# Forces on Magnetic Dipoles

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# 1 Problem

Deduce expressions for the force on a permanent Ampèrian magnetic dipole with moment $\mathbf{m}_{\rm A}$ (due to a loop of electrical current), and for the force on a permanent Gilbertian magnetic dipole $\mathbf{m}_{\rm G}$ (due to a pair of opposite magnetic charges), when the dipoles are (instantaneously at rest in an external electromagnetic field.

Experiments have been performed[1] to determine whether a neutron [6] has an Ampèrian or a Gilbertian magnetic dipole moment, by studying the interaction of neutrons with magnetized foils. The conclusion is that the neutron has an Ampèrian magnetic moment. To accommodate this case, your analysis should consider that the test moment $\mathbf{m}$ is inside a magnetic material.

Consider also the case that the external fields include (possibly time-dependent) electric fields, and comment on "hidden momentum" [7] of the system.

# 2 Solution

## 2.1 Electrodynamics with Both Electric and Magnetic Charges

*This section follows [8].*

When Heaviside [9] first presented Maxwell's equations [10] in vector notation he assumed that in addition to electric charge and current densities, $\rho_e$ and $\mathbf{J}_e$, there existed magnetic charge and current densities, $\rho_m$ and $\mathbf{J}_m$, although there remains no experimental evidence for the latter.[2] Maxwell's equations for microscopic electrodynamics are then (in SI units),

$$\boldsymbol{\nabla}\cdot\epsilon_0\mathbf{E} = \rho_e, \quad \boldsymbol{\nabla}\cdot\frac{\mathbf{B}}{\mu_0} = \rho_m, \quad -c^2\boldsymbol{\nabla}\times\epsilon_0\mathbf{E} = \frac{\partial}{\partial t}\frac{\mathbf{B}}{\mu_0} + \mathbf{J}_m, \quad \boldsymbol{\nabla}\times\frac{\mathbf{B}}{\mu_0} = \frac{\partial\epsilon_0\mathbf{E}}{\partial t} + \mathbf{J}_e, \tag{1}$$

where $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum. In macroscopic electrodynamics we consider media that contain volume densities of electric- and Ampèrian magnetic-dipole moments, $\mathbf{P}_e$ and $\mathbf{M}_e$, respectively (often called the densities of polarization and magnetization). Supposing that magnetic charges exist, the media could also contain volume densities of (Gilbertian) electric- and magnetic-dipole moments, $\mathbf{P}_m$ and $\mathbf{M}_m$, respectively. These densities can be associated with bound charge and current densities, which together with the "free" charge and current densities $\tilde\rho_e$, $\tilde{\mathbf{J}}_e$, $\tilde\rho_m$ and $\tilde{\mathbf{J}}_m$ comprise the total charge and current densities, and are related by,

$$\rho_e = \tilde\rho_e - \boldsymbol{\nabla}\cdot\mathbf{P}_e, \qquad \mathbf{J}_e = \tilde{\mathbf{J}}_e + \frac{\partial\mathbf{P}_e}{\partial t} + \boldsymbol{\nabla}\times\mathbf{M}_e, \tag{2}$$

[1] The pioneering experiments are [1]-[4]. For a review that emphasizes the physics principles, see [5].
[2] Heaviside seems to have regarded magnetic charges as "fictitious", as indicated on p. 25 of [11].

$$\rho_m = \tilde{\rho}_m - \boldsymbol{\nabla} \cdot \mathbf{M}_m, \qquad \mathbf{J}_m = \tilde{\mathbf{J}}_m + \frac{\partial \mathbf{M}_m}{\partial t} - c^2 \boldsymbol{\nabla} \times \mathbf{P}_m. \tag{3}$$

It is customary in macroscopic electrodynamics to use versions of Maxwell's equations in which only "free" charge and current densities appear. For this we introduce the fields[3]

$$\mathbf{D}_e = \epsilon_0 \mathbf{E} + \mathbf{P}_e, \qquad \mathbf{H}_e = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}_e, \qquad \mathbf{D}_m = \frac{\mathbf{E}}{\mu_0} - c^2 \mathbf{P}_m, \qquad \mathbf{H}_m = \frac{\mathbf{B}}{\mu_0} + \mathbf{M}_m, \tag{4}$$

such that $\mathbf{D}_e$ and $\mathbf{H}_m$, and also $\mathbf{H}_e$ and $\mathbf{D}_m$, have similar forms, and,

$$\boldsymbol{\nabla} \cdot \mathbf{D}_e = \tilde{\rho}_e, \qquad \boldsymbol{\nabla} \cdot \mathbf{H}_m = \tilde{\rho}_m, \qquad -\boldsymbol{\nabla} \times \mathbf{D}_m = \frac{\partial \mathbf{H}_m}{\partial t} + \tilde{\mathbf{J}}_m, \qquad \boldsymbol{\nabla} \times \mathbf{H}_e = \frac{\partial \mathbf{D}_e}{\partial t} + \tilde{\mathbf{J}}_e. \tag{5}$$

where in the absence of magnetic charges $\mathbf{D}_e$ and $\mathbf{H}_e$ are the familiar fields $\mathbf{D}$ and $\mathbf{H}$.[4]

In static situations with no "free" currents $\tilde{\mathbf{J}}_e$ or $\tilde{\mathbf{J}}_m$ the curls of both $\mathbf{D}_m$ and $\mathbf{H}_e$ are zero and these fields can be deduced from scalar potentials $V_e$ and $V_m$,

$$\boldsymbol{\nabla} \times \mathbf{D}_m = 0 \quad \Leftrightarrow \quad \mathbf{D}_m = -\boldsymbol{\nabla} V_e, \qquad \boldsymbol{\nabla} \times \mathbf{H}_e = 0 \quad \Leftrightarrow \quad \mathbf{H}_e = -\boldsymbol{\nabla} V_m. \tag{6}$$

We can associate potential energies,

$$U_e = \mu_0 q_e V_e, \qquad U_m = \mu_0 q_m V_m, \tag{7}$$

with electric and magnetic "test" charges $q_e$ and $q_m$ in scalar potentials due to other charges. If those other charges are held fixed, the forces on the "test" charges can be written as,

$$\mathbf{F}_e = -q_e \boldsymbol{\nabla} V_e = \mu_0 q_e \mathbf{D}_m, \qquad \mathbf{F}_m = -q_m \boldsymbol{\nabla} V_m = \mu_0 q_m \mathbf{H}_e. \tag{8}$$

The magnetic version of eq. (8) was introduced by Poisson [17], and Maxwell [16] reflected this tradition by calling $\mathbf{H}_e = \mathbf{H}$ the magnetic force (per unit magnetic charge) and $\mathbf{B}$ the magnetic induction. Note that eq. (8) holds in media with nonzero, static densities $\mathbf{P}_e$, $\mathbf{P}_m$, $\mathbf{M}_e$ and $\mathbf{M}_m$; the forces on charges inside static electromagnetic media are not $q_e\mathbf{E}$ or $q_m\mathbf{B}/\mu_0$.[5,6] This contrasts with force calculations for the effective magnetic-charge density,

[3] The forms (4) were suggested to the author by David Griffiths in a comment on an early draft of the note [8]. Such "double" $\mathbf{D}$ and $\mathbf{H}$ fields were anticipated by Heaviside [12], who wrote $\mathbf{H}$ for $\mathbf{H}_e$ and $\mathbf{h}_0$ for $\mathbf{H}_m$ near his eq. (88). Our eq. (4) appears as eq. (5.9) of [13], in Gaussian units, where $\mathbf{P} \to \mathbf{P}_e$, $\mathbf{M} \to \mathbf{M}_e$, $\mathbf{P}^\star \to \mathbf{P}_m$, $\mathbf{M}^\star \to \mathbf{M}_m$, $\mathbf{D} \to \mathbf{D}_e$, $\mathbf{H} \to \mathbf{H}_e$, $\mathbf{E}^\star \to \mathbf{D}_m$, $\mathbf{B}^\star \to \mathbf{H}_m$. See also in sec. 4 of [14], with the identifications that $\bar{\mathbf{e}} \to \mathbf{E}$, $\bar{\mathbf{b}} \to \mathbf{B}$, $\mathbf{p} \to \mathbf{P}_e$, $\mathbf{m} \to \mathbf{M}_e$, $\mathbf{m}^\star \to \mathbf{P}_m$, $\mathbf{p}^\star \to \mathbf{M}_m$, $\mathbf{D} \to \mathbf{D}_e$, $\mathbf{H} \to \mathbf{H}_e$, $\mathbf{E} \to \mathbf{D}_m$, $\mathbf{B} \to \mathbf{H}_m$.

[4] The relation $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ (or $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ in Gaussian units) seems to have been first introduced by W. Thomson in 1871, eq. (r), p. 401 of [15], and appears in sec. 399 of Maxwell's *Treatise* [16].

[5] The notion of the force on a static "test" charge inside a macroscopic medium is somewhat contradictory, in that the macroscopic fields are based on averages over volumes larger than atoms/molecules. People often suppose the test charge to be inside a cavity whose volume is at least as large as an atom/molecule, but then the magnitude of the force depends on the shape of the cavity (pp. 290-293 of [17], sec. 517 of [15], Arts. 395-400 of [16]). A more meaningful issue is the force on a "test" charge that moves through the medium, thereby sampling the microscopic fields in a way that can be well approximated in terms of the macroscopic fields. See also sec. 8 of [18].

[6] In sec. 400 of [16], Maxwell noted that (in Gaussian units) the $\mathbf{H}$ field inside a disk-shaped cavity with axis parallel to $\mathbf{B}$ and $\mathbf{H}$ inside a magnetic medium has $\mathbf{H}_{\text{cavity}} = \mathbf{B}_{\text{cavity}} = \mathbf{B}_{\text{medium}} = \mathbf{H}_{\text{medium}} + 4\pi\mathbf{M}$, so that in this case one could say that the force on a magnetic charge $q_m$ in the cavity is $\mathbf{F}_m = q_m\mathbf{H}_{\text{cavity}} = q_m\mathbf{B}_{\text{medium}}$. The led Maxwell to the characterization of $\mathbf{B}$ as the "actual magnetic force", which this author finds misleading.

$\rho_{m,\text{eff}} = -\boldsymbol{\nabla}\cdot\mathbf{M}_e$, which represent effects of Ampèrian currents, as discussed in Appendix A of [8].

*As noted in [19] and on p. 429 of [20], if a magnetic charge $q_m$ could be made to move around a loop some or all of which lies inside an Ampèrian magnetic material where $\mathbf{B}$ does not equal $\mu_0\mathbf{H}_e$ (and hence $\boldsymbol{\nabla}\times\mathbf{B}$ is nonzero around the loop), then energy could be extracted from the system each cycle if the force were $q_m\mathbf{B}/\mu_0$, and we would have a perpetual-motion machine. Similarly, if an electric charge $q_e$ could be made to move around a loop some or all of which lies inside a Gilbertian magnetic material where $\mathbf{E}$ does not equal $\mathbf{D}_m/\epsilon_0$ (and hence $\boldsymbol{\nabla}\times\mathbf{E}$ is nonzero around the loop), then energy could be extracted from the system each cycle if the force were $q_e\mathbf{E}$, and we would again have a perpetual-motion machine.*

The electromagnetic force on a moving electric charge $q_e$ and magnetic charge $q_m$, each with velocity $\mathbf{v}$, is, in microscopic electrodynamics,[7,8,9,10]

$$\begin{aligned}\mathbf{F}_e &= q_e(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \mu_0 q_e\left(\mathbf{D}_m+\mathbf{v}\times\mathbf{H}_m\right), &(9)\\ \mathbf{F}_m &= q_m\left(\mathbf{B}-\frac{\mathbf{v}}{c^2}\times\mathbf{E}\right) = \mu_0 q_m(\mathbf{H}_e-\mathbf{v}\times\mathbf{D}_e). &(10)\end{aligned}$$

It has been verified [32] that the deflection of high-energy electrically charged particles as they pass through magnetized iron depends on $\mathbf{B}$ and not $\mu_0\mathbf{H}_e$ (assuming the magnetization of iron is Ampèrian), nor on $\mu_0\mathbf{H}_m$ (assuming the magnetization of iron is Gilbertian, which confirms eq. (9) and that the magnetization of iron, *i.e.*, of electrons, is not Gilbertian (and hence is Ampèrian). See also [33, 34, 35].

In macroscopic electrodynamics the Lorentz force law for the force density $\mathbf{f}$ on "free" charge and current densities takes the forms,[11,12]

$$\mathbf{f}_e = \tilde\rho_e\mathbf{E}+\tilde{\mathbf{J}}_e\times\mathbf{B}, \qquad [\text{and not}\quad \mu_0(\tilde\rho_e\mathbf{D}_m+\tilde{\mathbf{J}}_e\times\mathbf{H}_m)], \tag{11}$$

$$\mathbf{f}_m = \mu_0(\tilde\rho_m\mathbf{H}_e-\tilde{\mathbf{J}}_m\times\mathbf{D}_e), \qquad [\text{and not}\quad \tilde\rho_m\mathbf{B}-\frac{\tilde{\mathbf{J}}_m}{c^2}\times\mathbf{E}], \tag{12}$$

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[7] Lorentz advocated the form $\mathbf{F}_e = \mu_0 q_e(\mathbf{D}_e+\mathbf{v}\times\mathbf{H}_e)$ in eq. (V), sec. 12 of [21], although he seems mainly to have considered its use in vacuum. See also eq. (23) of [22]. That is, Lorentz considered $\mathbf{D}_e$ and $\mathbf{H}_e$, rather than $\mathbf{E}$ and $\mathbf{B}$, to be the microscopic electromagnetic fields.

[8] Maxwell discussed the "Lorentz" force law in Arts. 599-603 of [16], but made almost no use of it. It is generally considered that Heaviside first gave the Lorentz force law (9) for electric charges in [23], but the key insight is already visible for the electric case in [9] and for the magnetic case in [24]. The form of $\mathbf{F}_m$ in terms of $\mathbf{B}$ and $\mathbf{E}$ is implicit in eq. (7) of [25] and explicit in sec. 28B of [26]. See also [27, 28].

[9] For the macroscopic equations to appear as in eq. (5), as given, for examples, in sec. 7.3.4 and prob. 7.60 of [29], the Lorentz force law must have the form (10) for magnetic charges. One can also redefine the strength of magnetic charges, $\rho_m \to \rho_m/\mu_0$, $\mathbf{J}_m \to \mathbf{J}_m/\mu_0$, which leads to the forms given, for example, in sec. 6.11 of [30]. These alternative definitions echo a debate initiated by Clausius in 1882 [31].

[10] Consistency of the Lorentz force law with special relativity requires that either $\mathbf{E}$ and $\mathbf{B}$ or $\mathbf{D}_e$ and $\mathbf{H}_e$ or $\mathbf{D}_m$ and $\mathbf{H}_m$ appear in $\mathbf{F}_e$ and in $\mathbf{F}_m$ (see Appendix B of [8]).

[11] A subtlety is that the field $\mathbf{B}$ in the first form of eq. (11) is not the total field, but rather the field at the location of the free current that would exist in its absence. See, for example, [36], especially sec. 4.

[12] In 1908-10, Einstein argued that the Lorentz force law should take the form $\mathbf{f}_e = \mu_0(\tilde\rho_e\mathbf{D}_e+\tilde{\mathbf{J}}_e\times\mathbf{H}_e)$ inside materials [37, 38, 39, 40], perhaps based on a misunderstanding discussed in [41], or that discussed in sec. 2.3.1 below. This misunderstanding underlies the recent "paradox" of Mansuripur [42].

based on a consistency argument involving Poynting's theorem [8] (as also required not to have magnetic perpetual-motion machines).[13,14,15,16,17]

*In a search for an isolated magnetic charge $q_m$ in media that otherwise contain only electric charges and currents,* $\mathbf{D}_e \to \mathbf{D}$, $\mathbf{D}_m \to \mathbf{E}/\mu_0$, $\mathbf{H}_e \to \mathbf{H}$, $\mathbf{H}_m \to \mathbf{B}/\mu_0$, *the Lorentz force law reduces to,*

$$\mathbf{F}_e = q_e(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}), \qquad \mathbf{F}_m \;=\; \mu_0 q_m(\mathbf{H} - \mathbf{v}_m \times \mathbf{E}). \tag{15}$$

## 2.2 Force on an Ampèrian Magnetic Dipole at Rest

We can now evaluate the force on an Ampèrian magnetic dipole at rest, taking this to be a loop of free electrical current density $\tilde{\mathbf{J}}_e$ that is electrically neutral, perhaps due to a "lattice" of opposite electrical charges at rest. Then, according to eq. (11) the force density on this current is $\tilde{\mathbf{J}}_e \times \mu_0 \mathbf{H}_{m,i}\, d\text{Vol}$, where $\mathbf{H}_{m,i}$ is the "initial" field in the medium in the absence of the magnetic dipole [43]. Ignoring variations of the external field over the thickness of

[13]The form (12) is also affirmed in [14] via considerations of a magnetic current in a "wire" surrounded by a dielectric medium. The issues here are somewhat different from those for the force on individual moving charges, but are similar to those considered in [43] for an electrical current in a wire inside a magnetic medium.

[14]It is argued in [13] that a slowly moving magnetic charge perturbs electric polarization of a dielectric medium in such a way that the velocity-dependent force is $-q_m \mathbf{v} \times \epsilon_0 \mathbf{E}$, where $\mathbf{E} = \mathbf{D}/\epsilon$ is the electric field in the absence of the moving magnetic charge. The argument of [13] seems to this author to be a variant of sec. 400 of [16] in which it is supposed that the charge resides in a "cavity" whose surface details affect the fields experienced by the charge. Such arguments assume that the charge occupies a volume at least equal to one atom/molecule of the medium, which might have seemed reasonable to Maxwell but is not consistent with our present understanding of the size of elementary charges. The results of [32] show that a moving electric charges does not create a "cavity" inside a magnetic medium wherein the average **B** field differs from the macroscopic average **B** field in the absence of the charge. See also [33, 34, 35]. We infer that a moving magnetic charge would experience an average **D** inside a dielectric medium equal to the macroscopic average **D** field in the absence of the charge.

[15]The Poynting vector is,

$$\mathbf{S} = \mu_0 \mathbf{D}_m \times \mathbf{H}_e \qquad \text{(all media)}, \tag{13}$$

and the density $u$ of stored energy associated with the electromagnetic fields is,

$$u = \mu_0 \frac{\mathbf{D}_e \cdot \mathbf{D}_m + \mathbf{H}_e \cdot \mathbf{H}_m}{2} \qquad \text{(linear media)}. \tag{14}$$

That Poynting's theorem retains its usual form when magnetic charges are present is discussed by Heaviside in sec. 19 of [12]. That the form of the Lorentz force law for magnetic charge and current densities is given by eqs. (11)-(12) is consistent with Heaviside's argument; for example, his eq. (88), but is not explicitly stated. See also sec. 50, p. 49 of [11].

[16]A peculiar argument that the "ordinary" form of Poynting's theorem implies the existence of magnetic charges is given in sec. 7.10 of [44]; thus misunderstanding is clarified in [45].

[17]The extension of Poynting's theorem to momentum flow, with the implication that $\mu_0 \mathbf{D}_e \times \mathbf{H}_m$ is the density of stored momentum, as argued by Minkowski [46], remains valid if the Lorentz force law for magnetic charges is given by eqs. (11)-(12), but not for other forms, as discussed in sec. V of [13]. See also Appendix C of [8].

the "wire" of the loop, we write the force density in the Biot-Savart form $I\,d\mathbf{l} \times \mu_0 \mathbf{H}_{m,i}$, and note that the magnetic dipole moment is $\mathbf{m}_{\rm A} = I\,\mathbf{Area}$. Then, the force on this Ampèrian loop of current is,[18]

$$\begin{aligned}
\mathbf{F}_{\rm A} &= I \oint d\mathbf{l} \times \mu_0 \mathbf{H}_{m,i} = I \int (d\mathbf{Area} \times \boldsymbol{\nabla}) \times \mu_0 \mathbf{H}_{m,i} \\
&= I \int \boldsymbol{\nabla}(d\mathbf{Area} \cdot \mu_0 \mathbf{H}_{m,i}) - I \int d\mathbf{Area}\,(\boldsymbol{\nabla} \cdot \mu_0 \mathbf{H}_{m,i}) \\
&= \boldsymbol{\nabla}(\mathbf{m}_{\rm A} \cdot \mu_0 \mathbf{H}_{m,i}) - \mathbf{m}_{\rm A}(\boldsymbol{\nabla} \cdot \mu_0 \mathbf{H}_{m,i}) = \boldsymbol{\nabla}(\mathbf{m}_{\rm A} \cdot \mu_0 \mathbf{H}_{m,i}) - \mu_0 \mathbf{m}_{\rm A} \tilde{\rho}_{m,i} \\
&= (\mathbf{m}_{\rm A} \cdot \boldsymbol{\nabla}) \mu_0 \mathbf{H}_{m,i} + \mathbf{m}_{\rm A} \times (\boldsymbol{\nabla} \times \mu_0 \mathbf{H}_{m,i}) - \mu_0 \mathbf{m}_{\rm A} \tilde{\rho}_{m,i} \qquad (16) \\
&= (\mathbf{m}_{\rm A} \cdot \boldsymbol{\nabla}) \mu_0 \mathbf{H}_{m,i} + \mathbf{m}_{\rm A} \times \mu_0 \frac{\partial \mathbf{D}_{e,i}}{\partial t} + \mu_0 \mathbf{m}_{\rm A} \times [\tilde{\mathbf{J}}_{e,i} + \boldsymbol{\nabla} \times (\mathbf{M}_{m,i} + \mathbf{M}_{e,i})] - \mu_0 \mathbf{m}_{\rm A} \tilde{\rho}_{m,i},
\end{aligned}$$

via a variant of Stokes' theorem [48], and eqs. (1)-(5).

If we suppose that the medium surrounding the magnetic dipole $\mathbf{m}_{\rm A}$ has no electric polarization $\mathbf{P}_e$, no free electrical current density $\tilde{\mathbf{J}}_e$, and no free magnetic charge density $\tilde{\rho}_m$, then,[19]

$$\begin{aligned}
\mathbf{F}_{\rm A} &= \boldsymbol{\nabla}(\mathbf{m}_{\rm A} \cdot \mu_0 \mathbf{H}_{m,i}) \\
&= (\mathbf{m}_{\rm A} \cdot \boldsymbol{\nabla}) \mu_0 \mathbf{H}_{m,i} + \mathbf{m}_{\rm A} \times \frac{1}{c^2} \frac{\partial \mathbf{E}_i}{\partial t} + \mathbf{m}_{\rm A} \times \mu_0 [\boldsymbol{\nabla} \times (\mathbf{M}_{m,i} + \mathbf{M}_{e,i})]\,. \qquad (17)
\end{aligned}$$

If we also suppose that no magnetic charges/dipoles exist, *i.e.*, that all magnetization is Ampèrian, then (writing $\mathbf{M}_{\rm A}$ for $\mathbf{M}_e$),[20]

$$\mathbf{F}_{\rm A} = \boldsymbol{\nabla}(\mathbf{m}_{\rm A} \cdot \mathbf{B}_i) = (\mathbf{m}_{\rm A} \cdot \boldsymbol{\nabla}) \mathbf{B}_i + \mathbf{m}_{\rm A} \times \frac{1}{c^2} \frac{\partial \mathbf{E}_i}{\partial t} + \mathbf{m}_{\rm A} \times \mu_0 (\boldsymbol{\nabla} \times \mathbf{M}_{{\rm A},i})\,, \qquad (18)$$

We consider the force on a moving Ampèrian magnetic dipole in sec. 2.6.

## 2.3 Force on a Gilbertian Magnetic Dipole at Rest

We consider a Gilbertian magnetic dipole $\mathbf{m}_{\rm G}$ at position $\mathbf{x}$ consisting of free magnetic charges $\pm q_m$ at positions $\mathbf{x} \pm \mathbf{s}/2$ such that $\mathbf{m}_{\rm G} = q_m\,\mathbf{s}$. The dipole is at rest, but the

[18]A more general argument would consider the magnetic dipole as due to a current density $\tilde{\mathbf{J}}_e$ as in sec. 12.4.1 of [47]. Then, the first term in the third line of eq. (16) would become $m_{{\rm A},k} \boldsymbol{\nabla} \mu_0 H_{m,i,k}$. For a constant magnetic moment $\mathbf{m}_{\rm A}$ this term readily transforms to $\boldsymbol{\nabla}(\mathbf{m}_{\rm A} \cdot \mu_0 \mathbf{H}_{m,i})$, but for, say, a diamagnetic moment, the alternative form should be used, leading to $\boldsymbol{\nabla}(\mathbf{m}_{\rm A} \cdot \mu_0 \mathbf{H}_{m,i}/2)$.

[19]The form (17) is implicit in the quantum argument of Schwinger [49] (at age 19), but eq. (16) may have first been given in eq. (5) of [45].

[20]While this note is primarily about permanent magnetic moments, we can also consider moments that are affected by the magnetic field, such as $\mathbf{m}_{\rm A} = \chi \mathbf{B}_i$, where $\chi < 0$ corresponds to a diamagnetic moment and $\chi > 0$ is a paramagnetic moment. From the use of Stokes' theorem in eq. (16), we see that the operator $\boldsymbol{\nabla}$ is not meant to act on the dipole $\mathbf{m}_{\rm A}$ in $\boldsymbol{\nabla}(\mathbf{m}_{\rm A} \cdot \mathbf{B}_i) = (\mathbf{m}_{\rm A} \cdot \boldsymbol{\nabla})\mathbf{B}_i + \mathbf{m}_{\rm A} \times (\boldsymbol{\nabla} \times \mathbf{B}_i)$. If $\mathbf{m}_{\rm A} = \chi \mathbf{B}_i$ then $\boldsymbol{\nabla}(\mathbf{m}_{\rm A} \cdot \mathbf{B}_i) = \chi \boldsymbol{\nabla}(\mathbf{B}_i^2) = 2\chi[(\mathbf{B}_i \cdot \boldsymbol{\nabla})\mathbf{B}_i + \mathbf{B}_i \times (\boldsymbol{\nabla} \times \mathbf{B}_i)] = 2[(\mathbf{m}_{\rm A} \cdot \boldsymbol{\nabla})\mathbf{B}_i + \mathbf{m}_{\rm A} \times (\boldsymbol{\nabla} \times \mathbf{B}_i)] = 2\mathbf{F}_{\rm A}$.

If we write $\mathbf{F}_{\rm A} = -\boldsymbol{\nabla} U$, then $U = -\mathbf{m}_{\rm A} \cdot \mathbf{B}_i$ for fixed $|\mathbf{m}_{\rm A}|$, but $U = -\mathbf{m}_{\rm A} \cdot \mathbf{B}_i/2 = -\chi B_i^2/2$ when $\mathbf{m}_{\rm A} = \chi \mathbf{B}_i$. This last result is deduced by rather different arguments in eq. (32.8) of [50], where $\mathbf{B}_i$ is written as $\mathbf{H}$.

separation $\mathbf{s}$ could be changing with time (in magnitude and/or direction) such that,

$$\frac{d\mathbf{m}_{\rm G}}{dt} = q_m \left[\frac{d(\mathbf{x}+\mathbf{s}/2)}{dt} - \frac{d(\mathbf{x}-\mathbf{s}/2)}{dt}\right] = q_m \frac{d\mathbf{s}}{dt}\,. \tag{19}$$

Associated with this time dependence is the free magnetic current density $\tilde{\mathbf{J}}_m = q_m\, d\mathbf{s}/dt = d\mathbf{m}_{\rm G}/dt$. The Lorentz force (12) on the Gilbertian dipole is then,

$$\begin{aligned}
\mathbf{F}_{\rm G} &= \mu_0 q_m[\mathbf{H}_{e,i}(\mathbf{x}+\mathbf{s}/2) - \mathbf{H}_{e,i}(\mathbf{x}-\mathbf{s}/2)] - \mu_0 \frac{d\mathbf{m}_{\rm G}}{dt} \times \mathbf{D}_{e,i} \\
&= (\mathbf{m}_{\rm G}\cdot\boldsymbol{\nabla})\mu_0\mathbf{H}_{e,i} - \frac{d\mathbf{m}_{\rm G}}{dt} \times \frac{\mathbf{E}_i}{c^2} \\
&= \boldsymbol{\nabla}(\mathbf{m}_{\rm G}\cdot\mu_0\mathbf{H}_{e,i}) - \mathbf{m}_{\rm G}\times(\boldsymbol{\nabla}\times\mu_0\mathbf{H}_{e,i}) - \frac{d\mathbf{m}_{\rm G}}{dt} \times \frac{\mathbf{E}_i}{c^2} \\
&= \boldsymbol{\nabla}(\mathbf{m}_{\rm G}\cdot\mu_0\mathbf{H}_{e,i}) - \mu_0\mathbf{m}_{\rm G}\times\tilde{\mathbf{J}}_{e,i} - \mathbf{m}_{\rm G}\times\frac{1}{c^2}\frac{\partial\mathbf{E}_i}{\partial t} - \frac{d\mathbf{m}_{\rm G}}{dt} \times \frac{\mathbf{E}_i}{c^2} \\
&= \boldsymbol{\nabla}(\mathbf{m}_{\rm G}\cdot\mu_0\mathbf{H}_{e,i}) - \frac{d}{dt}\frac{\mathbf{m}_{\rm G}\times\mathbf{E}_i}{c^2}\,,
\end{aligned} \tag{20}$$

supposing that the medium surrounding the magnetic dipole has no electric polarization $\mathbf{P}_e$ and no free electrical current $\tilde{\mathbf{J}}_{e,i}$.[21] If we also suppose that the last term in eq. (20) can be ignored, then[22]

$$\mathbf{F}_{\rm G} = \boldsymbol{\nabla}(\mathbf{m}_{\rm G}\cdot\mu_0\mathbf{H}_{e,i}). \tag{21}$$

If we keep the last term in eq. (20) but suppose that all magnetization is due to Gilbertian dipoles, such that $\mathbf{M}_e = \mathbf{M}_{\rm A} = 0$ and $\mu_0\mathbf{H}_e = \mathbf{B}$, then,

$$\begin{aligned}
\mathbf{F}_{\rm G} &= (\mathbf{m}_{\rm G}\cdot\boldsymbol{\nabla})\mathbf{B}_i - \frac{d\mathbf{m}_{\rm G}}{dt}\times\frac{\mathbf{E}_i}{c^2} \\
&= \boldsymbol{\nabla}(\mathbf{m}_{\rm G}\cdot\mathbf{B}_i) - \frac{d}{dt}\frac{\mathbf{m}_{\rm G}\times\mathbf{E}_i}{c^2}\,.
\end{aligned} \tag{22}$$

### 2.3.1 The Einstein-Laub Force on Magnetic Dipoles

In 1908 Einstein and Laub gave the force density on magnetization $\mathbf{M}$ in a static case as eq. (7) of [38],

$$\mathbf{f}_{\rm E-L} = (\mathbf{M}\cdot\boldsymbol{\nabla})\mu_0\mathbf{H}. \tag{23}$$

After their eq. (3) Einstein and Laub said: *We think of electric and magnetic polarizations, respectively, as consisting of spatial displacement of electric and magnetic mass particles of dipoles that are bound to equilibrium positions.* This appears to imply that in 1908 they considered that magnetization is Gilbertian, *i.e.*, associated with pairs of opposite, true magnetic charges, even though they use $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ with its implication that isolated, true

[21]The first form of eq. (20) was given as eq. (6) of [45].

[22]The form (21) is implicit in the papers of Bloch [51, 52] that started the debate as to the relation between character of the magnetic moment of the neutron and forces on it.

magnetic charges do not exist. Their expression (23) assumes the force on a magnetic charge $q_m$ (at rest) is $\mathbf{F}_m = \mu_0 q_m \mathbf{H}$ as holds for true (Gilbertian) magnetic charges, rather than $\mathbf{F}_m = q_{m,\text{eff}}\mathbf{B}$ as holds for effective (fictitious) magnetic charges associated with Ampèrian magnetization $\mathbf{M}_e$ according to $\rho_{m,\text{eff}} = -\boldsymbol{\nabla} \cdot \mathbf{M}_e$, as reviewed in the Appendix. However, they seem to have missed the insight of Heaviside [12] that Gilbertian magnetization $\mathbf{M}_m$ would be associated with a different field $\mathbf{H}$ field, $\mathbf{H}_m$ of eq. (4), than the "ordinary" field $\mathbf{H}_e$ associated with Ampèrian magnetization $\mathbf{M}_e$.

As such, their force law (23) for magnetization agrees with none of eqs. (17)-(18) or (22) in the static limit. It does agree with the second line of eq. (20) under the physically doubtful hypothesis that the test magnetization is Gilbertian, but the $\mathbf{H}$ field is that ($\mathbf{H}_e$) associated with Ampèrian magnetization.[23]

## 2.4 Interactions of Neutrons with Magnetized Foils

The experiments on neutron scattering by magnetized foils [1]-[4] were interpreted via quantum scattering computations [49, 51]-[54] in the Born approximation, which are based on interaction potentials rather than forces. If we ignore the possibility of spin-flip interactions, the forces (17) and (21) correspond to interaction potentials,

$$U_\text{A} = -\mathbf{m}_\text{A} \cdot \mu_0 \mathbf{H}_{m,i}, \qquad \text{and} \qquad U_\text{G} = -\mathbf{m}_\text{G} \cdot \mu_0 \mathbf{H}_{e,i}. \tag{24}$$

Most discussions of the experiments tacitly assume that the magnetization of the foils is Ampèrian, in which case,

$$U_\text{A} = -\mathbf{m}_\text{A} \cdot \mathbf{B}_i, \qquad U_\text{G} = -\mathbf{m}_\text{G} \cdot \mu_0 \mathbf{H}_{e,i} = -\mathbf{m}_\text{G} \cdot \mu_0 \mathbf{H}_i \qquad \text{(Ampèrian magnetization)}, \tag{25}$$

and the debate is often characterized as whether the neutron magnetic moment couples to $\mathbf{B}$ or to $\mathbf{H}$. However, it could be that the magnetization is Gilbertian, in which case,

$$U_\text{A} = -\mathbf{m}_\text{A} \cdot \mu_0 \mathbf{H}_{m,i}, \qquad U_\text{G} = -\mathbf{m}_\text{G} \cdot \mathbf{B}_i \qquad \text{(Gilbertian magnetization)}. \tag{26}$$

Thus, it is not immediately obvious whether the character of the neutron magnetic moment can be determined by the scattering experiments unless the character of the magnetization known.

### 2.4.1 The Scattering Experiments

One of the neutron-scattering experiments involved reflection [1, 2], and the other involved transmission [3, 4]. In both experiments the neutrons were randomly polarized, *i.e.*, the direction of the magnetic momentum $\mathbf{m}$ varied randomly from event to event. The foils were placed inside a large external magnetic field $\mathbf{B}_0 = \mu_0 \mathbf{H}_0$, parallel to the plane of the foil, such that the magnetization $\mathbf{M}$ of the foils was in the plane of the foils. Hence, we expect that $\mathbf{B}$, $\mathbf{H}_e$ and $\mathbf{H}_m$ inside the foil all lie in its plane. The direction of $\mathbf{B}_0/\mathbf{H}_0$, and hence of $\mathbf{M}$, could be varied within the plane of the foils.

[23]The form (23) would be a useful hypothesis in a search for isolated Gilbertian dipoles inside a magnetic material that consists predominantly of Ampèrian dipoles.

Neutron scattering was observed for various directions of $\mathbf{M}$, and various scattering angles, all of which were small in the reflection experiment [1, 2]. The interpretation of the results in terms of forces, rather than via Born-approximation integrals over the interaction potentials, is more straightforward for the reflection experiment (as noted in [5]).

#### 2.4.2 Ampèrian Magnetization

If the magnetization of the foil is Ampèrian, $\mathbf{M}_i = \mathbf{M}_{e,i}$, and $\boldsymbol{\nabla} \times \mathbf{H}_{e,i} = 0$ according to eq. (5), so the tangential component of $\mathbf{H}_{e,i} = \mathbf{H}_i$ is continuous across the surface of the foil. Outside the foil, $\mathbf{H}_i \approx \mathbf{H}_0$, which is tangential, so we infer that $\mathbf{H}_i \approx \mathbf{H}_0$ and $\mathbf{B}_i = \mu_0(\mathbf{H}_0 + \mathbf{M}_{e,i})$ inside the foil. Thus, $\mathbf{H}_i = \mathbf{H}_0$ everywhere, while $\mathbf{B}_i = \mu_0 \mathbf{H}_{m,i}$ is different inside and outside the foil. According to eq. (17) a neutron with Ampèrian moment $\mathbf{m}_{\mathrm{A}}$ would be affected by such magnetization, whereas according to eq. (21) neutrons with Gilbertian moment $\mathbf{m}_{\mathrm{G}}$ would show no effect.

A simplified interpretation of the data [1, 2] is that the magnetization of the foil does have an effect on the small-angle reflective scattering, so if the magnetization is Ampèrian, the magnetic moment of the neutron is also Ampèrian.

#### 2.4.3 Gilbertian Magnetization

If the magnetization of the foil is Gilbertian, such that $\mathbf{M}_{e,i} = 0$, then $\mathbf{B}_i = \mu_0 \mathbf{H}_{e,i}$, and $\boldsymbol{\nabla} \times \mathbf{B}_i = \boldsymbol{\nabla} \times \mu_0 \mathbf{H}_{e,i} = 0$. Hence, the tangential component of $\mathbf{B}_i$ is continuous at the foil surface, and since outside the foil $\mathbf{B}_i = \mathbf{B}_0$, which is tangential, we infer that $\mathbf{B}_i = \mu_0 \mathbf{H}_{e,i} = \mathbf{B}_0$ everywhere. In this case, there would again be no force on a Gilbertian magnetic moment $\mathbf{m}_{\mathrm{G}}$ according to eq. (21), but since $\mathbf{H}_{m,i} = \mathbf{B}_0/\mu_0 - \mathbf{M}_{m,i}$ is different inside and outside of the foil, there would again be a force on an Ampèrian magnetic moment $\mathbf{m}_{\mathrm{A}}$ according to eq. (17).

Since the data show that the neutron was affected by the magnetization of the foil, we again conclude that the magnetic moment of the neutron is Ampèrian, even if the magnetization of the foil is Gilbertian.

Thus, from the reflection experiment alone, we infer that the neutron has an Ampèrian magnetic moment, but we cannot decide whether the magnetization of the foil is Ampèrian or Gilbertian.

#### 2.4.4 Comments

Fermi [55] gave a quantum argument in 1930 that details of the hyperfine interaction imply that the magnetic moment of nuclei (including the proton) is Ampèrian.[24] This preceded the "discovery" of the neutron [6], and apparently most people did not make the connection that Fermi's argument implies that the magnetic moment of the neutron is Ampèrian.

[24]Fermi's argument is reviewed in secs. 5.6-7 of [30], and in greater detail in [56] (which also reviews neutron scattering assuming the foils have Ampèrian magnetization). Doubt as to the validity of Fermi's argument is expressed in [57], using a classical model of electrons in atoms as moving in loops, which seems irrelevant to the S-wave hyperfine interaction.

As noted in [56], Fermi's argument can also be applied to positronium ($e^+e^-$) and to muonium ($e^{\pm}\mu^{\mp}$), in which cases the "nucleus" is an electron or muon, such that the data indicate the magnetic moments of electrons and muons to be Ampèrian.

The experiment of Rasetti [32] (a former student of Fermi) in 1944 indicated that the force on a moving electric charge is $q\mathbf{v} \times \mathbf{B}_i$ inside magnetized iron, with the implication that magnetization of iron is Ampèrian (as noted on p. 3, after eq. (10)), although this appears to have been little recognized at the time.[25] See also [33, 34, 35].

## 2.5 "Hidden" Momentum

It remains to discuss the versions of eqs. (17) and (20) that exhibit effects of time-varying electric fields (which are irrelevant to the experiments with static, magnetized foils). We first consider the medium surrounding the dipole to be vacuum, in which case $\mathbf{H}_{e,i} = \mathbf{H}_{m,i} = \mathbf{B}_i$, and eqs. (17) and (20) become,

$$\mathbf{F}_{\mathrm{A}} = \boldsymbol{\nabla}(\mathbf{m}_{\mathrm{A}} \cdot \mathbf{B}_i) = (\mathbf{m}_{\mathrm{A}} \cdot \boldsymbol{\nabla})\mathbf{B}_i + \mathbf{m}_{\mathrm{A}} \times \frac{1}{c^2}\frac{\partial \mathbf{E}_i}{\partial t}\,, \tag{27}$$

and,

$$\mathbf{F}_{\mathrm{G}} = \boldsymbol{\nabla}(\mathbf{m}_{\mathrm{G}} \cdot \mathbf{B}_i) - \frac{d}{dt}\frac{\mathbf{m}_{\mathrm{G}} \times \mathbf{E}_i}{c^2} = (\mathbf{m}_{\mathrm{G}} \cdot \boldsymbol{\nabla})\mathbf{B}_i - \frac{d\mathbf{m}_{\mathrm{G}}}{dt} \times \frac{\mathbf{E}_i}{c^2}\,. \tag{28}$$

According to Newton, force is related to time rate of change of momentum, so we are led to suppose that the difference between eqs. (27) and (28), $d/dt)(\mathbf{m} \times \mathbf{E}_i/c^2$, is associated with the difference in the electromagnetic momentum of the Ampèrian and the Gilbertian magnetic dipoles in the external fields $\mathbf{E}_i$ and $\mathbf{B}_i$, as remarked in [45] (after eq. (6) there).

In 1904, J.J. Thomson [58, 59, 60, 61], deduced via two different methods that the field momentum of an Ampèrian magnetic dipole is,

$$\mathbf{P}_{\mathrm{EM,A}} = \frac{\mathbf{E}_i \times \mathbf{m}_{\mathrm{G}}}{c^2}\,, \tag{29}$$

and that the field momentum of a Gilbertian dipole is zero,[26]

$$\mathbf{P}_{\mathrm{EM,G}} = 0. \tag{30}$$

However, these results were little noticed until 1969 [63], after the notion of "hidden momentum" had been introduced in 1967 by Shockley [7]. That is, an Ampèrian magnetic dipole at rest in a static electric field has nonzero field momentum according to eq. (29), so Shockley inferred that this system must also contain a "hidden" mechanical momentum,

$$\mathbf{P}_{\mathrm{mech,hidden}} = \frac{\mathbf{m}_{\mathrm{A}} \times \mathbf{E}_i}{c^2}\,, \tag{31}$$

[25] An otherwise thoughtful text by Fano, Chu and Adler [44] claimed that no experiment could decide whether permanent magnetism is Ampèrian or Gilbertian, and recommends the student to consider it to be Gilbertian.

[26] As emphasized in [62], electromagnetic field momentum is nonzero only in examples in which there are moving (electric and/or magnetic) charges. The case of a Gilbertian magnetic dipole at rest in a static electric field has no moving charges, and no field momentum.

such that the total momentum of the system is zero when "at rest".

In this view, the force on a dipole equals the rate of change if its total mechanical momentum, such that for an Ampèrian magnetic dipole of mass $M_{\rm G}$ and velocity $\mathbf{v}_{\rm G}$,

$$\frac{d\mathbf{P}_{{\rm mech},A}}{dt} = \frac{d}{dt}\left(M_{\rm A}\mathbf{v}_{\rm A} + \frac{\mathbf{m}_{\rm A}\times\mathbf{E}_i}{c^2}\right) = \mathbf{F}_{\rm A} = \boldsymbol{\nabla}(\mathbf{m}_{\rm A}\cdot\mathbf{B}_i) = (\mathbf{m}_{\rm A}\cdot\boldsymbol{\nabla})\mathbf{B}_i + \mathbf{m}_{\rm A}\times\frac{1}{c^2}\frac{\partial\mathbf{E}_i}{\partial t}\,, \tag{32}$$

$$M_{\rm A}\frac{d\mathbf{v}_{\rm A}}{dt} = \boldsymbol{\nabla}(\mathbf{m}_{\rm A}\cdot\mathbf{B}_i) - \frac{d}{dt}\left(\frac{\mathbf{m}_{\rm A}\times\mathbf{E}_i}{c}\right) = (\mathbf{m}_{\rm A}\cdot\boldsymbol{\nabla})\mathbf{B}_i - \frac{d\mathbf{m}_{\rm A}}{dt}\times\frac{\mathbf{E}_i}{c^2}\,, \tag{33}$$

while for a Gilbertian magnetic dipole,

$$\frac{d\mathbf{P}_{{\rm mech},G}}{dt} = M_{\rm G}\frac{d\mathbf{v}_{\rm G}}{dt} = \mathbf{F}_{\rm G} = \boldsymbol{\nabla}(\mathbf{m}_{\rm G}\cdot\mathbf{B}_i) - \frac{d}{dt}\frac{\mathbf{m}_{\rm G}\times\mathbf{E}_i}{c^2} = (\mathbf{m}_{\rm G}\cdot\boldsymbol{\nabla})\mathbf{B}_i - \frac{d\mathbf{m}_{\rm G}}{dt}\times\frac{\mathbf{E}_i}{c^2}\,. \tag{34}$$

That is, the "visible" effect, $d\mathbf{v}/dt$, of the forces (27) and (28) on the motion of Ampèrian and Gilbertian magnetic dipoles has the same form, eqs. (33) and (34), and the difference between these forces affects only the "hidden" mechanical momentum of the Ampèrian dipole.[27,28,29]

### 2.5.1 Magnetic Media

We now return to the case of a magnetic dipole inside a magnetic medium, as in the classic neutron-scattering experiments [1]-[4].

In view of the preceding discussion about the electromagnetic field momentum (29) of an Ampèrian magnetic dipole $\mathbf{m}_{\rm A}$ at rest in an electric field $\mathbf{E}_i$, we rewrite eq. (17) as,

$$\begin{aligned}\mathbf{F}_{\rm A} &= (\mathbf{m}_{\rm A}\cdot\boldsymbol{\nabla})\mu_0\mathbf{H}_{m,i} + \frac{d}{dt}\frac{\mathbf{m}_{\rm A}\times\mathbf{E}_i}{c^2} - \frac{d\mathbf{m}_{\rm A}}{dt}\times\frac{\mathbf{E}_i}{c^2} + \mu_0\mathbf{m}_{\rm A}\times[\boldsymbol{\nabla}\times(\mathbf{M}_{m,i}+\mathbf{M}_{e,i})] \\ &= \frac{d}{dt}\left(M_{\rm A}\mathbf{v}_{\rm A} + \frac{\mathbf{m}_{\rm A}\times\mathbf{E}_i}{c^2}\right),\end{aligned} \tag{35}$$

whence,

$$M_{\rm A}\frac{d\mathbf{v}_{\rm A}}{dt} = (\mathbf{m}_{\rm A}\cdot\boldsymbol{\nabla})\mu_0\mathbf{H}_{m,i} - \frac{d\mathbf{m}_{\rm A}}{dt}\times\frac{\mathbf{E}_i}{c^2} + \mathbf{m}_{\rm A}\times\mu_0[\boldsymbol{\nabla}\times(\mathbf{M}_{m,i}+\mathbf{M}_{e,i})]. \tag{36}$$

Meanwhile, the corresponding relation for a Gilbertian magnetic dipole reverts to,

$$M_{\rm G}\frac{d\mathbf{v}_{\rm G}}{dt} = (\mathbf{m}_{\rm G}\cdot\boldsymbol{\nabla})\mu_0\mathbf{H}_{e,i} - \frac{d\mathbf{m}_{\rm G}}{dt}\times\frac{\mathbf{E}_i}{c^2}\,. \tag{37}$$

[27] The similarity of the forms for $Md\mathbf{v}/dt = M\mathbf{a}$ for Ampèrian and Gilbertian magnetic dipoles was perhaps first noted in [64], where, however, $M\mathbf{a}$ was called the force $\mathbf{F}$. A version of this argument also appears on pp. 49-55 of [65].

[28] For a broader view of "hidden" momentum, see [66].

[29] In the Aharonov-Casher experiment [67], a neutron moves past a static electric line charge in zero magnetic field, with quantum interference effects between passage to the "left" and "right" of the line charge. In the frame of the neutron, the electric field is time dependent, so eq. (32) suggests that there is a classical force that would affect the motion $Md\mathbf{v}/dt$ [68, 69], whereas eq. (33) indicates that the classical force does not affect the motion (while it does affect the "hidden" mechanical motion of the system, as noted in [70]).

The form of eq. (36) for an Ampèrian dipole differs from eq. (34) for a Gilbertian dipole by the presence in the former of the term,

$$\mathbf{m}_{\mathrm{A}} \times \mu_0[\boldsymbol{\nabla} \times (\mathbf{M}_{m,i} + \mathbf{M}_{e,i})]. \tag{38}$$

In, for example, the magnetized foils of the neutron-scattering experiments, the magnetization is uniform inside them, so $\boldsymbol{\nabla} \times \mathbf{M} = 0$ there, but the discontinuity of the magnetization at the foil surface leads to the term (38) being nonzero. As noted in eq. (2), the term $\boldsymbol{\nabla} \times \mathbf{M}_e$ is part of the total Ampèrian current density $\mathbf{J}_e$. However, the term $\boldsymbol{\nabla} \times \mathbf{M}_m$ does not correspond to a piece of the Gilbertian current density $\mathbf{J}_m$ of eq. (3).[30]

## 2.6 Force on a Moving Ampèrian Magnetic Dipole

In this section we consider the lab-frame force $\mathbf{F}$ on an electrically neutral object that has mass $M$, Ampèrian magnetic moment $\mathbf{m}_0$ and (Gilbertian) electric dipole moment $\mathbf{p}_0$ (both moments of fixed magnitude)[31] in its rest frame. We restrict the discussion to the case that the surrounding medium has zero electric and magnetic polarization densities, and that the velocity $\mathbf{v}$ of the object in the lab frame has magnitude much less than the speed of light $c$.

Gaussian units are employed in this section, and rest-frame quantities (other than the dipole moments $\mathbf{m}_0$ and $\mathbf{p}_0$) are denoted with a $'$, while lab-frame quantities are unprimed.

### 2.6.1 $\mathbf{F} = d\mathbf{P}_{\mathrm{mech,total}}/dt$

The lab-frame force $\mathbf{F}$ does not equal mass times acceleration, but rather equals the rate of change of the total mechanical momentum of the object,

$$\begin{aligned}\mathbf{F} &= \frac{d\mathbf{P}_{\mathrm{mech,total}}}{dt} = \frac{d(M\mathbf{v})}{dt} + \frac{d\mathbf{P}_{\mathrm{mech,hidden}}}{dt} = M\mathbf{a} + \frac{d}{dt}\frac{\mathbf{m} \times \mathbf{E}_i}{c} \\ &= M\mathbf{a} + \frac{\dot{\mathbf{m}} \times \mathbf{E}_i}{c} + \frac{\mathbf{m}}{c} \times \left(\frac{\partial \mathbf{E}_i}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{E}_i\right).\end{aligned} \tag{39}$$

noting that in Gaussian units the "hidden" mechanical momentum (31) has the form,

$$\mathbf{P}_{\mathrm{mech,hidden}} = \frac{\mathbf{m}_0 \times \mathbf{E}'_i}{c} \approx \frac{1}{c}\left(\mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p}\right) \times \left(\mathbf{E}_i + \frac{\mathbf{v}}{c} \times \mathbf{B}_i\right) \approx \frac{\mathbf{m} \times \mathbf{E}_i}{c}, \tag{40}$$

which holds for $v \ll c$ in quasistatic electric field $\mathbf{E}_i$.[32] The precession $\dot{\mathbf{m}} = d\mathbf{m}/dt$ of the magnetic moment (whose magnitude $m_0$ is constant in the rest frame of the object) is discussed in sec. 2.6.5.

[30] Comparison of eqs. (36)-(37) does not provide special insight into the issue of Ampèrian *vs.* Gilbertian moments in magnetic materials, although as noted in sec. 2.4, the difference between these forms has been used to show that the magnetic moment of a neutron is Ampèrian.

[31] Strictly, the discussion is limited to dipoles whose moment does not depend on the position of their center of mass, such that the gradient operator with respect to this position does not act on the moment. I believe that this means that the moments must have constant magnitude in their rest frame, but they can rotate. In particular, the electric and magnetic polarizabilities are taken to be zero; $\alpha = 0 = \alpha_m$ in the relations $\mathbf{p}_0 = \alpha\mathbf{E}'_i$ and $\mathbf{m}_0 = \alpha_m\mathbf{B}'_i$.

[32] See sec. 4.1.4 of [66].

### 2.6.2 Force on a Dipole "at Rest"

The force $\mathbf{F}'$ on a magnetic dipole $\mathbf{m}_0$ in its instantaneous rest frame[33] follows from eq. (18) as,

$$\mathbf{F}'_{\mathbf{m}} = \boldsymbol{\nabla}'(\mathbf{m}_0 \cdot \mathbf{B}'_i), \tag{41}$$

even if the magnetic moment is changing in magnitude or direction. In this, the operator $\boldsymbol{\nabla}'$ acts only on the magnetic fields.

The usual argument[34] for an electric dipole $\mathbf{p}_0 = q\mathbf{d}$ "at rest" with electric charges $\pm q$ at $\mathbf{x}'_\pm = \mathbf{x}' \pm \mathbf{d}/2$ is that,

$$\mathbf{F}'_{\mathbf{p}} = q\mathbf{E}'_i(\mathbf{x}' + \mathbf{d}/2) - q\mathbf{E}'_i(\mathbf{x}' - \mathbf{d}/2) = q[(\mathbf{d}/2 - (-\mathbf{d}/2)) \cdot \boldsymbol{\nabla}']\mathbf{E}'_i = (\mathbf{p}_0 \cdot \boldsymbol{\nabla}')\mathbf{E}'_i. \tag{42}$$

This argument is typically made before the magnetic field and the Lorentz force law have been introduced, so no consideration is given to the possibility that the charges $\pm q$ are in motion even if the center of mass of the dipole is at rest. But, if "at rest" means only that the center of mass of the dipole is at rest, while the charges $\pm q$ can be in motion relative to the center of mass, there also exists the Lorentz force,[35]

$$\mathbf{F}''_{\mathbf{p}} = \frac{q\mathbf{v}'_+ - q\mathbf{v}'_-}{c} \times \mathbf{B}'_i = \frac{q}{c}\frac{\partial \mathbf{d}}{\partial t'} \times \mathbf{B}'_i = \frac{\partial \mathbf{p}_0}{\partial t'} \times \frac{\mathbf{B}'_i}{c}\,, \tag{43}$$

such that the total force on an electric dipole "at rest" is actually,[36]

$$\begin{aligned} \mathbf{F}'_{\mathbf{p}} &= (\mathbf{p}_0 \cdot \boldsymbol{\nabla}')\mathbf{E}'_i + \frac{\partial \mathbf{p}_0}{\partial t'} \times \frac{\mathbf{B}'_i}{c} = \boldsymbol{\nabla}'(\mathbf{p}_0 \cdot \mathbf{E}'_i) - \mathbf{p}_0 \times (\boldsymbol{\nabla}' \times \mathbf{E}'_i) + \frac{\partial \mathbf{p}_0}{\partial t'} \times \frac{\mathbf{B}'_i}{c} \\ &= \boldsymbol{\nabla}'(\mathbf{p}_0 \cdot \mathbf{E}'_i) + \mathbf{p}_0 \times \frac{1}{c}\frac{\partial \mathbf{B}'_i}{\partial t'} + \frac{\partial \mathbf{p}_0}{\partial t'} \times \frac{\mathbf{B}'_i}{c} = \boldsymbol{\nabla}'(\mathbf{p}_0 \cdot \mathbf{E}'_i) + \frac{\partial}{\partial t'}\frac{\mathbf{p}_0 \times \mathbf{B}'_i}{c}\,. \end{aligned} \tag{44}$$

The force on an object "at rest" with both electric and magnetic dipole moments is then,

$$\mathbf{F}' = \boldsymbol{\nabla}'(\mathbf{m}_0 \cdot \mathbf{B}'_i) + \boldsymbol{\nabla}'(\mathbf{p}_0 \cdot \mathbf{E}'_i) + \frac{\partial}{\partial t'}\frac{\mathbf{p}_0 \times \mathbf{B}'_i}{c}\,. \tag{45}$$

$$M\mathbf{a}' = \boldsymbol{\nabla}'(\mathbf{m}_0 \cdot \mathbf{B}'_i) + \boldsymbol{\nabla}'(\mathbf{p}_0 \cdot \mathbf{E}'_i) + \frac{\partial}{\partial t'}\left(\frac{\mathbf{p}_0 \times \mathbf{B}'_i}{c} - \frac{\mathbf{m}_0 \times \mathbf{E}'_i}{c}\right). \tag{46}$$

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[33]Strictly, by rest frame we mean the inertial frame with velocity $\mathbf{v}$ in the lab frame, and axes parallel to those of the lab frame. It is sometimes convenient to consider the comoving rest frame whose $z'$ axis is parallel to $\mathbf{v}$, as discussed, for example, in [71, 72], but we do not use this frame here.

[34]See, for example, sec. 4.1.3 of [29].

[35]The equivalent of relation (43) for a Gilbertian magnetic dipole was deduced in eq. (6) of [45] (as given in eqs. (19)-(20) of sec. 2.3), but its application to an electric dipole moment was not discussed.

[36]The assumption that $|\mathbf{p}_0|$ is constant implies that the operator $\boldsymbol{\nabla}'$ does not act on $\mathbf{p}_0$. For example, if $\mathbf{p}_0 = \alpha\mathbf{E}'_i$, where the constant $\alpha$ is the polarizability, then $(\mathbf{p}_0 \cdot \boldsymbol{\nabla})\mathbf{E}'_i = \alpha(\mathbf{E}'_i \cdot \boldsymbol{\nabla}')\mathbf{E}'_i = \boldsymbol{\nabla}'(\mathbf{p}_0 \cdot \mathbf{E}'_i)/2 + \mathbf{p}_0/c \times \partial\mathbf{B}'_i/\partial t$.

### 2.6.3 Transformation of the Rest-Frame Force to the Lab Frame

In the low-velocity approximation, $v \ll c$, the transformations of the various rest-frame quantities in eq. (45) to the lab frame are,

$$\mathbf{F}' = \mathbf{F}, \quad \boldsymbol{\nabla}' = \boldsymbol{\nabla}, \quad \frac{\partial}{dt'} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} = \frac{d}{dt}, \quad \mathbf{B}_i' = \mathbf{B}_i - \frac{\mathbf{v}}{c}\times\mathbf{E}_i, \quad \mathbf{E}_i' = \mathbf{E}_i + \frac{\mathbf{v}}{c}\times\mathbf{B}_i, \tag{47}$$

and the lab-frame dipole moments $\mathbf{m}$ and $\mathbf{p}$ are related by[37]

$$\mathbf{m}_0 = \mathbf{m} + \frac{\mathbf{v}}{c}\times\mathbf{p}, \quad \mathbf{p}_0 = \mathbf{p} - \frac{\mathbf{v}}{c}\times\mathbf{m}, \quad \mathbf{m} = \mathbf{m}_0 - \frac{\mathbf{v}}{c}\times\mathbf{p}_0, \quad \mathbf{p} = \mathbf{p}_0 + \frac{\mathbf{v}}{c}\times\mathbf{m}_0. \tag{48}$$

The physical significance of the lab-frame dipole moments $\mathbf{m}$ and $\mathbf{p}$ of a moving object is not crisp [73], and a good practice is to consider the lab-frame force in terms of the lab-frame fields but the rest-frame moments $\mathbf{m}_0$ and $\mathbf{p}_0$. Nonetheless, we can now rewrite eq. (45) entirely in terms of lab-frame quantities, to order $1/c$,[38] as[39]

$$\begin{aligned}
\mathbf{F} &= \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_i) - \boldsymbol{\nabla}\left(\mathbf{m}\cdot\frac{\mathbf{v}}{c}\times\mathbf{E}_i\right) + \boldsymbol{\nabla}\left(\frac{\mathbf{v}}{c}\times\mathbf{p}\cdot\mathbf{B}_i\right) \\
&\quad \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}_i) + \boldsymbol{\nabla}\left(\mathbf{p}\cdot\frac{\mathbf{v}}{c}\times\mathbf{B}_i\right) - \boldsymbol{\nabla}\left(\frac{\mathbf{v}}{c}\times\mathbf{m}\cdot\mathbf{E}_i\right) \\
&\quad + \frac{d}{dt}\frac{\mathbf{p}\times\mathbf{B}_i}{c} \\
&= \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_i) + \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}_i) + \frac{d}{dt}\frac{\mathbf{p}\times\mathbf{B}_i}{c}\,,
\end{aligned} \tag{49}$$

The equation of motion for the center of mass of the object can now be written as

$$M\mathbf{a} = \mathbf{F} - \frac{d\mathbf{P}_{\rm mech,hidden}}{dt} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_i) + \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}_i) - \frac{d}{dt}\frac{\mathbf{m}\times\mathbf{E}_i}{c} + \frac{d}{dt}\frac{\mathbf{p}\times\mathbf{B}_i}{c}\,. \tag{50}$$

We also note that if either $\mathbf{p}$ or $\mathbf{B}_i$ vanishes, then the force (49) can be deduced from a potential,[40]

$$\mathbf{F} = -\boldsymbol{\nabla}U, \qquad U = -\mathbf{m}\cdot\mathbf{B}_i - \mathbf{p}\cdot\mathbf{E}_i \qquad (\mathbf{p} \text{ or } \mathbf{B}_i = 0). \tag{51}$$

There is an erratic history of derivations of the lab-frame force $\mathbf{F}$ on moving dipoles.

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[37] See, for example, eq. (2) of [73].

[38] Strictly, considering terms only to order $1/c$ is distinct from the low-velocity approximation that $v \ll c$. Further, in our expressions "to order $1/c$", we suppose that magnetic fields and moments are of order $c^0$. This procedure has been called the "semirelativistic approximation" in [74].

[39] As remarked on p. 32 of [75], the term $(d/dt)(\mathbf{p}\times\mathbf{B}_i/c)$ in their eq. (4.22), our eq. (49), suggests that $\mathbf{p}\times\mathbf{B}_i/c$ is part of the momentum of the system. Indeed, it is related to the electromagnetic field momentum of the electric dipole plus external magnetic field. However, this term is not a "hidden" mechanical momentum of the dipole, and does not contribute to $\mathbf{P}_{\rm mech,total}$ of eq. (39).

[40] The term $-\mathbf{p}\cdot\mathbf{E}_i$ in the potential $U$ holds only assuming that the magnitude of $\mathbf{p}$ does not depend on $\mathbf{E}_i$. For example, if $\mathbf{p} = \alpha\mathbf{E}_i$, where the constant $\alpha$ is the polarizability, then $(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}_i = \alpha(\mathbf{E}_i\cdot\boldsymbol{\nabla})\mathbf{E}_i = \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}_i)/2 + \mathbf{p}/c\times\partial\mathbf{B}_i/\partial t$, with the implication that the term in the potential is $-\mathbf{p}\cdot\mathbf{E}_i/2$.

DeGroot and Suttorp [74] gave a lengthy discussions of forces on moving dipoles, starting from the basic definition of dipoles in their rest frame, and then considering transformations to the lab frame. Their low-velocity result, eq. (A118), p. 233, agrees with our eq. (46) in the rest frame, but differs slightly from our lab-frame expression (50).

Schwinger *et al.* [75] worked only in the lab frame and considered a collection of electric charges with electric dipole moment $\mathbf{p} = \sum q_i \mathbf{r}_i$ and magnetic dipole moment $\mathbf{m} = \sum q_i(\mathbf{r}_i - \mathbf{r}) \times (\mathbf{v}_i - \mathbf{v})$, where $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity of the center of mass of the system. Schwinger's $\mathbf{p}$ and $\mathbf{m}$ are not quite the same as transforms (48) of the moments $\mathbf{p}_0$ and $\mathbf{m}_0$ in the rest frame of the system, and he arrives at his eq. (4.22), our eq. (49), only with the neglect of a term $\mathbf{B}_i/c \times (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{p}$. *A difficulty with the use of Schwinger's definition of the lab-frame magnetic moment is discussed in sec. 2.1 of [73]. Such difficulties may have first been discussed in [76].*

Vekstein [77] worked in the lab frame, and defined the electric and magnetic dipoles as integrals of the lab-frame densities $\mathbf{P}$ and $\mathbf{M}$ of electric and magnetic polarization (which were considered to be the Lorentz transformations of the rest-frame densities $\mathbf{P}_0$ and $\mathbf{M}_0$), *i.e.*, $\mathbf{p} = \int \mathbf{P}\, d\text{Vol}$ and $\mathbf{m} = \int \mathbf{M}\, d\text{Vol}$. Vekstein made the unreasonable assumption (which contradicts the existence of Thomas precession [78, 79]), that the dipole moments cannot change direction (or magnitude) in their rest frame, leading to his eq. (9) instead of our eq. (49),

$$\mathbf{F}_{\text{Vekstein}} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}_i) + (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}_i + \frac{\mathbf{p}}{c} \times (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B}_i = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}_i) + \boldsymbol{\nabla}(\mathbf{p} \cdot \mathbf{E}_i) + \frac{\mathbf{p}}{c} \times \frac{d\mathbf{B}_i}{dt} . \tag{52}$$

Kholmetskii *et al.* [80] argued that Vekstein's expression (52) was wrong, and offered their version,

$$\mathbf{F}_{\text{Kholmetskii}} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}_i) + \boldsymbol{\nabla}(\mathbf{p} \cdot \mathbf{E}_i) + \frac{\partial}{\partial t} \frac{\mathbf{p} \times \mathbf{B}_i}{c} , \tag{53}$$

which agrees with our eq. (49) if the $\partial/\partial t$ is replaced by $d/dt$, as was done by Kholmetskii *et al.* in eq. (5) of [81] in response to a criticism by Hnizdo [82].

Hnizdo [82] argued that Kholmetskii *et al.* were wrong, and offered an expression only in the case of static, lab-frame, external fields,

$$\mathbf{F}_{\text{Hnizdo}} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}_i) + \boldsymbol{\nabla}(\mathbf{p} \cdot \mathbf{E}_i) + \frac{d}{dt} \frac{\mathbf{p}_0 \times \mathbf{B}_i}{c} , \tag{54}$$

noting that Hnizdo's eq. (4) is for $M\mathbf{a}$ rather than the lab-frame force $\mathbf{F}$. To order $1/c$ the rest-frame moment $\mathbf{p}_0$ in eq. (54) can be replaced by the lab-frame moment $\mathbf{p}$, leading to agreement with with our eq. (49). Despite that fact that the stated assumption of Hnizdo's method (that the lab-frame interaction energy can be written as in eq. (51) is not generally valid, correct equations of motion were deduced, as will be clarified in sec. 2.6.4.

Zangwill [47] gave an expression for the lab-frame force on a medium with polarization densities $\mathbf{M}$ and $\mathbf{P}$ in sec. 15.9, p. 526, and then considered lab-frame dipole moments $\mathbf{m}$ and $\mathbf{p}$ as in the manner of chap. 4 of [75], arriving at eq. (15.137), which includes the extra term deftly ignored by Schwinger in his eq. (4.22). As such, Zangwill's eq. (15.137) differs slightly from our eq. (49).

Zangwill [47] also presented a Lagrangian derivation based on the interaction energy $\rho\,\varphi_i - \mathbf{J} \cdot \mathbf{A}_i/c$ in Ex. 24.2, p. 922, where $\rho$ is the electric charge density, $\varphi_i$ is the external electric scalar potential and $\mathbf{A}_i$ is the external magnetic vector potential. The derivation appears to be performed in the lab frame, and the result agrees with his eq. (15.137) (although this is not stated). Zangwill supposed that the electric dipole moment can have time dependence, but implied that the magnetic moment does not, which is physically implausible. And, while Zangwill noted that a moving (electric) polarization is associated with an apparent magnetization, he omitted that a moving magnetization is associated with an apparent polarization. The author's version of Zangwill's derivation will be given in sec. 2.6.4, which will confirm our eq. (49).

### 2.6.4 Lagrangian Approach

An alternative derivation of the force on a moving dipole, based on a Lagrangian approach in the lab frame can be given following [47].[41]

The derivation is based on the lab-frame interaction energy,

$$U_{\rm int} = \int \left( \rho\, \varphi_i - \frac{\mathbf{J}}{c} \cdot \mathbf{A}_i \right) d{\rm Vol}. \tag{55}$$

In the rest frame of the object, taken to be a "point" dipole, its electric charge density is,

$$\rho_0 = -\boldsymbol{\nabla}' \cdot \mathbf{P}_0 = -\boldsymbol{\nabla}' \cdot \mathbf{p}_0\, \delta^3(\mathbf{r}' - \mathbf{r}_0') = -\mathbf{p}_0 \cdot \boldsymbol{\nabla}' \delta^3(\mathbf{r}' - \mathbf{r}_0'), \tag{56}$$

and its electric current density is,

$$\begin{aligned} \mathbf{J}_0 &= \frac{\partial \mathbf{P}_0}{\partial t'} + c \boldsymbol{\nabla}' \times \mathbf{M}_0 = \frac{\partial \mathbf{p}_0}{\partial t'}\, \delta^3(\mathbf{r}' - \mathbf{r}_0') + c \boldsymbol{\nabla}' \times \mathbf{m}_0\, \delta^3(\mathbf{r}' - \mathbf{r}_0') \\ &= \frac{\partial \mathbf{p}_0}{\partial t'}\, \delta^3(\mathbf{r}' - \mathbf{r}_0') - c\, \mathbf{m}_0 \times \boldsymbol{\nabla}' \delta^3(\mathbf{r}' - \mathbf{r}_0'), \end{aligned} \tag{57}$$

In the low-velocity approximation, the lab-frame charge and current densities are, to order $1/c$, and recalling eq. (48),

$$\begin{aligned} \rho &= \rho_0 + \frac{\mathbf{v} \cdot \mathbf{J}_0}{c^2} = -\mathbf{p}_0 \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0) - \frac{\mathbf{v}}{c} \cdot \mathbf{m}_0 \times \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0') \\ &= -\left( \mathbf{p} - \frac{\mathbf{v}}{c} \times \mathbf{m} \right) \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0) - \frac{\mathbf{v}}{c} \cdot \mathbf{m} \times \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0') = -\mathbf{p} \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0), \end{aligned} \tag{58}$$

$$\begin{aligned} \mathbf{J} &= \mathbf{J}_0 + \rho_0 \mathbf{v} = \frac{\partial \mathbf{p}_0}{\partial t}\, \delta^3(\mathbf{r} - \mathbf{r}_0) - c \mathbf{m}_0 \times \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0) - (\mathbf{p}_0 \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0)) \mathbf{v} \\ &= \frac{\partial}{\partial t} \left( \mathbf{p} - \frac{\mathbf{v}}{c} \times \mathbf{m} \right) \delta^3(\mathbf{r} - \mathbf{r}_0) - c \left( \mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p} \right) \times \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0) - (\mathbf{p} \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0)) \mathbf{v}. \end{aligned} \tag{59}$$

Then, the lab-frame interaction energy is, to order $1/c$,

$$\begin{aligned} U_{\rm int} &= \int \left( \rho\, \varphi_i - \frac{\mathbf{J}}{c} \cdot \mathbf{A}_i \right) d{\rm Vol} \\ &= -\int \varphi_i\, \mathbf{p} \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0)\, d{\rm Vol} - \int \frac{\mathbf{A}_i}{c} \cdot \frac{\partial \mathbf{p}}{\partial t}\, \delta^3(\mathbf{r} - \mathbf{r}_0)\, d{\rm Vol} \\ &\quad + \int \mathbf{A}_i \cdot \left( \mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p} \right) \times \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0)\, d{\rm Vol} + \int \left( \frac{\mathbf{v}}{c} \cdot \mathbf{A}_i \right) (\mathbf{p} \cdot \boldsymbol{\nabla} \delta^3(\mathbf{r} - \mathbf{r}_0))\, d{\rm Vol} \\ &= \mathbf{p} \cdot \boldsymbol{\nabla} \varphi_i - \frac{\mathbf{A}_i}{c} \cdot \frac{\partial \mathbf{p}}{\partial t} - \boldsymbol{\nabla} \cdot \left[ \mathbf{A}_i \times \left( \mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p} \right) \right] - \boldsymbol{\nabla} \cdot \left[ \left( \frac{\mathbf{v}}{c} \cdot \mathbf{A}_i \right) \mathbf{p} \right] \\ &= \mathbf{p} \cdot \boldsymbol{\nabla} \varphi_i - \frac{\mathbf{A}_i}{c} \cdot \frac{\partial \mathbf{p}}{\partial t} - \left( \mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p} \right) \cdot \mathbf{B}_i - \mathbf{p} \cdot \boldsymbol{\nabla} \left( \frac{\mathbf{v}}{c} \cdot \mathbf{A}_i \right) \\ &= \mathbf{p} \cdot \boldsymbol{\nabla} \varphi_i - \frac{\mathbf{A}_i}{c} \cdot \frac{\partial \mathbf{p}}{\partial t} - \left( \mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p} \right) \cdot \mathbf{B}_i - \frac{\mathbf{p}}{c} \cdot [(\mathbf{v} \cdot \boldsymbol{\nabla}) \mathbf{A}_i + \mathbf{v} \times \mathbf{B}_i] \end{aligned}$$

[41] The approach of Ex. 24.2 of [47] is valid, although the derivation there contained some errors, in this author's view.

$$
\begin{aligned}
&= -\mathbf{m}\cdot\mathbf{B}_i + \mathbf{p}\cdot\boldsymbol{\nabla}\varphi_i - \frac{\mathbf{A}_i}{c}\cdot\frac{\partial\mathbf{p}}{\partial t} + \mathbf{p}\cdot\frac{1}{c}\frac{\partial\mathbf{A}_i}{\partial t} - \mathbf{p}\cdot\frac{1}{c}\frac{\partial\mathbf{A}_i}{\partial t} - \mathbf{p}\cdot\left[(\mathbf{v}\cdot\boldsymbol{\nabla})\frac{\mathbf{A}_i}{c}\right] \\
&= -\mathbf{m}\cdot\mathbf{B}_i - \mathbf{p}\cdot\mathbf{E}_i - \frac{\mathbf{A}_i}{c}\cdot\frac{\partial\mathbf{p}}{\partial t} - \mathbf{p}\cdot\frac{1}{c}\frac{\partial\mathbf{A}_i}{\partial t} - (\mathbf{v}\cdot\boldsymbol{\nabla})\frac{\mathbf{p}\cdot\mathbf{A}_i}{c} \\
&= -\mathbf{m}\cdot\mathbf{B}_i - \mathbf{p}\cdot\mathbf{E}_i - \frac{d}{dt}\frac{\mathbf{p}\cdot\mathbf{A}_i}{c} \to -\mathbf{m}\cdot\mathbf{B}_i - \mathbf{p}\cdot\mathbf{E}_i, \qquad (60)
\end{aligned}
$$

noting that the total (convective) derivative of a function $f$ associated with the moving object is $df/dt = \partial f/\partial t + (\mathbf{v}\cdot\boldsymbol{\nabla})f$, that $\mathbf{E}_i = -\boldsymbol{\nabla}\varphi_i - (1/c)\partial\mathbf{A}_i/\partial t$ and $\mathbf{B}_i = \boldsymbol{\nabla}\times\mathbf{A}_i$, and that a total derivative term in a Lagrangian does not affect the equations of motion.[42]

The total derivative term in eq. (60) vanishes if either $\mathbf{p} = 0$ or $\mathbf{B}_i = 0$. Hence, as claimed at eq. (51), the interaction energy/potential has the form $U = -\mathbf{m}\cdot\mathbf{B}_i - \mathbf{p}\cdot\mathbf{E}_i$ only when these conditions are met.[43] Hence, even in the general case this potential can be used in a Lagrangian to deduce the equations of motion (as was done by Hnizdo [82] without detailed justification).

Thus, the general Lagrangian is effectively,

$$
\begin{aligned}
\mathcal{L} &= T - U = \frac{Mv^2}{2} + \mathbf{m}\cdot\mathbf{B}_i + \mathbf{p}\cdot\mathbf{E}_i \\
&= \frac{Mv^2}{2} + \mathbf{m}_0\cdot\mathbf{B}_i - \mathbf{v}\cdot\frac{\mathbf{p_0}\times\mathbf{B}_i}{c} + \mathbf{p}_0\cdot\mathbf{E}_i + \mathbf{v}\cdot\frac{\mathbf{m_0}\times\mathbf{E}_i}{c}, \qquad (61)
\end{aligned}
$$

and the equations of motion are,

$$
\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{v}} = \boldsymbol{\nabla}\mathcal{L} = \frac{d}{dt}\left(M\mathbf{v} - \frac{\mathbf{p_0}\times\mathbf{B}_i}{c} + \frac{\mathbf{m_0}\times\mathbf{E}_i}{c}\right) = -\boldsymbol{\nabla}U = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_i) + \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}_i), \quad (62)
$$

To order $1/c$, we can replace $\mathbf{m}_0$ and $\mathbf{p}_0$ by $\mathbf{m}$ and $\mathbf{p}$ in eq. (62). Then,

$$
\begin{aligned}
M\mathbf{a} &= \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_i) + \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}_i) + \frac{1}{c}\frac{d}{dt}(\mathbf{p}\times\mathbf{B}_i - \mathbf{m}\times\mathbf{E}_i), \qquad (63) \\
M\mathbf{a} + \frac{d}{dt}\frac{\mathbf{m}\times\mathbf{E}_i}{c} &= \frac{d\mathbf{P}_{\text{mech,total}}}{dt} = \mathbf{F} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_i) + \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}_i) + \frac{d}{dt}\frac{\mathbf{p}\times\mathbf{B}_i}{c}, \qquad (64)
\end{aligned}
$$

as found previously in eq. (50).

### 2.6.5 Precession of the Dipole Moments

The torque on a (moving) dipole, relative to a fixed origin in its rest frame, is given by standard arguments as,[44]

$$
\boldsymbol{\tau}' = \mathbf{r}'\times\mathbf{F}' + \mathbf{p}_0\times\mathbf{E}'_i + \mathbf{m}_0\times\mathbf{B}'_i. \qquad (65)
$$

---

[42] See, for example, p. 918 of [47].

[43] To order $1/c$, $\mathbf{p}\cdot\mathbf{A}_i/c = \mathbf{p}_0\cdot\mathbf{A}_i/c$, so it suffices that either $\mathbf{p}_0$ or $\mathbf{B}_i$ for eq. (51) to be an effective potential.

[44] Textbooks typically compute the torque about the center of mass of the dipole. See, for example, secs. 4.1.3 and 6.1.2 of [29].

The effect of this torque is to change the mechanical angular momentum of the dipole,

$$\boldsymbol{\tau}' = \frac{d\mathbf{L}'_{\rm mech}}{dt'} \tag{66}$$

If the dipole can be described by an inertia tensor $\mathsf{I}'$ measured with respect to its center of mass (at position $\mathbf{r}'$), then its angular momentum can be written as,

$$\mathbf{L}' = \mathbf{r}' \times \mathbf{P}'_{\rm mech} + \mathsf{I}' \cdot \boldsymbol{\omega}' = \mathbf{r}' \times (M\mathbf{v}' + \mathbf{P}'_{\rm mech,hidden}) + \mathsf{I}' \cdot \boldsymbol{\omega}', \tag{67}$$

where $\mathbf{v}' = 0$ in the rest frame, $\boldsymbol{\omega}$ is the angular velocity of the dipole in its rest frame, and $\mathbf{P}'_{\rm mech,hidden} = m_0 \times \mathbf{E}'_i/c$ is the "hidden" mechanical momentum of the dipole. Recalling that $\mathbf{F}' = d\mathbf{P}'_{\rm mech}/dt'$, we combine eqs. (65)-(67) to find the rotational equation of motion in the rest frame of the dipole,

$$\frac{d}{dt'}\mathsf{I}' \cdot \boldsymbol{\omega}' = \mathbf{p}_0 \times \mathbf{E}'_i + \mathbf{m}_0 \times \mathbf{B}'_i. \tag{68}$$

To go further, we must have a model for the inertia tensor $\mathsf{I}'$. If the dipole is an extended object, and is mass distribution is known, such a model can be given, but in general there will be little relation between the inertia tensor and the dipole moments $\mathbf{m}_0$ and $\mathbf{p}_0$. So, we restrict our attention to "point" dipoles, particularly to electrons, and recall that "point" dipoles do not have intrinsic electric dipole moments, as far as is known.[45] That is, we only consider the precession of charged particle with an intrinsic magnetic moment in this section.

As first deduced by Larmor [85] in 1897, an Ampèrian magnetic moment $\mathbf{m}_0$ due to a current of electric charges $e$ with rest mass $M$ has (orbital) angular momentum $\mathbf{s}_0$ about the center of mass of the moment in its rest frame related by,

$$\mathbf{m}_0 = \frac{ge}{2Mc}\,\mathbf{s}_0, \tag{69}$$

where $g = 1$ in a classical model.[46]

Apparently, Voigt [87] (1902) was the first to consider that an electron would have a magnetic moment if it is a rotating, charged object. That such an intrinsic magnetic moment would affect atomic spectroscopy was first discussed by Ritz [88] (1907), and pursued more by Weiss [89, 90], who coined the term magneton and noted the emerging awareness that the electron magnetic moment has something to do with $\hbar$ (three

[45] "Point" electric dipoles do not appear to exist in Nature, while electrons can be considered as examples of "point", Ampèrian magnetic dipoles. Objects with finite size (water molecules, for example) can have electric dipole moments in their rest frame.

As remarked by Purcell and Ramsey [83], the only vector associated with an elementary particle at rest is its spin $\mathbf{s}$, and the magnetic moment $\mathbf{m}_0$ of the particle is along this direction. If the particle were to have an electric dipole moment this would have to be parallel to the spin, $\mathbf{p}_0 \propto \mathbf{s}$. But then, an interaction energy of the form $-\mathbf{p}_0 \cdot \mathbf{E} \propto \mathbf{s} \cdot \mathbf{E}$ violates the discrete symmetries of time reversal (T) and space inversion (P = parity), whereas the interaction $-\mathbf{m}_0 \cdot \mathbf{B} \propto \mathbf{s} \cdot \mathbf{B}$ is P and T invariant. As such, it is unlikely that elementary particles will have electric dipole moments.

The present limit [84] on the electric dipole moment $p_0 = ed$ of the electron (of charge $e$) is that $d < 10^{-28}$ cm, some 15 orders of magnitude smaller than the Compton radius of the electron.

[46] The $g$-factor was introduced by Landé [86] in 1921, in the context of the Bohr-Sommerfeld quantum theory.

years before Bohr [91] identified this as the quantum of orbital angular momentum). However, in the 1910's most consideration was given to the magnetic moment associated with orbital motion of an electron, rather than to its possible intrinsic magnetic moment. Compton [92] (1921) revived the notion that an electron might have an intrinsic magnetic moment equal to the (orbital) Bohr magneton $g = 1$ and $s_0 = \hbar$. In 1925, Uhlenbeck and Goudsmit [93] proposed that $s_0 = \hbar/2$ and $g = 2$ for an electron. A theoretical justification of this was given by Dirac [94] in 1928 via his famous 4-spinor equation for a quantum electron.

The precession of the magnetic moment of an electron in an external electromagnetic field was famously considered by Thomas [78, 79] in 1926. A textbook review of this topic is given in secs. 11.8 and 11.11 of [30]. The subtle result is that the direction of the lab-frame magnetic moment (and spin vector) has an additional precession beyond that due to the torque on the dipole moment. In the low-velocity approximation the rate of change of the spin angular momentum of an electron is,

$$\frac{d\mathbf{s}}{dt} = \mathbf{m} \times \left(\mathbf{B}_i - \frac{\mathbf{v}}{c} \times \mathbf{E}_i\right) + \frac{\mathbf{m}}{2} \times \left(\frac{\mathbf{v}}{c} \times \mathbf{E}_i\right). \tag{70}$$

The first term on the right side of eq. (70) is the nominal torque on the magnetic dipole, and the second term is the Thomas precession. Finally,

$$\frac{d\mathbf{m}}{dt} = \frac{ge}{2Mc}\frac{d\mathbf{s}}{dt} \approx \mathbf{m} \times \frac{ge}{2Mc}\mathbf{B}_i = \boldsymbol{\omega} \times \mathbf{m}, \qquad \boldsymbol{\omega} = -\frac{ge}{2Mc}\mathbf{B}_i, \tag{71}$$

where the approximation holds to order $1/c$, and $\boldsymbol{\omega}$ is the Larmor frequency for an electron.

# A Appendix: Effective Magnetic Charge Density $\rho_{m,\mathrm{eff}} = -\boldsymbol{\nabla} \cdot \mathbf{M}$

So far as is presently known, magnetic charges do not exist, and all magnetic effects can be associated with electrical currents, as first advocated by Ampère [95]. For materials with magnetization density $\mathbf{M}_e = \mathbf{M}$ the associated (macroscopic) electrical current density is,

$$\mathbf{J}_e = \boldsymbol{\nabla} \times \mathbf{M}, \tag{72}$$

and on the surface of such materials there is the surface current density,

$$\mathbf{K}_e = \hat{\mathbf{n}} \times \mathbf{M}, \tag{73}$$

where $\hat{\mathbf{n}}$ is the outward unit vector normal to the surface.

Alternatively, we can suppose the magnetization is associated with densities of effective magnetic charges. Some care is required to use this approach, since a true (Gilbertian) magnetic charge density $\rho_m$ would obey $\boldsymbol{\nabla}\cdot\mathbf{B} = \mu_0\rho_m$ as in eq. (1), and the static force density on these charges would be $\mathbf{F}_m = \mu_0\rho_m\mathbf{H}_e$. However, in Nature $\boldsymbol{\nabla} \cdot \mathbf{B} = 0 = \boldsymbol{\nabla} \cdot \mu_0(\mathbf{H} + \mathbf{M})$, so we can write,

$$\boldsymbol{\nabla} \cdot \mathbf{H} = -\boldsymbol{\nabla} \cdot \mathbf{M} = \rho_{m,\mathrm{eff}}, \tag{74}$$

and identify,

$$\rho_{m,\mathrm{eff}} = -\boldsymbol{\nabla} \cdot \mathbf{M} \tag{75}$$

as the volume density of effective (Ampèrian) magnetic charges.

Inside linear magnetic media, where $\mathbf{B} = \mu\mathbf{H}$, the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ then implies that $\rho_{m,\text{eff}} = 0$. However, a surface density $\sigma_{m,\text{eff}}$ of effective magnetic charges can exist on an interface between two media, and we see that Gauss' law for the field $\mathbf{H}$ implies that,

$$\sigma_{m,\text{eff}} = (\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{n}}, \tag{76}$$

where unit normal $\hat{\mathbf{n}}$ points across the interface from medium 1 to medium 2. The magnetic surface charge density can also be written in terms of the magnetization $\mathbf{M} = \mathbf{B}/\mu_0 - \mathbf{H}$ as,

$$\sigma_{m,\text{eff}} = (\mathbf{M}_1 - \mathbf{M}_2) \cdot \hat{\mathbf{n}}, \tag{77}$$

since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ insures that the normal component of $\mathbf{B}$ is continuous at the interface.

The force on the surface density of effective magnetic charges is,

$$\mathbf{F} = \sigma_{m,\text{eff}}\mathbf{B}, \tag{78}$$

since the effective magnetic charges, which are a representation of effects of electrical currents, couple to the magnetic field $\mathbf{B}$, as in eq. (9).[47]

The total force on a linear medium is, in this view, the sum of the force on the conduction current plus the force on the effective magnetic surface charges. Care is required to implement such a computation of the force, as discussed in [43], where eq. (78) is affirmed by example.

The key result of this Appendix is that while "true" (Gilbertian, and nonexistent in Nature) magnetic charges $p$ obey the force law $\mathbf{F}_{m,\text{true}} = \mu_0 p\mathbf{H}$, the effective (Ampèrian) magnetic charges (which are a representation of effects of electrical currents) obey $\mathbf{F}_{m,\text{eff}} = p_{\text{eff}}\mathbf{B}$.

For "effective" Ampèrian magnetic charges the magnetic fields obey $\boldsymbol{\nabla} \cdot \mathbf{B}/\mu_0 = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{H}_e = \rho_{m,\text{eff}}$ inside magnetic materials, while for "true" Gilbertian magnetic charges the fields obey $\boldsymbol{\nabla} \cdot \mathbf{B}/\mu_0 = \rho_{m,\text{true}}$ and $\boldsymbol{\nabla} \cdot \mathbf{H}_m = 0$ inside magnetic materials where there are no "free", "true" magnetic charges. Hence, the roles of $\mathbf{B}/\mu_0$ and $\mathbf{H}$ are reversed in magnetic materials that contain "true" or "effective" magnetic charges. We illustrate this below for the fields of a uniformly magnetized sphere.

## A.1 Fields of a Uniformly Magnetized Sphere

In this subappendix we deduce the static magnetic fields associated with uniform spheres of radius $a$ with either uniform Gilbertian magnetization density $M_m$ or uniform Ampèrian (effective) magnetization density $M_e$.

[47] Equation (78) is in agreement with prob. 5.20 of [30], recalling the different convention for factors of $\mu_0$ used there. However, the Coulomb Committee in their eq. (1.3-4) [96], and Jefimenko in his eq. (14-9.9a,b) [97], recommends that the field $\mathbf{H}/\mu_0$ be used rather than $\mathbf{B}$ when using the method of effective magnetic charges, which would imply a force $\mu_0/\mu$ times that of eq. (78) for linear media.

### A.1.1 Uniform Ampèrian Magnetization Density $M_e$

The total magnetic moment of the sphere is,

$$\mathbf{m}_e = \frac{4\pi\mathbf{M}_e a^3}{3}. \tag{79}$$

We speed up the derivation by noting that the fields inside the sphere are uniform, and the fields outside the sphere are the same at those of a point magnetic dipole of strength $\mathbf{m}_e$,

$$\frac{\mathbf{B}(r>a)}{\mu_0} = \mathbf{H}_e(r>a) = \mathbf{H}_m(r>a) = \frac{3(\mathbf{m}_e\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_e}{4\pi r^3} = \frac{M_e a^3(2\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})}{3r^3}, \tag{80}$$

in a spherical coordinate system with origin at the center of the sphere and $z$-axis parallel to $\mathbf{M}_e$.

To characterize the fields inside the sphere, we note use the method of effective magnetic charges (Appendix A). Since $\mathbf{M}_e$ is constant inside the sphere, there is no net effective magnetic charge density there, $\rho_{e,\text{eff}}(r<a) = -\boldsymbol{\nabla}\cdot\mathbf{M}_e(r<a) = 0$, while there is a nonzero surface density of effective magnetic charge,

$$\sigma_{e,\text{eff}}(r=a) = \mathbf{M}_e\cdot\hat{\mathbf{r}} = M_e\cos\theta. \tag{81}$$

The boundary condition on the magnetic field $\mathbf{H}_e$ at the surface of the sphere is that,

$$H_{e,r}(r=a^+) - H_{e,r}(r=a^-) = \sigma_{e,\text{eff}}(r=a), \tag{82}$$

and hence,

$$\begin{aligned} H_{e,r}(r=a^-) &= H_{e,r}(r<a) = H_e(r<a)\cos\theta = H_{e,r}(r=a^+) - \sigma_{e,\text{eff}}(r=a) \\ &= \frac{2M_e\cos\theta}{3} - M_e\cos\theta = -\frac{M_e\cos\theta}{3}, \end{aligned} \tag{83}$$

$$\mathbf{H}_e(r<a) = -\frac{\mathbf{M}_e}{3}, \qquad \mathbf{H}_m(r<a) = \frac{\mathbf{B}(r<a)}{\mu_0} = \mathbf{H}_e(r<a) + \mathbf{M}_e(r<a) = \frac{2\mathbf{M}_e}{3}. \tag{84}$$

The result (84) for $\mathbf{B}/\mu_0$ implies that the magnetic field for the idealization of a "point", "effective" (Ampèrian) magnetic dipole $\mathbf{m}_e$ would be,

$$\frac{\mathbf{B}}{\mu_0} = \frac{3(\mathbf{m}_e\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_e}{4\pi r^3} + \frac{2\mathbf{m}_e}{3}\delta^3(\mathbf{r}). \tag{85}$$

### A.1.2 Uniform Gilbertian Magnetization Density $M_m$

The total magnetic moment of the sphere for this case is,

$$\mathbf{m}_m = \frac{4\pi\mathbf{M}_m a^3}{3}. \tag{86}$$

As in sec. D.1.1,we speed up the derivation by noting that the fields inside the sphere are uniform, and the fields outside the sphere are the same at those of a point magnetic dipole of strength $\mathbf{m}_m$,

$$\frac{\mathbf{B}(r > a)}{\mu_0} = \mathbf{H}_e(r > a) = \mathbf{H}_m(r > a) = \frac{3(\mathbf{m}_m \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_e}{4\pi r^3} = \frac{M_m a^3(2\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})}{3r^3}. \quad (87)$$

To characterize the fields inside the sphere, we note use the method of effective magnetic charges (Appendix A). Since $\mathbf{M}_m$ is constant inside the sphere, there is no net true magnetic charge density there, $\rho_m(r < a) = -\boldsymbol{\nabla} \cdot \mathbf{M}_m(r < a) = 0$, while there is a nonzero surface density of true magnetic charge,

$$\sigma_m(r = a) = \mathbf{M}_m \cdot \hat{\mathbf{r}} = M_m \cos\theta. \quad (88)$$

The boundary condition on the magnetic field $\mathbf{B}$ at the surface of the sphere is that,

$$B_r(r = a^+) - B_r(r = a^-) = \mu_0 \sigma_m(r = a), \quad (89)$$

and hence,

$$\begin{aligned} \frac{B_r(r = a^-)}{\mu_0} &= \frac{B_r(r < a)}{\mu_0} = \frac{B(r < a)\cos\theta}{\mu_0} = \frac{B_r(r = a^+)}{\mu_0} - \sigma_r(r = a) \\ &= \frac{2M_m\cos\theta}{3} - M_m\cos\theta = -\frac{M_m\cos\theta}{3}, \end{aligned} \quad (90)$$

$$\frac{\mathbf{B}(r < a)}{\mu_0} = \mathbf{H}_e(r < a) = -\frac{\mathbf{M}_m}{3}, \qquad \mathbf{H}_m(r < a) = \frac{\mathbf{B}(r < a)}{\mu_0} + \mathbf{M}_e(r < a) = \frac{2\mathbf{M}_e}{3}. \quad (91)$$

Comparing with eqs. (83)-(84) we see that the roles of $\mathbf{B}$ and $\mathbf{H}$ are reversed in the case of uniform true and effective magnetization. In particular, the sign of $\mathbf{B}$ inside the magnetized sphere is opposite for the cases of Ampèrian and Gilbertian magnetization, although $\mathbf{B}$ is the same outside the sphere in the two cases.[48]

The result (91) for $\mathbf{B}/\mu_0$ implies that the magnetic field for the idealization of a "point", "true" (Gilbertian) magnetic dipole $\mathbf{m}_m$ would be,

$$\frac{\mathbf{B}}{\mu_0} = \frac{3(\mathbf{m}_m \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_m}{4\pi r^3} - \frac{\mathbf{m}_m}{3}\delta^3(\mathbf{r}). \quad (92)$$

# B Appendix: Lab-Frame Fields to order $1/c$

*Gaussian units are used in this Appendix.*

We consider an object that has total charge $Q$, dipole moments $\mathbf{m}_0$ and $\mathbf{p}_0$ in its rest frame, and no other multipole moments. At the time of interest, we take the object to be at $\mathbf{r}'_0$.[49]

[48]For the case of a cylinder with uniform transverse magnetization, see [98], where the interior $\mathbf{B}$ field is equal and opposite for Ampèrian and Gilbertian magnetization.

[49]In this Appendix, quantities other than $Q$, $\mathbf{m}_0$ and $\mathbf{p}_0$ are indicated with a prime.

The magnetic dipole moment is taken to be Ampèrian [95], while the electric dipole moment is Gilbertian [99],[50] meaning that for "point" dipoles "at rest" at the origin, their electromagnetic fields can be written as,

$$\mathbf{B}'(\mathbf{r}') = \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{R}}')\hat{\mathbf{R}}' - \mathbf{m}_0}{R'^3} + \frac{8\pi\mathbf{m}_0}{3}\delta^3(\mathbf{R}'), \qquad \mathbf{E}'(\mathbf{r}') = \frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{R}}')\hat{\mathbf{R}}' - \mathbf{p}_0}{R'^3} - \frac{4\pi\mathbf{p}_0}{3}\delta^3(\mathbf{R}'), \tag{93}$$

where $\mathbf{R}' = \mathbf{r} - \mathbf{r}_0'$ and $\hat{\mathbf{R}}' = |bfR'/R'$. That is, the electric dipole is equivalent to a pair of electric charges $\pm q$ separated by distance $d'$ where $p_0 = qd'$, while the magnetic dipole is equivalent to a small loop of electrical current $I$, with area $A$, such that $m_0 = IA/c$. The electric field between the charges $\pm q$ is opposite to the direction of $\mathbf{p}_0$, while the magnetic field in the interior of the current loop is in the same direction as $\mathbf{m}_0$.

While the center of mass of the object is, by definition, at rest in its rest frame, the center of mass may be acceleration, and the object may be rotating and/or deforming such that the rest-frame dipole moments have nonzero time derivatives.

To display the electromagnetic fields of the object in its rest frame (and in the lab frame) to order $1/c$, we first consider the potentials and fields of a single electric charge, Appendix B.1, and then.....

## B.1 The Potentials and Fields for a Single Electric Charge

The Liénard-Wiechert potentials [101, 102] in the Lorenz gauge [103] can be written for continuous charge and current densities as,

$$\varphi = \int \frac{\rho(\mathbf{r}, t - R/c)}{R}\, d\text{Vol}, \qquad \mathbf{A} = \int \frac{\mathbf{J}(\mathbf{r}, t - R/c)}{cR}\, d\text{Vol}, \tag{94}$$

where distance $\mathbf{R}$ points from the source to the observer. If we are interested in the fields only to order $1/c$, the potentials (94) can be approximated as (see, for example, sec. 65 of [104]),

$$\varphi \approx \int \frac{\rho}{R}\, d\text{Vol} - \frac{1}{c}\frac{d}{dt}\int \rho\, d\text{Vol} = \int \frac{\rho}{R}\, d\text{Vol}, \qquad \mathbf{A} = \int \frac{\mathbf{J}}{cR}\, d\text{Vol}, \tag{95}$$

noting that the total charge, $\int \rho\, d\text{Vol}$, is conserved.

Specializing to the case of a single charge $e$ at position $\mathbf{r}$ with velocity $\mathbf{v} = d\mathbf{r}/dt$, the Lorenz-gauge potentials to order $1/c$ are,

$$\varphi = \frac{e}{R}, \qquad \mathbf{A} = \frac{e\mathbf{v}}{cR}. \tag{96}$$

The lab-frame electric and magnetic fields to order $1/c$ follow as,

$$\mathbf{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial \mathbf{A}}{\partial ct} = -\boldsymbol{\nabla}\varphi = \frac{e\,\hat{\mathbf{R}}}{R^2}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{e\mathbf{v} \times \hat{\mathbf{R}}}{cR^2}, \tag{97}$$

where unit vector $\hat{\mathbf{R}} = \mathbf{R}/R$ is directed from the charge to the observer.

Of course, the results of this section also hold in the rest frame of the charge, where $\mathbf{v}' = 0$ so that $\mathbf{A}'$ and $\mathbf{B}'$ vanish, while $\varphi' = e/R'$ and $\mathbf{E}' = e\,\hat{\mathbf{R}}'/R'^2$.

[50] The notion that a magnet consists of a pair of opposite "poles" (= magnetic charges) was perhaps first expressed by Peregrinus [100] in 1269.

## B.2 The Potentials of a Localized Collection of Charges

The fields of a collection of charges $e_i$ with masses $M_i$ at positions $\mathbf{r}'_i + \mathbf{r}'_0$, where $\mathbf{r}_0$ is the position of the center of mass of the system, with velocities $\mathbf{v}'_i$ in the rest frame of the object are, of course, the superpositions of the fields of the individual charges. The fields can be deduced from the potentials, so we emphasize the latter here.

The distance $\mathbf{R}'_i$ from charge $i$ to the observer at $\mathbf{r}'$ is related by $\mathbf{R}'_i = \mathbf{r}' - \mathbf{r}'_i - \mathbf{r}'_0 \equiv \mathbf{R}' - \mathbf{r}'$, where $\mathbf{R}' = \mathbf{r}' - \mathbf{r}_0$ is the distance from the center of mass of the object to the observer. Then,

$$\begin{aligned} \frac{1}{R'_i} &= \frac{1}{|\mathbf{R}' - \mathbf{r}'_i|} = \frac{1}{(R'^2 - 2\mathbf{R}' \cdot \mathbf{r}'_i + r'^2_i} = \frac{1}{R'(1 - 2\hat{\mathbf{R}}' \cdot \mathbf{r}'_i/R' + r'^2_i/R'^2)^{1/2}} \\ &\approx \frac{1}{R'} + \frac{\hat{\mathbf{R}}' \cdot \mathbf{r}'_i}{R'^2} + \frac{3(\hat{\mathbf{R}}' \cdot \mathbf{r}'_i)\,\hat{\mathbf{R}}' - \mathbf{r}'_i}{2R'^3} + \cdots \end{aligned} \tag{98}$$

The rest-frame scalar potential (in the Lorenz gauge) is just,

$$\varphi' = \sum \frac{e_i}{R'_i} \approx \frac{\sum e_i}{R'} + \hat{\mathbf{R}}' \cdot \frac{\sum e_i \mathbf{r}'_i}{R'^2} + \cdots = \frac{Q}{R'} + \frac{\hat{\mathbf{R}}' \cdot \mathbf{p}_0}{R'^2} + \cdots, \tag{99}$$

where $Q = \sum e_i$ is the total charge, and,

$$\mathbf{p}_0 = \sum e_i \mathbf{r}'_i \tag{100}$$

is the electric dipole moment in the rest frame of the object.

Similarly, the rest-frame vector potential (in the Lorenz gauge) is,

$$\begin{aligned} \mathbf{A}' &= \sum \frac{e_i \mathbf{v}'_i}{cR'_i} \approx \frac{\sum e_i \mathbf{v}'_i}{cR'} + \frac{\sum e_i \mathbf{v}'_i(\hat{\mathbf{R}}' \cdot \mathbf{r}'_i)}{cR'^2} + \cdots \\ &= \frac{\sum e_i \mathbf{v}'_i}{cR'} + \frac{1}{cR'^2} \sum \left( \frac{e_i}{2}[\mathbf{v}'_i(\hat{\mathbf{R}}' \cdot \mathbf{r}'_i) - \mathbf{r}'_i(\hat{\mathbf{R}}' \cdot \mathbf{v}'_i)] + \frac{e_i}{2}[\mathbf{v}'_i(\hat{\mathbf{R}}' \cdot \mathbf{r}'_i) + \mathbf{r}'_i(\hat{\mathbf{R}}' \cdot \mathbf{v}'_i)] \right) + \cdots \\ &= \frac{1}{cr'}\frac{\partial \mathbf{p}_0}{\partial t'} + \frac{1}{cR'^2} \sum \left( \frac{\mathbf{r}'_i \times e_i \mathbf{v}'_i}{2} \times \hat{\mathbf{R}}' + \frac{e_i}{2}[\mathbf{v}'_i(\hat{\mathbf{R}}' \cdot \mathbf{r}'_i) + \mathbf{r}'_i(\hat{\mathbf{R}}' \cdot \mathbf{v}'_i)] \right) + \cdots, \end{aligned} \tag{101}$$

noting that for an object at rest there is no difference between the operations $d/dt'$ and $\partial/\partial t'$.

The second term in the last line of eq. (101) can be rewritten as,

$$\frac{1}{cR'^2} \sum \frac{\mathbf{r}'_i \times e_i \mathbf{v}'_i}{2} \times \hat{\mathbf{R}}' = \sum \frac{\mathbf{r}'_i \times e_i \mathbf{v}'_i}{2c} \times \frac{\hat{\mathbf{R}}'}{R'^2} = \frac{\mathbf{m}_0 \times \hat{\mathbf{R}}'}{R'^2}, \tag{102}$$

where we identify the rest-frame magnetic moment as,

$$\mathbf{m}_0 = \sum \frac{\mathbf{r}'_i \times e_i \mathbf{v}'_i}{2c}. \tag{103}$$

Thus, the rest-frame vector potential (in the Lorenz gauge) of an object with no multipole moments higher than dipoles is,[51]

$$\mathbf{A}' = \frac{1}{cR'}\frac{\partial \mathbf{p}_0}{\partial t'} + \frac{\mathbf{m}_0 \times \hat{\mathbf{R}}'}{R'^2} + \frac{\sum e_i[\mathbf{v}'_i(\hat{\mathbf{R}}' \cdot \mathbf{r}'_i) + \mathbf{r}'_i(\hat{\mathbf{R}}' \cdot \mathbf{v}'_i)]}{2cR'^2} \equiv \frac{1}{cR'}\frac{\partial \mathbf{p}_0}{\partial t'} + \frac{\mathbf{m}_0 \times \hat{\mathbf{R}}'}{R'^2} + \mathbf{A}_s \tag{104}$$

We now consider the electromagnetic fields in the rest frame of the object to order $1/c$. Here, we encounter a famous difficulty in classical electrodynamics, that some objects in Nature, such as electrons, have an intrinsic magnetic moment. In particular, a semiclassical model of an electron as a rotating sphere of charge of radius roughly the Compton wavelength $\lambda\!\!\!^{-} = \hbar/Mc$ of the electron requires the equatorial velocity to exceed the speed of light.[52] Without committing to such a model, we infer that the rest frame magnetic moment $\mathbf{m}_0$ should not be considered as of order $1/c$, despite the result (103). Rather, in the following we consider the magnetic moment $\mathbf{m}_0$ to be of order $c^0$, while the center-of-mass velocities of all charges have $v_i \ll c$. This is called the semirelativistic approximation in [74].

The rest-frame electric and magnetic fields to order $1/c$ (in the semirelativistic approximation), and for observers at distance $R'$ large compared to the spatial size of the source object, follow as,

$$\mathbf{E}' = -\boldsymbol{\nabla}'\varphi' - \frac{\partial \mathbf{A}'}{\partial ct'} = \frac{e\,\hat{\mathbf{R}}'}{R'^2} + \frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{R}}')\hat{\mathbf{R}}' - \mathbf{p}_0}{R'^3} + \frac{\partial}{\partial t'}\frac{\mathbf{m}_0 \times \hat{\mathbf{R}}'}{cR'^2}\,, \tag{105}$$

$$\mathbf{B}' = \boldsymbol{\nabla}' \times \mathbf{A}' = \frac{\partial \mathbf{p}_0}{\partial t'} \times \frac{\hat{\mathbf{R}}'}{cR'^2} + \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{R}}')\hat{\mathbf{R}}' - \mathbf{m}_0}{R'^3} + \mathbf{A}_s \times \frac{\hat{\mathbf{R}}'}{R'}\,. \tag{106}$$

## B.3 Transformation of the Rest-Frame Fields to the Lab Frame

In the lab frame the object has center-of-mass velocity $\mathbf{v} = \mathbf{v}_0 = d\mathbf{r}_0/dt$, where $\mathbf{r}_0$ is the position of its center of mass. To order $1/c$, position and time are the same in the lab frame as in the rest frame, $\mathbf{r}' = \mathbf{r}$ and $t' = t$, while the derivative operators are related by $\boldsymbol{\nabla}' = \boldsymbol{\nabla}$ and $\partial/\partial t' = \partial/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla} = d/dt$. Then, recalling (the inverse of) eq. (47), the lab-frame fields in terms of the rest-frame dipole moments are,

$$\mathbf{E} = \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}' = \frac{e\,\hat{\mathbf{R}}}{R^2} + \frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{p}_0}{R^3} + \frac{d}{dt}\frac{\mathbf{m}_0 \times \hat{\mathbf{R}}}{cR^2} - \frac{\mathbf{v}}{c} \times \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{m}_0}{R^3}, \tag{107}$$

$$\begin{aligned}\mathbf{B} &= \mathbf{B}' + \frac{\mathbf{v}}{c} \times \mathbf{E}' = \frac{d\mathbf{p}_0}{dt} \times \frac{\hat{\mathbf{R}}}{cR^2} + \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{m}_0}{R^3} + \mathbf{A}_s \times \frac{\hat{\mathbf{R}}}{R} \\ &\quad + \frac{\mathbf{v}}{c} \times \left(\frac{e\,\hat{\mathbf{R}}}{R^2} + \frac{3(\mathbf{p}_0 \cdot \hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{p}_0}{R^3}\right),\end{aligned} \tag{108}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ is the distance from the center of mass of the object to the fixed observer.

We can also express the rest-frame dipole moments in terms of their lab-frame values using the Lorentz transformation (48) in the semirelativistic approximation,

$$\mathbf{m}_0 = \mathbf{m} + \frac{\mathbf{v}}{c} \times \mathbf{p}, \quad \mathbf{p}_0 = \mathbf{p} - \frac{\mathbf{v}}{c} \times \mathbf{m}, \quad \mathbf{m} = \mathbf{m}_0 - \frac{\mathbf{v}}{c} \times \mathbf{p}_0, \quad \mathbf{p} = \mathbf{p}_0 + \frac{\mathbf{v}}{c} \times \mathbf{m}_0, \tag{48}$$

[51] The last term, $\mathbf{A}_s$, in eq. (104) is symmetric. Decomposition of the vector potential into symmetric and antisymmetric pieces is advocated in prob. 6.22 of [30]. See also [73].

[52] See, for example, prob. 12 of [105].

leading to,

$$\mathbf{E} = \frac{e\,\hat{\mathbf{R}}}{R^2} + \frac{3(\mathbf{p}\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{p}}{R^3} - \frac{2\mathbf{v}}{c} \times \frac{3(\mathbf{m}\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{m}}{R^3} + \frac{d}{dt}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{cr^2}\,, \tag{109}$$

$$\mathbf{B} = \frac{\mathbf{v}}{c} \times \frac{e\,\hat{\mathbf{R}}}{R^2} + \frac{3(\mathbf{m}\cdot\hat{\mathbf{R}})_0\hat{\mathbf{R}} - \mathbf{m}}{R^3} + \mathbf{A}_s \times \frac{\hat{\mathbf{R}}}{R} + \frac{2\mathbf{v}}{c} \times \frac{3(\mathbf{p}\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{p}}{R^3} + \frac{d\mathbf{p}}{dt} \times \frac{\hat{\mathbf{r}}}{cr^2}\,. \tag{110}$$

where to order $1/c$,

$$\mathbf{A}_s = \frac{\sum e_i[\mathbf{v}'_i(\hat{\mathbf{R}}'\cdot\mathbf{r}'_i) + \mathbf{r}'_i(\hat{\mathbf{R}}'\cdot\mathbf{v}'_i)]}{2cR'^2} = \frac{\sum e_i[\mathbf{v}'_i(\hat{\mathbf{R}}\cdot\mathbf{r}'_i) + \mathbf{r}'_i(\hat{\mathbf{R}}\cdot\mathbf{v}'_i)]}{2cR^2}\,. \tag{111}$$

Note that the dipole contributions to the lab-frame fields are not simply the forms $[3(\mathbf{p}\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{p}]/R^3$ and $[3(\mathbf{m}\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{m}]/R^3$ as might have been expected naïvely. Nor are they the forms $[3(\mathbf{p}_0\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{p}_0]/R^3$ and $[3(\mathbf{m}_0\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} - \mathbf{m}_0]/R^3$ as might have been expected in a slightly more sophisticated view.

*A general conclusion is that the interpretation of dipole (and higher multipole) moments is only straightforward in the rest frame of a system of charges.*

For completeness, we also display the transformation of the Lorenz-gauge potentials into the lab frame, in the semirelativistic approximation:

$$\begin{aligned}
\varphi &= \varphi' + \frac{\mathbf{v}}{c}\cdot\mathbf{A}' = \frac{Q}{R} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}_0}{R^2} + \frac{\mathbf{v}}{c}\cdot\mathbf{m}_0 \times \frac{\hat{\mathbf{R}}}{R^2} \\
&= \frac{Q}{R} + \frac{\hat{\mathbf{R}}\cdot\mathbf{p}}{R^2}\,,
\end{aligned} \tag{112}$$

$$\begin{aligned}
\mathbf{A} &= \mathbf{A}' + \frac{\mathbf{v}}{c}\varphi' = \frac{1}{cR}\frac{d\mathbf{p}_0}{dt} + \frac{\mathbf{m}_0\times\hat{\mathbf{R}}}{R^2} + \mathbf{A}_s + \frac{Q\mathbf{v}}{cR} + \frac{(\hat{\mathbf{R}}\cdot\mathbf{p}_0)\mathbf{v}}{cR^2} \\
&= \frac{1}{cR}\frac{d\mathbf{p}}{dt} + \frac{\mathbf{m}\times\hat{\mathbf{R}}}{R^2} + \left(\frac{\mathbf{v}}{c}\times\mathbf{p}\right) \times \frac{\hat{\mathbf{R}}}{R^2} + \mathbf{A}_s + \frac{Q\mathbf{v}}{cR} + \frac{(\hat{\mathbf{R}}\cdot\mathbf{p})\mathbf{v}}{cR^2} \\
&= \frac{Q\mathbf{v}}{cR} + \frac{1}{cR}\frac{d\mathbf{p}}{dt} + \frac{\mathbf{m}\times\hat{\mathbf{r}}}{R^2} + \frac{(\hat{\mathbf{R}}\cdot\mathbf{v})\mathbf{p}}{cR^2} + \mathbf{A}_s\,.
\end{aligned} \tag{113}$$

While the lab-frame scalar potential (112) has an appealing form, the lab-frame vector potential (113) has three terms in addition to the first two that might have been naïvely expected.

## B.4 Lab-Frame Analysis

It is irresistibly tempting to perform an analysis like that of Appendix B.2 in the lab frame rather than in the rest frame.

Taking the center of mass of the object to be at $\mathbf{r}_0$ at the time under consideration, the distance $\mathbf{R}_i$ from charge $i$ at $\mathbf{r}_i = \mathbf{r}_0 + \Delta\mathbf{r}_i$ to the observer at $\mathbf{r}$ is related by $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i = \mathbf{r} - \mathbf{r}_0 - \Delta\mathbf{r}_i = \mathbf{R} - \mathbf{r}'_i$, where $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$ is the distance from the center of mass of the object to the observer. To order $1/c$, the separation between charge $i$ and the center of mass of the object is the same in the lab frame as in the rest frame of the object; that is, $\Delta\mathbf{r}_i = \mathbf{r}'_i$. Then,

$$\begin{aligned}\frac{1}{R_i} &= \frac{1}{|\mathbf{R}-\mathbf{r}'_i|} = \frac{1}{(R^2 - 2\mathbf{R}\cdot\mathbf{r}') + r'^2_i} = \frac{1}{R(1-2\hat{\mathbf{R}}\cdot\mathbf{r}'_i/R + r'^2_i/R^2)^{1/2}} \\ &\approx \frac{1}{R} + \frac{\hat{\mathbf{R}}\cdot\mathbf{r}'_i}{R^2} + \frac{3(\hat{\mathbf{R}}\cdot\mathbf{r}'_i)\,\hat{\mathbf{R}} - \mathbf{r}'_i}{2R^3} + \cdots \end{aligned} \tag{114}$$

The lab-frame scalar potential (in the Lorenz gauge, and to order $1/c$ recalling the argument of Appendix B.1) is just,

$$\varphi = \sum \frac{e_i}{R_i} \approx \frac{\sum e_i}{R} + \hat{\mathbf{R}}\cdot\frac{\sum e_i\mathbf{r}'_i}{R^2} + \cdots = \frac{Q}{R} + \hat{\mathbf{R}}\cdot\frac{\sum e_i\mathbf{r}'_i}{R^2} + \cdots = \frac{Q}{R} + \frac{\hat{\mathbf{R}}\cdot\tilde{\mathbf{p}}}{R^2} + \cdots, \tag{115}$$

where $Q = \sum e_i$ is the total charge, and,

$$\tilde{\mathbf{p}} = \sum e_i\mathbf{r}'_i = \sum e_i\Delta\mathbf{r}_i \tag{116}$$

seems to be a "natural" definition of the electric dipole moment in the lab frame. However, since $\Delta\mathbf{r}_i = \mathbf{r}'_i$, we see that $\tilde{\mathbf{p}} = \mathbf{p}_0$, with the implication that the lab-frame electric dipole moment is that same as that in the rest frame. This contradicts the Lorentz transformation (48), so the apparently straightforward analysis of the lab-frame scalar potential is to be regarded with suspicion.

Surprisingly, the lab-frame electric dipole moment is sometimes defined as,

$$\breve{\mathbf{p}} = \sum e_i\mathbf{r}_i = \sum e_i(\mathbf{r}_0 + \mathbf{r}'_i) = Q\mathbf{r} + \tilde{\mathbf{p}}, \tag{117}$$

which implies that a single charge $Q$ has an electric dipole moment when not at the origin. See, for example, the bottom right column of p. 1528 of [106], and eq. (15.129) of [47].

Similarly, the lab-frame vector potential (in the Lorenz gauge) is,

$$\begin{aligned}\mathbf{A} &= \sum \frac{e_i\mathbf{v}_i}{cR_i} \approx \frac{\sum e_i\mathbf{v}_i}{cR} + \frac{\sum e_i\mathbf{v}_i(\hat{\mathbf{R}}\cdot\mathbf{r}_i)}{cR^2} + \cdots \\ &= \frac{1}{cR}\frac{d\breve{\mathbf{p}}}{dt} + \frac{1}{cR^2}\sum\left(\frac{e_i}{2}[\mathbf{v}_i(\hat{\mathbf{R}}\cdot\mathbf{r}_i) - \mathbf{r}_i(\hat{\mathbf{R}}\cdot\mathbf{v}_i)] + \frac{e_i}{2}[\mathbf{v}_i(\hat{\mathbf{R}}\cdot\mathbf{r}_i) + \mathbf{r}_i(\hat{\mathbf{R}}\cdot\mathbf{v}_i)]\right) + \cdots \\ &= \frac{1}{cR}\frac{d\breve{\mathbf{p}}}{dt} + \sum\frac{\mathbf{r}_i\times e_i\mathbf{v}_i}{2c}\times\frac{\hat{\mathbf{R}}}{R^2} + \frac{\sum e_i[\mathbf{v}_i(\hat{\mathbf{R}}\cdot\mathbf{r}_i) + \mathbf{r}_i(\hat{\mathbf{R}}\cdot\mathbf{v}_i)]}{2cR^2}. + \cdots \\ &= \frac{1}{cR}\frac{d\breve{\mathbf{p}}}{dt} + \breve{\mathbf{m}}\times\frac{\hat{\mathbf{R}}}{R^2} + \mathbf{A}_s + \cdots \end{aligned} \tag{118}$$

where $\mathbf{A}_s$ is given in eq. (111), and some people define the lab-frame magnetic dipole moment as[53,54]

$$\breve{\mathbf{m}} = \sum\frac{\mathbf{r}_i\times e_i\mathbf{v}_i}{2c}, \tag{121}$$

[53]See, for example, eq. (15.134) of [47]. The definition (121) seems to be implied by eq. (44.2) of [104], but Landau introduces his sec. 44 with the restriction that it applies only to "a system of charges in stationary motion". The center of mass of a "system in stationary motion" is at rest, so Landau's eq. (44.2) holds only in the rest frame of the system, and not in the lab frame.

[54]In sec. III of [106] it is claimed that the magnetic moment of a point electric dipole $\mathbf{p}_0$ (and $\mathbf{m}_0 = 0$)

although this attributes a nonzero magnetic moment to a single electric charge moving along a straight line.

Noting that the velocity $\mathbf{v}_i$ of charge $i$ can be written as $\mathbf{v}_i = \mathbf{v} + \mathbf{v}'_i$, where $\mathbf{v}'_i$ is the velocity of the charge relative to the center of mass of the object (and, of course, also equal to the velocity of the charge in the rest frame of the object), we can write,

$$\begin{aligned} \breve{\mathbf{m}} &= \sum \frac{\mathbf{r}_i \times e_i \mathbf{v}_i}{2c} = \sum \frac{(\mathbf{r}_0 + \mathbf{r}'_i) \times e_i(\mathbf{v} + \mathbf{v}'_i)}{2c} \\ &= \tilde{\mathbf{m}} - \frac{\mathbf{v}}{2c} \times \tilde{\mathbf{p}} + \mathbf{r}_0 \times \frac{Q\mathbf{v}}{2c} + \frac{\mathbf{r}_0}{2c} \times \frac{d\tilde{\mathbf{p}}}{dt}, \end{aligned} \tag{122}$$

where,

$$\tilde{\mathbf{m}} = \sum \frac{\mathbf{r}'_i \times e_i \mathbf{v}'_i}{2c} = \mathbf{m}_0, \tag{123}$$

is the lab-frame magnetic dipole moment defined in terms of quantities relative to the center of mass of the charge distribution.[56] Neither $\breve{\mathbf{m}}$ nor $\tilde{\mathbf{m}}$ are related to the lab-frame dipole moments according to the Lorentz transformation (48).

Then, recalling eq. (117), the lab-frame vector potential (in the Lorenz gauge and in the semirelativistic approximation) of an object with no multipole moments higher than dipoles is,

$$\begin{aligned} \mathbf{A} &= \frac{Q\mathbf{v}}{cR} + \frac{1}{cR}\frac{d\tilde{\mathbf{p}}}{dt} + \left(\tilde{\mathbf{m}} - \frac{\mathbf{v}}{2c} \times \tilde{\mathbf{p}} + \mathbf{r}_0 \times \frac{Q\mathbf{v}}{2c} + \frac{\mathbf{r}_0}{2c} \times \frac{d\tilde{\mathbf{p}}}{dt}\right) \times \frac{\hat{\mathbf{R}}}{R^2} + \mathbf{A}_s \\ &= \frac{Q\mathbf{v}}{cR} + \frac{1}{cR}\frac{d\tilde{\mathbf{p}}}{dt} + \frac{\tilde{\mathbf{m}} \times \hat{\mathbf{R}}}{R^2} + \frac{(\hat{\mathbf{R}} \cdot \tilde{\mathbf{p}})\mathbf{v} - (\hat{\mathbf{R}} \cdot \mathbf{v})\tilde{\mathbf{p}}}{2cR^2} + \left(\mathbf{r}_0 \times \frac{Q\mathbf{v}}{2c} + \frac{\mathbf{r}_0}{2c} \times \frac{d\tilde{\mathbf{p}}}{dt}\right) \times \frac{\hat{\mathbf{R}}}{R^2} + \mathbf{A}_s. \end{aligned} \tag{124}$$

However, this form does not agree well with eq. (113).

In sum, it is misleading to perform multipole expansions in the lab-frame of a system of charges, rather than in its rest frame, despite the apparent elegance of a lab-frame analysis.

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which moves with lab-frame velocity $\mathbf{v}$ can be calculated according to,

$$\breve{\mathbf{m}} = \frac{1}{2c}\int \mathbf{r} \times \mathbf{J}\, d\mathrm{Vol} = \frac{1}{2c}\int \mathbf{r} \times \rho_0 \mathbf{v}\, d\mathrm{Vol} = -\frac{\mathbf{v}}{2c} \times \int \rho_0 \mathbf{r}\, d\mathrm{Vol} = -\frac{\mathbf{v}}{2c} \times \mathbf{p}_0, \tag{119}$$

which differs from eq. (48) by a factor of 2.

Furthermore, support for this result appears to be given in probs. 6.21, 6.22 and 11.27 of [30].

However, we should recall that the origin of the first equality in (119) is a multipole expansion of the (quasistatic) vector potential of a current distribution. See, for example, sec. 5.6 of [30]. That form depends on the current density $\mathbf{J}$ having zero divergence. In general,

$$\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}, \tag{120}$$

where $\rho$ is the electric charge density.[55] The charge density of a moving electric dipole is time dependent, such that $\boldsymbol{\nabla} \cdot \mathbf{J} \neq 0$, and we cannot expect the analysis of eq. (119) to be valid. *Thanks to Grigory Vekstein for pointing this out.*

See [73] for additional discussion.

[56] See, for example, eq. (4.19) of [75].

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