Volume and Surface Area of a Sphere in N Dimensions

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(February 4, 2003)

1 Problem

Deduce expressions for the volume and surface area of a sphere in Euclidean N-space for positive integer N.

A sphere in N-space is called an $N-1$ sphere.

2 Solution

This solution follows http://db.uwaterloo.ca/~alopez-o/math-faq/node75.html *by A. Lopez-Ortiz. Who first gave this solution?*

We expect that the volume V_N of an $N-1$ sphere varies with its radius r as,

$$V_N = C_N r^N, \tag{1}$$

where the C_N are constants to be determined.

If we consider the $N-1$sphere to be made up of a set of concentric shells, then the volume dV_N of a shell of radius r and thickness dr is related the surface area A_{N-1} of the shell by,

$$dV_N = A_{N-1}\, dr. \tag{2}$$

Thus,

$$A_{N-1} = \frac{dV_N}{dr} = N C_N r^{N-1}. \tag{3}$$

A clever method to evaluate the C_N is to consider the integral of e^{-r^2} in both rectangular and "spherical" coordinates,

$$\begin{aligned}
\int e^{-r^2}\, dV_N &= \int_{-\infty}^{\infty} dx_1 ... \int_{-\infty}^{\infty} dx_N\, e^{-x_1^2-...-x_N^2} = \left(\int_{-\infty}^{\infty} dx\, e^{-x^2}\right)^N = \pi^{N/2} \\
&= \int_0^{\infty} e^{-r^2}\, A_{N-1}\, dr = N C_N \int_0^{\infty} e^{-r^2}\, r^{N-1} dr = \frac{N C_N}{2} \int_0^{\infty} e^{-s}\, s^{N/2-1} ds \\
&= \frac{N C_N}{2} \Gamma(N/2) = \frac{N C_N}{2} (N/2-1)! = \frac{N C_N}{2} \frac{(N/2)!}{N/2} = C_N (N/2)!
\end{aligned} \tag{4}$$

Thus,

$$C_N = \frac{\pi^{N/2}}{(N/2)!}, \tag{5}$$

so the volume and surface area of an $N-1$ sphere are,

$$V_N = \frac{\pi^{N/2}}{(N/2)!} r^N, \qquad A_{N-1} = N \frac{\pi^{N/2}}{(N/2)!} r^{N-1}. \tag{6}$$

An expression for $(N/2)!$ for odd integer N can be deduced from the fact that,

$$\Gamma(1/2) = \int_0^\infty e^{-s} s^{-1/2}\, ds = 2\int_0^\infty e^{-r^2} dr = \sqrt{\pi} \tag{7}$$

and the recurrence relation,

$$\Gamma(x+1) = x\Gamma(x). \tag{8}$$

Thus,

$$\begin{aligned} (N/2)! &= \Gamma(N/2+1) = \sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{N}{2} = \sqrt{\pi}\cdot\frac{1}{2}\cdot\frac{2}{2\cdot 1}\cdot\frac{3}{2}\cdot\frac{4}{2\cdot 2}\cdots\frac{N}{2}\cdot\frac{N+1}{2\cdot\frac{N+1}{2}} \\ &= \sqrt{\pi}\frac{(N+1)!}{(\frac{N+1}{2})!\ 2^{N+1}} \qquad (\text{odd } N). \end{aligned} \tag{9}$$

With this, we find the first few C_N to be,

$$C_1 = 2, \quad C_2 = \pi, \quad C_3 = \frac{4}{3}\pi, \quad C_4 = \frac{\pi^2}{2} \quad C_5 = \frac{8\pi^2}{15}, \quad C_6 = \frac{\pi^3}{6}, \quad C_7 = \frac{16\pi^3}{105}, \quad C_8 = \frac{\pi^4}{24}. \tag{10}$$

For large N, Stirling's approximation yields,

$$C_N \approx \frac{1}{\sqrt{N\pi}}\left(\frac{2e\pi}{N}\right)^{N/2} \qquad (N \gg 1). \tag{11}$$