Currents in a Conducting Sheet with a Hole

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(February 22, 2006; sec. 3 added June 23, 2012)

1 Problem

Current enters an infinite plane conducting sheet at some point p and leaves at infinity. A circular hole, exclusive of p, is cut anywhere in the sheet and filled with a material of a different conductivity. Show that the potential difference between any two points on the edge of the hole is twice that between the same two points before the hole was cut, if the conductivity of the material in the hole is zero [1].

Consider also a three-dimensional version of this problem.

2 Solution

We take the hole to have radius a, which is less than the distance b from the center of the hole to the point p where the current enters the sheet.

We use a polar coordinate system with its origin at the center of the hole, and point p at $(r, \phi) = (b, 0)$.

The electric scalar potential V is taken to be $V = V_0 + V_1$, where V_0 is the potential in the absence of the hole, and V_1 is a correction that vanishes at large radius r (and is finite at $r = 0$).

In the absence of the hole, the current density $\mathbf{J}_0$ flows radially outward from point p. Let $\rho = \sqrt{r^2 + b^2 - 2rb\cos\phi}$ be the distance from point p to the point (r, ϕ), and $\hat{\boldsymbol{\rho}}$ be a unit vector pointing away from p. Then the current density $\mathbf{J}_0$ is given by,

$$\mathbf{J}_0 = \frac{I}{2\pi\rho}\hat{\boldsymbol{\rho}} = \sigma\mathbf{E}_0 = -\sigma\boldsymbol{\nabla}V_0, \tag{1}$$

where σ is the electrical conductivity of the sheet at $r > a$. Solving for the potential V_0, we find,

$$V_0 = -\frac{I}{2\pi\sigma}\ln\rho + K. \tag{2}$$

Note that the potential V_0 is the same as that due to a line of linear charge density $\lambda = I/4\pi\sigma$ (in Gaussian units) perpendicular to the (r, ϕ) plane at point p. Indeed, following sec. 4-10 of [1] or sec. 4.02 of [2], we can rewrite V_0 (ignoring the constant K) as,

$$V_0(r, \phi) = \frac{I}{2\pi\sigma}\left\{\begin{array}{ll} \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{b}\right)^n\cos n\phi - \ln b, & r < b, \\ \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{b}{r}\right)^n\cos n\phi - \ln r, & r > b. \end{array}\right. \tag{3}$$

The hole perturbs the potential V_0 by the addition of potential V_1. This correction to the potential vanishes for large r, is finite at $r = 0$, and is independent of angle ϕ. Hence, we can write (ignoring the constant term in the potential),

$$V_1(r,\phi) = \begin{cases} \sum_{n=1}^{\infty} A_n \left(\frac{r}{a}\right)^n \cos n\phi, & r < a, \\ \sum_{n=1}^{\infty} A_n \left(\frac{a}{r}\right)^n \cos n\phi, & r > a. \end{cases} \tag{4}$$

The radial component of the (steady) current density is continuous across the boundary $r = a$, which implies that the radial component of the electric field obeys $\sigma E_r(r = a^+) = \sigma' E_r(r = a^-)$, where σ' is the electrical conductivity of the material at $r < a$.[1] This constrains the radial derivative of the potential at $r = a$,

$$\begin{aligned} \sigma \frac{\partial V(r = a^+)}{\partial r} &= \sigma \frac{I}{2\pi\sigma} \sum_{n=1}^{\infty} \frac{1}{a} \left(\frac{a}{b}\right)^n \cos n\phi - \sigma \sum_{n=1}^{\infty} \frac{n}{a} A_n \cos n\phi \\ = \quad \sigma' \frac{\partial V(r = a^-)}{\partial r} &= \sigma' \frac{I}{2\pi\sigma} \sum_{n=1}^{\infty} \frac{1}{a} \left(\frac{a}{b}\right)^n \cos n\phi + \sigma' \sum_{n=1}^{\infty} \frac{n}{a} A_n \cos n\phi \end{aligned} \tag{5}$$

using the forms (3)-(4). Hence, the Fourier coefficients A_n are given by,

$$A_n = \frac{\sigma - \sigma'}{\sigma + \sigma'} \frac{I}{2n\pi\sigma} \left(\frac{a}{b}\right)^n. \tag{6}$$

The potential V_1 is therefore given by,

$$V_1(r,\phi) = \frac{\sigma - \sigma'}{\sigma + \sigma'} \frac{I}{2\pi\sigma} \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{b}\right)^n \cos n\phi, & r < a, \\ \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{a^2/b}{r}\right)^n \cos n\phi, & r > a. \end{cases} \tag{7}$$

In particular, on the rim of the hole (as well as inside the hole) the potential V_1 is given by,

$$V_1(r \le a, \phi) = \frac{\sigma - \sigma'}{\sigma + \sigma'} V_0(r \le a, \phi) + \frac{\sigma - \sigma'}{\sigma + \sigma'} \frac{I}{2\pi\sigma} \ln b, \tag{8}$$

and so,

$$V(r \le a, \phi) = \frac{2\sigma}{\sigma + \sigma'} V_0(r \le a, \phi) - \frac{2\sigma'}{\sigma + \sigma'} \frac{I}{2\pi\sigma} \ln b. \tag{9}$$

Hence, the potential difference between any two points on the rim of the hole (or within the hole) is $2\sigma/(\sigma + \sigma')$ times that in the absence of the hole. If $\sigma' = 0$ then potential difference is twice that in the absence of the hole.

Inside the hole, the functional form of the potential is that same (to within an additive constant) as that in its absence, so the equipotentials are circles centered on point p. Hence, the electric field lines (and lines of current density $\mathbf{J} = \sigma'\mathbf{E}$) for $r < a$ radiate from point p. Since the electric field is not continuous at $r = a$, there must be a distribution of fixed charges along the rim to support the field lines that cross the hole.

[1]If $\sigma' = 0$ then the radial electric field vanishes at $r = a^+$, while E_r can be nonzero for $r < a$.

The potential and fields within the sheet at $r > a$ can be understood via an image model with a source and a sink at $r < a$. From eq. (3) we can write,

$$V_1(r > a, \phi) = \frac{I}{2\pi\sigma}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a^2/b}{r}\right)^n \cos n\phi - \ln r\right] + \frac{I}{2\pi\sigma}\ln r. \tag{10}$$

Comparing with eq. (3) for $r > b$, we see that the potential $V_1(r > a)$ can be thought of a due to a current source located at point $p' = (a^2/b, 0)$ in a conducting sheet with no hole, plus a current sink at the origin. The image source and sink have magnitudes $(\sigma - \sigma')/(\sigma + \sigma')$ that of the current source at point p. The electric field at an arbitrary point at $r > a$ is therefore given by,

$$\mathbf{E}(r > a) = \frac{I}{2\pi\sigma}\left(\frac{\hat{\boldsymbol{\rho}}}{\rho} + \frac{\sigma - \sigma'}{\sigma + \sigma'}\frac{\hat{\boldsymbol{\rho}}'}{\rho'} - \frac{\sigma - \sigma'}{\sigma + \sigma'}\frac{\hat{\mathbf{r}}}{r}\right), \tag{11}$$

where $\hat{\boldsymbol{\rho}}'$ is the unit vector along the line from point p' to the arbitrary point, and ρ' is the distance between these two points.[2]

2.1 Solution Using Eq. (2) Rather Than (3)

If we did not recall the expansion (3) of the logarithmic potential (2), we could evaluate the constraint (5) at the rim of the hole as,

$$0 = E_r(r = a^+) = -\frac{\partial V(r = a^+)}{\partial r} = \frac{I}{2\pi\sigma}\frac{a - b\cos\phi}{a^2 + b^2 - 2ab\cos\phi} + \sum_{n=1}^{\infty}\frac{n}{a}A_n \cos n\phi. \tag{12}$$

The Fourier coefficients A_n are then given by,

$$\begin{aligned} A_n &= -\frac{aI}{2n\pi^2\sigma}\int_0^{2\pi}\frac{a - b\cos\theta}{a^2 + b^2 - 2ab\cos\phi}\cos n\theta\; d\phi \\ &= -\frac{I}{n\pi^2\sigma}\frac{a^2}{a^2 + b^2}\int_0^{\pi}\frac{\cos n\theta - \frac{b}{2a}\cos(n+1)\phi - \frac{b}{2a}\cos(n-1)\phi}{1 - \frac{2ab}{a^2+b^2}\cos\theta}\; d\phi \\ &= \frac{I}{2n\pi\sigma}\left(\frac{a}{b}\right)^n, \end{aligned} \tag{13}$$

using Dwight 858.536 [4], which tells us that for $b > a$,

$$\int_0^{\pi}\frac{\cos n\phi}{1 - \frac{2ab}{a^2+b^2}\cos\phi}\, d\phi = \frac{\pi}{\sqrt{1 - \left(\frac{2ab}{a^2+b^2}\right)^2}}\left(\frac{1 - \sqrt{1 - \left(\frac{2ab}{a^2+b^2}\right)^2}}{\frac{2ab}{a^2+b^2}}\right)^n = \pi\frac{b^2 + a^2}{b^2 - a^2}\left(\frac{a}{b}\right)^n. \tag{14}$$

We now wish to show that the azimuthal electric field at $r = a^+$ due to the potential V_1 is the same as that due to potential V_0, *i.e.*,

$$-\frac{\partial V_0(a, \phi)}{\partial\phi} = -\frac{\partial V_1(a, \phi)}{\partial\phi}. \tag{15}$$

[2] The solution found here is identical to that for a charged wire in vacuum outside a cylinder of (relative) dielectric constant $\epsilon = \sigma'/\sigma$. See sec. 2.2 of [3].

From eq. (2) we have,

$$-\frac{\partial V_0(a,\phi)}{\partial\phi} = \frac{I}{2\pi\sigma}\frac{b\sin\phi}{a^2+b^2-2ab\cos\phi}\,, \tag{16}$$

and from eq. (4) we have,

$$-\frac{\partial\phi_1(a,\phi)}{\partial\phi} = \sum_{n=0}^{\infty}\frac{n}{a}A_n\sin n\phi. \tag{17}$$

If eq. (15) is to be valid, eqs. (16) and (17) imply that the Fourier coefficients A_n can also be calculated according to,

$$\begin{aligned} A_n &= \frac{aI}{2n\pi^2\sigma}\int_0^{2\pi}\frac{b\sin\phi}{a^2+b^2-2ab\cos\phi}\sin n\phi\, d\phi \\ &= \frac{I}{2n\pi^2\sigma}\frac{ab}{a^2+b^2}\int_0^{\pi}\frac{\cos(n-1)\phi-\cos(n+1)\phi}{1-\frac{2ab}{a^2+b^2}\cos\phi}\, d\phi \\ &= \frac{I}{2n\pi\sigma}\left(\frac{a}{b}\right)^n. \end{aligned} \tag{18}$$

Since calculations (13) and (18) give the same results for A_n we again conclude that the potential between any two points on the rim of the hole is twice that in its absence.

3 Three-Dimensional Version of the Problem

In a three-dimensional version of this problem the hole is a sphere of radius a centered on the origin and the current enters at $(r,\theta,\phi) = (b,0,0)$ in a spherical coordinate system. The fields and potentials in this problem are independent of the azimuth ϕ.

In the absence of the hole, the current density $\mathbf{J}_0$ flows radially outward from point p. Let $\rho = \sqrt{r^2+b^2-2rb\cos\phi}$ be the distance from point p to the point (r,θ,ϕ), and $\hat{\boldsymbol{\rho}}$ be a unit vector pointing away from p. The current density $\mathbf{J}_0$ is now given by,

$$\mathbf{J}_0 = \frac{I}{4\pi\rho^2}\hat{\boldsymbol{\rho}} = \sigma\mathbf{E}_0 = -\sigma\boldsymbol{\nabla}V_0, \tag{19}$$

where σ is the electrical conductivity of the medium outside the hole. Solving for the potential V_0, we find,

$$V_0 = \frac{I}{4\pi\sigma\rho} + K. \tag{20}$$

Note that the potential V_0 is the same as that due to a charge $q = I/4\pi\sigma$ at point p. Following sec. 5-2 of [1] or sec. 5.16 of [2], we can rewrite V_0 (ignoring the constant K) as,

$$V_0(r,\theta) = \frac{I}{4\pi\sigma b}\begin{cases} \sum_{n=1}^{\infty}\left(\frac{r}{b}\right)^n P_n(\cos\theta), & r<b, \\ \sum_{n=1}^{\infty}\left(\frac{b}{r}\right)^{n+1} P_n(\cos\theta), & r>b. \end{cases} \tag{21}$$

The hole perturbs the potential V_0 by the addition of potential V_1. This correction to the potential vanishes for large r, is finite at $r = 0$, and is independent of angle ϕ. Hence, we can write (ignoring the constant term in the potential),

$$V_1(r,\theta) = \begin{cases} \sum_{n=1}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta), & r < a, \\ \sum_{n=1}^{\infty} A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta), & r > a. \end{cases} \tag{22}$$

The radial component of the (steady) current density is continuous across the boundary $r = a$, which implies that the radial component of the electric field obeys $\sigma E_r(r = a^+) = \sigma' E_r(r = a^-)$, where σ' is the electrical conductivity of the material at $r < a$. This constrains the radial derivative of the potential at $r = a$,

$$\begin{aligned} \sigma \frac{\partial V(r = a^+)}{\partial r} &= \sigma \frac{I}{4\pi\sigma b} \sum_{n=1}^{\infty} \frac{n}{a} \left(\frac{a}{b}\right)^n P_n(\cos\theta) - \sigma \sum_{n=1}^{\infty} \frac{n+1}{a} A_n P_n(\cos\theta) \\ = \sigma' \frac{\partial V(r = a^-)}{\partial r} &= \sigma' \frac{I}{4\pi\sigma b} \sum_{n=1}^{\infty} \frac{n}{a} \left(\frac{a}{b}\right)^n P_n(\cos\theta) + \sigma' \sum_{n=1}^{\infty} \frac{n}{a} A_n P_n(\cos\theta), \end{aligned} \tag{23}$$

using the forms (21)-(22). Hence, the Fourier coefficients A_n are given by,

$$A_n = \frac{n(\sigma - \sigma')}{(n+1)(\sigma + \sigma')} \frac{I}{4\pi\sigma b} \left(\frac{a}{b}\right)^n . \tag{24}$$

The potential V_1 is therefore given by,

$$V_1(r,\theta) = \frac{(\sigma - \sigma')I}{4\pi\sigma b} \begin{cases} \sum_{n=1}^{\infty} \frac{n}{(n+1)(\sigma+\sigma')} \left(\frac{r}{b}\right)^n P_n(\cos\theta), & r < a, \\ \frac{b}{a} \sum_{n=1}^{\infty} \frac{n}{(n+1)(\sigma+\sigma')} \left(\frac{a^2/b}{r}\right)^{n+1} P_n(\cos\theta), & r > a. \end{cases} \tag{25}$$

This completes a series expansion of the potential for arbitrary conductivity σ' at $r < a$, but the form of V_1 does not admit interpetation via an image method.[3]

References

[1] Problem 7-3 of W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, 1962), http://kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

[2] W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill), 1968), http://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf

[3] K.T. McDonald, *Dielectric Image Methods* (Nov. 21, 2009), http://kirkmcd.princeton.edu/examples/image.pdf

[4] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4th ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

[3]The solution found here is identical to that for a point charge in vacuum outside a sphere of (relative) dielectric constant $\epsilon = \sigma'/\sigma$. See sec. 2.3 of [3].