difference in the computation of the oscillation length (15). That factor of 2 is then argued away in a manner that is not easy to follow. Rather, it seems to this author that if the antilepton in the reaction (29) never interacted (and so was never observed), then the neutrino oscillation length would indeed be different by a factor ≈ 2 from the standard version give above. See the Appendix.

The fact that the standard analysis of neutrino oscillations without entanglement gives an excellent understanding is somewhat fortuitous, in that it is essential that entanglement exists to conserve energy and momentum, while, if neutrino oscillations are to occur, interaction of the antilepton with the environment is needed so the phase factors of the antilepton energy can be ignored, but this interaction must not destroy the "Schrödinger cat" character of the wavefunction of the system. The latter behavior is unusual for entangled quantum systems that interact with their environment. The difference between the case of neutrinos and, say, qbits, is that, for neutrinos, the energy difference between the (antilepton) quantum states that interact with the environment is so small that these interactions do not, in effect, determine/measure the energy well enough to resolve the energy difference; the interactions of the antileptons with the environment do not decohere the quantum state of the neutrino.

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