Pendulum in Orbit

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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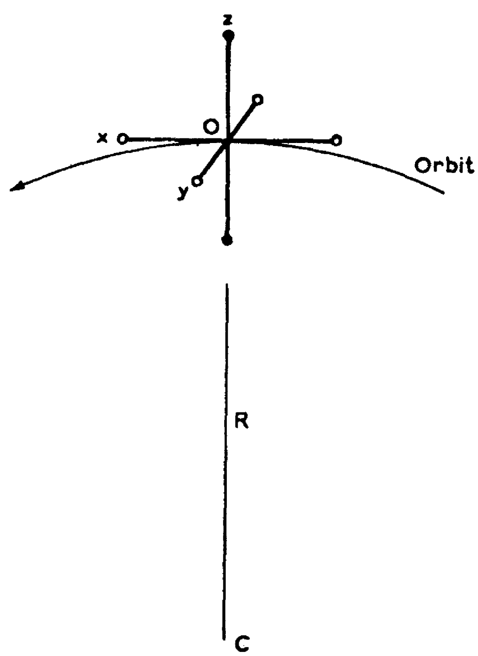
1 Problem

Discuss the frequency of small oscillations of a simple pendulum in orbit, say, about the Earth, supposing that the point of support of the pendulum is much more massive than the bob of the pendulum, and the support point is in a circular orbit.

2 Solution

The support point is taken to be at radius R from the center of the (spherical) Earth whose mass is M. Then, this point moves with angular velocity $\Omega = \sqrt{GM/R^3} = \sqrt{g/R}$ with respect to an inertial frame which the center of the Earth is at rest, where $g = GM/R^2$ is the acceleration due to gravity at radius R.[1]

We are perhaps most interested in the motion as would be reported by an observer at the point of support of the pendulum, so we work in a rotating frame, centered on the Earth, whose angular velocity is $\mathbf{\Omega}$, which is perpendicular to the plane of the orbit of the support point. Of course, the support point is at rest in this frame.



We use a rectangular coordinate system centered on the support point (O in the figure above, from [1]), with the z-axis along the vector $\mathbf{R}$ from the center of the Earth to the

[1]Note that the angular velocity of small oscillations of a pendulum of length l is $\omega = \sqrt{g/l}$ if its support point is at rest in an inertial frame with acceleration g due to gravity.

support point, and the x-axis in the plane of the orbit of the latter. Then, the angular velocity of the rotating frame is $\mathbf{\Omega} = \Omega\,\hat{\mathbf{y}}$.

The (simple) pendulum has length l and a bob of mass m, and is at position $\mathbf{x} = (x, y, z)$ where $x^2 + y^2 + z^2 = l^2$. The distance from the center of the Earth to the bob is,

$$\mathbf{x}_E = \mathbf{R} + \mathbf{x} = (x, y, z + R), \qquad x_E \approx R + z. \tag{1}$$

The forces in the rotating frame on the bob are that due to gravity,

$$\begin{aligned} -\frac{GMm\,\mathbf{x}_E}{x_E^3} &= -\frac{m\,\Omega^2 R^3 \mathbf{x}_E}{x_E^3} \approx -\frac{m\,\Omega^2 \mathbf{x}_E}{(1+z/R)^3} \approx -m\,\Omega^2\left(1 - \frac{3z}{R}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + R(1+z/R)\,\hat{\mathbf{z}}) \\ &\approx -m\,\Omega^2(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + (R - 2z)\,\hat{\mathbf{z}}), \end{aligned} \tag{2}$$

to the centrifugal force,

$$-m\,\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{x}_E) = m\,\Omega^2[x\,\hat{\mathbf{x}} + (R + z)\,\hat{\mathbf{z}}], \tag{3}$$

to the Coriolis force,

$$2m\,\mathbf{\Omega} \times \dot{\mathbf{x}} = 2m\,\Omega(\dot{z}\,\hat{\mathbf{x}} - \dot{x}\,\hat{\mathbf{z}}), \tag{4}$$

and to the tension $\mathbf{T} = -T\mathbf{x}$ in the massless rod/string of the pendulum.

To avoid need for knowledge of the constraint force $\mathbf{T}$, we consider the torque, $\boldsymbol{\tau} = \mathbf{x} \times \mathbf{F}_{\text{total}}$, and angular momentum, $\mathbf{L} = \mathbf{x} \times m\dot{\mathbf{x}}$ of the bob, about the support point,

$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= m\mathbf{x} \times \ddot{\mathbf{x}} = m[(y\ddot{z} - z\ddot{y})\,\hat{\mathbf{x}} + (z\ddot{x} - x\ddot{z})\,\hat{\mathbf{y}} + (x\ddot{y} - y\ddot{x})\,\hat{\mathbf{z}}] \\ &= \boldsymbol{\tau} = \mathbf{x} \times m[-\Omega^2 y\,\hat{\mathbf{y}} + 3\Omega^2 z\,\hat{\mathbf{z}} + 2\Omega(\dot{z}\,\hat{\mathbf{x}} - \dot{x}\,\hat{\mathbf{z}}) - T\,\hat{\mathbf{x}}] \\ &= m[\Omega^2(yz\,\hat{\mathbf{x}} - xy\,\hat{\mathbf{z}}) + 3\Omega^2(yz\,\hat{\mathbf{x}} - xz\,\hat{\mathbf{y}}) - 2\Omega(y\dot{x}\,\hat{\mathbf{x}} + (x\dot{x} - z\dot{z})\,\hat{\mathbf{y}} + y\dot{z}\,\hat{\mathbf{z}})]. \end{aligned} \tag{5}$$

Hence, the equations of motion can be written as,

$$y\ddot{z} - z\ddot{y} = 4\Omega^2 yz - 2\Omega\,y\dot{x}, \tag{6}$$

$$z\ddot{x} - x\ddot{z} = -3\Omega^2 xz - 2\Omega(x\dot{x} - z\dot{z}), \tag{7}$$

$$x\ddot{y} - y\ddot{x} = -\Omega^2 xy - 2\Omega\,y\dot{z}. \tag{8}$$

The conditions for equilibrium, at which all time derivatives vanish, are $xy = yz = xz = 0$. These are satisfied at the six locations $(\pm l, 0, 0)$, $(0, \pm l, 0)$ and $(0, 0, \pm l)$ of the bob, as shown in the figure on p. 1.

2.1 The Equilibria at $(\pm l, 0, 0)$ are Unstable

For motion near these equilibrium points, both $\dot{x}$ and $\ddot{x}$ are small.

For motion in the x-y plane (with $z = 0$), eq. (8) implies that $\ddot{y} = -\Omega^2 y$, so small oscillations can exist in y.

However, for motion in the x-z plane (with $y = 0$), eq. (7) implies that $\ddot{z} = 4\Omega^2 z$, so any small perturbation in z would grow exponentially with time.

Hence, these equilibria are unstable.

2.2 The Equilibria at $(0, \pm l, 0)$ are Unstable

For motion near these equilibrium points, both $\dot{y}$ and $\ddot{y}$ are small.

For motion in the x-y plane (with $z = 0$), eq. (8) implies that $\ddot{x} = \Omega^2 x$, so any small perturbation in x would grow exponentially with time.

For motion in the y-z plane (with $x = 0$), eq. (6) implies that $\ddot{z} = 4\Omega^2 z$, so any small perturbation in z would grow exponentially with time.

Hence, these equilibria are unstable.

2.3 The Equilibria at $(0, 0, \pm l)$ are Stable

For motion near these equilibrium points, both $\dot{z}$ and $\ddot{z}$ are small.

For motion in the x-z plane (with $y = 0$), eq. (7) implies that $\ddot{x} = -3\Omega^2 x$,[2] so small oscillations in x can exist with angular frequency $\boxed{\omega_{xz} = \sqrt{3}\Omega = \sqrt{3g/R}}$.

For motion in the y-z plane (with $x = 0$), eq. (6) implies that $\ddot{y} = -4\Omega^2 y$, so small oscillations in y can exist with angular frequency $\boxed{\omega_{yz} = 2\Omega = 2\sqrt{g/R}}$.

That the two frequencies ω_{xz} and ω_{yz} are different is a consequence of the different symmetries of the gravitational and centrifugal forces; the former is spherically symmetric while the latter is axially symmetric (about y).

The periods $2\pi/\sqrt{3}\Omega$ and π/Ω of these oscillations are independent of the length l of the pendulum, and are of the same order as the period $2\pi/\Omega$ (≈ 90 min) of the (low-Earth-) orbital motion.[3] Hence, astronauts in a space station would tend to say that a pendulum does not oscillate (according to their expectations of period $2\pi\sqrt{l/g} = \sqrt{l/R}\,2\pi/\Omega \approx 1$ s from experience on Earth).[4]

The equations of motion for oscillations in the x-z or y-z planes have no terms (at first order in $\dot{x}$ or $\dot{y}$) associated with the Coriolis force, so the small oscillations of a pendulum in orbit do not exhibit the precession first discussed by Foucault [3]. However, since the frequencies of oscillation in x-z and y-z planes are incommensurate, the general motion of the pendulum over long times would be considered as "chaotic" by an astronaut, even for small oscillations (unless the oscillation were purely in the x-z or in the y-z planes).

A Appendix: Shorter Derivation of Motion in the Plane of the Orbit of the Support Point

A somewhat briefer derivation was given in Appendix 17 of [4], using conservation of energy, $E = T + V$, to deduce the motion in the plane of the orbit of the support point from the time derivative $\dot{E} = \dot{T} + \dot{V} = 0$.

[2]There is also a second-order, Coriolis term $2\Omega(\dot{z} - x\dot{x}/l)$ that we neglect for small oscillations in the x-z plane about $(0, 0, \pm l)$.

[3]These results agree with those found in [1], where a clever variant of Lagrange's method was employed using a Lagrange multiplier, considering the relation $x^2 + y^2 + z^2 = l^2$ to be a constraint. Lagrange's method was also used in [2], for spherical coordinates.

[4]See, for example, https://www.quora.com/Would-a-pendulum-swing-in-orbit.

The potential energy V of the bob of mass m at distance $\mathbf{r}$ from the center of the Earth is, in the rotating frame where the centrifugal can be related to the centrifugal potential,[5]

$$V = -\frac{GMm}{r} - \frac{m(\boldsymbol{\Omega} \times \mathbf{r})^2}{2} \approx -m\Omega^2 \left(\frac{r^2}{2} + \frac{R^3}{r}\right), \qquad \dot{V} \approx -m\Omega^2 \dot{r} \left(r - \frac{R^3}{r^2}\right), \tag{9}$$

where the approximation holds for a pendulum of length $l \ll R$.

Anticipating that the pendulum might oscillate about the "vertical" from the center of the Earth to the support point, we use a spherical coordinate system (ρ, θ, ϕ) (in the rotating frame) whose origin is at the support point, whose z-axis points away from the center of the Earth, and with $\phi = 0$ and π in the plane of the orbit of the support point. Then, the kinetic energy of the bob of the pendulum of length l is,

$$T = \frac{ml^2}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta), \qquad \dot{T} = ml^2(\dot{\theta}\ddot{\theta} + \dot{\phi}\ddot{\phi}\sin^2\theta + \dot{\phi}^2\dot{\theta}\sin 2\theta). \tag{10}$$

Furthermore, to a very good approximation, $r = R + l\cos\theta$, so $\dot{r} = -l\dot{\theta}\sin\theta$, and,

$$\dot{V} \approx m\Omega^2 R l\dot{\theta}\sin\theta \left\{1 + (l/R)\cos\theta + \frac{1}{[1 + (l/R)\cos\theta]^2}\right\} \approx 3m\Omega^2 l^2 \sin\theta\cos\theta. \tag{11}$$

Then, for motion in the plane of the orbit of the support point, $\dot{\phi} = 0$,

$$0 = \dot{T} + \dot{V} = ml^2\dot{\theta}(\ddot{\theta} + 3\Omega^2 \sin\theta\cos\theta). \tag{12}$$

For $\theta = \epsilon$ or $\pi + \epsilon$ and small ϵ, we have that,

$$0 \approx \ddot{\epsilon} + 3\Omega^2\epsilon. \tag{13}$$

The the angular velocity of small oscillations of the pendulum in the plane of the orbit of the support point, about either $\theta = 0$ or π, is $\sqrt{3}\Omega = \sqrt{3g/R}$, as found previously in sec. 2.3.

Note how eq. (12) also shows that $\theta = \pi$ and $3\pi/2$ correspond to the unstable equilibria of sec. 2.1.

References

[1] J.L. Synge, *On the Behaviour, According to Newtonian Theory, of a Plumb Line or Pendulum Attached to an Artificial Satellite*, Proc. Roy. Irish Acad. A **60**, 1 (1959), http://kirkmcd.princeton.edu/examples/mechanics/synge_priaa_60_1_59.pdf

[2] L. Blitzer, *Equilibrium and stability of a pendulum in an orbiting spaceship*, Am. J. Phys. **47**, 241 (1979), http://kirkmcd.princeton.edu/examples/mechanics/blitzer_ajp_47_241_79.pdf

[3] L. Foucault, *Démonstration physique du mouvement de rotation de la terre au moyen du pendule*, Comptes Rendus Acad. Sci. **32**, 135 (1851), http://kirkmcd.princeton.edu/examples/mechanics/foucault_cras_32_135_51.pdf

[5] Both the Coriolis force and the tension in the rod/string of the pendulum do no work, and so do not contribute to the potential energy in the rotating frame.

[4] E.J. van der Heide and M. Kruijff, *StarTrack: A swinging tether assisted re-entry for the International Space Station*, Eur. Space Agency, EWP 1883 (1996), http://kirkmcd.princeton.edu/examples/mechanics/vanderheide_ewp_96.pdf