"Hidden" Momentum in a Current Loop

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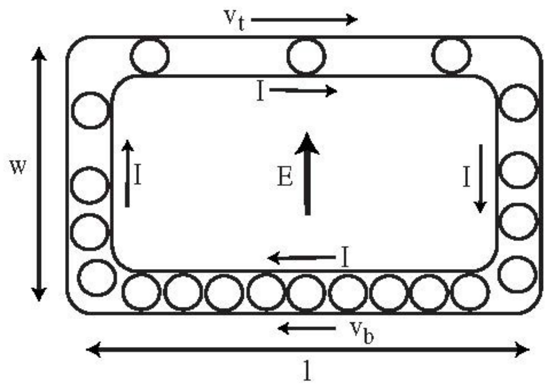
(June 30, 2012; updated May 9, 2018)

1 Problem

Discuss the momentum of a loop of electrical current I that is subject to a uniform external electric field $\mathbf{E}$ in the plane of the loop.[1]



Does the current contain hidden momentum, $\mathbf{P}_{\text{hidden}}$, defined for a subsystem by,[2]

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area}, \tag{1}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass", c is the speed of light in vacuum, U is its total energy, $\mathbf{x}_{\text{cm}}$ is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, and $\mathbf{v}_b$ is the velocity (field) of its boundary?

This problem was first considered by J.J. Thomson in 1904 [3]-[6]. The present version first appeared on p. 215 of [7]. See also [8]-[14].

2 Solution

If an electric charge q of rest mass m_q is moving with (variable) velocity $\mathbf{v}$ in an external electric field $\mathbf{E}$, its kinetic energy changes according to the work done on charge q,

$$q\int_i^f \mathbf{E} \cdot d\mathbf{l} = \Delta\text{KE} = \Delta U_{\text{mech}} = m_q(\gamma_f - \gamma_i) \qquad \text{where} \qquad \gamma = 1/\sqrt{1 - v^2/c^2}, \tag{2}$$

independent of the possible presence of an external magnetic field $\mathbf{B}$. In the present example, let $\hat{\mathbf{x}}$ point to the right in the figure above, with $\hat{\mathbf{y}}$ upwards, such that the electric field is

[1]The current is not "shielded" from the external electric field, and does not correspond to the current in either a resistive or superconducting wire.

[2]The definition (1) was inspired by discussions with Daniel Vanzella [1]. See also [2].

$\mathbf{E} = E\,\hat{\mathbf{y}}$. Then, the change in a charge's mechanical energy when moving distance w upwards through this field is,

$$\Delta U_{\rm mech} = m_q(\gamma_t - \gamma_b)c^2 = qEw, \tag{3}$$

where t and b refer to the top and bottom of the loop, w is its height.[3,4]

We now consider the system to contain three subsystems, the circulating charges, the electromagnetic fields (which include both the external electric field and the fields of the charges), and other mechanical apparatus at rest in the lab frame, including the sources of the external electric field and a nonconducting tube that constrains the charges, without friction, to move in the rectangular circuit.

The electrical current of the circulating charges in the bottom and top segments of the loop is given by,

$$I = qn_b v_b = qn_t v_t, \tag{4}$$

where n is the number of moving charges per unit length. The total momentum of the charges can now be written as,

$$\mathbf{P}_{\rm charges} = (n_t l m_q \gamma_t v_t - n_b l m_q \gamma_b v_b)\,\hat{\mathbf{x}} = \frac{lI\Delta U_{\rm mech}}{c^2 q}\,\hat{\mathbf{x}} = \frac{IlwE}{c^2}\,\hat{\mathbf{x}} = \frac{\mathbf{m}\times\mathbf{E}}{c}\,, \tag{5}$$

noting that the magnetic moment $\mathbf{m}$ of the loop of circulating charge is given (in Gaussian units) by,

$$\mathbf{m} = \int \frac{\mathbf{r}\times\mathbf{J}}{2c}\,d{\rm Vol} = \frac{I\,\mathbf{Area}}{c} = -\frac{Ilw}{c}\,\hat{\mathbf{z}}, \tag{6}$$

where $\hat{\mathbf{z}}$ is out of the paper.

The center-of-mass velocity of the charges is given by,

$$M_{\rm charges}\mathbf{v}_{\rm cm,charges} = (n_t l m_q \gamma_t v_t - n_b l m_q \gamma_b v_b)\,\hat{\mathbf{x}} = \mathbf{P}_{\rm charges}, \tag{7}$$

where $M_{\rm charges}$ is the sum of the masses of all charges. This result is peculiar in that the tube that constrains the charges is at rest, and the charge distribution within that tube is stationary, from a macroscopic perspective. Equation (7) corresponds to a microscopic view in which charges enter and leave each segment of the loop, such that the center of mass of the charges in, say, the top segment moves to the right at speed v_t until a charge leaves on the right and another enters from the left; at that moment the center of mass instantaneously

[3]The use of this argument in cases of time-dependent electric fields is considered in sec. 3 below for transient fields, and in sec. 4 below for wave fields.

[4]It could be that the electric field is related only to a scalar potential V, but this is not required in the present argument. In any case, the increase in the mechanical energy of the charges is compensated by a reduction in the energy of the electric field.

If the electric field is static, with $\mathbf{E} = -\boldsymbol{\nabla}V$, then the interference energy between the external field and the field of the energized charges is readily shown to be negative, as discussed in [15].

Also, as noted in footnote 9 of [16], in case of an electrostatic field $\mathbf{E}$, the total energy of electric charge q has the constant value $U = \gamma m_q c^2 + qV$. Then, the change in a charge's mechanical energy when moving distance w upwards through this field is $\Delta U_{\rm mech} = m_q(\gamma_t - \gamma_b)c^2 = q\Delta V = qEw$, as in eq. (3).

For additional remarks on the relation $U = \gamma m_q c^2 + qV$, see [17].

moves to the left by distance $d_t/n_t l$, where d_t is the spacing between charges in the top segment. That is, the motion of the microscopic center of mass is a Zitterbewegung.

It is more appropriate to provide a macroscopic description, in which we can speak of a steady current I, rather than a series of delta functions of current spaced at small time intervals. The macroscopic description is obtained by averaging over distances larger than the gaps d between the moving charges. In the macroscopic description the Zitterbewegung is ignored, and we set the macroscopic center-of-mass velocity $\bar{\mathbf{v}}_{\rm cm,charges}$ of the charges to zero,

$$\bar{\mathbf{v}}_{\rm cm,charges} = 0. \tag{8}$$

The subsystem of charges is then said to contain "hidden" momentum according to the definition (1),

$$\mathbf{P}_{\rm hidden,charges} = \mathbf{P}_{\rm charges} - M_{\rm charges}\bar{\mathbf{v}}_{\rm cm,charges} = \mathbf{P}_{\rm charges} = \frac{\mathbf{m}\times\mathbf{E}}{c}\,. \tag{9}$$

The subsystem of the electromagnetic fields has nonzero momentum,

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d{\rm Vol} = \int \frac{V\mathbf{J}}{c^2}\,d{\rm Vol} = \frac{\mathbf{E}\times\mathbf{m}}{c} = -\mathbf{P}_{\rm charges}, \tag{10}$$

where the second form of eq. (10) is due to Furry [8].[5] Thus, the total momentum of the system is zero.

The center of energy of the electromagnetic fields is at rest in the lab frame, so $\mathbf{v}_{\rm cm,EM} = 0$. Hence, according to definition (1), the electromagnetic subsystem has nonzero "hidden" momentum,

$$\mathbf{P}_{\rm hidden,EM} = \mathbf{P}_{\rm EM} - M_{\rm EM}\mathbf{v}_{\rm cm,EM} = \mathbf{P}_{\rm EM} = \frac{\mathbf{E}\times\mathbf{m}}{c} = -\mathbf{P}_{\rm charges} = -\mathbf{P}_{\rm hidden,charges}, \tag{11}$$

where $M_{\rm EM} \equiv U_{\rm EM}/c^2$.

The key result of this problem is the clarification that the center-of-mass velocity to be used in application of definition (2) is the macroscopic average velocity, in the same sense that the electrical current I is a macroscopic average. Then, mechanical momentum which is explicitly noted in a microscopic description can, in some cases, be characterized as "hidden" in the macroscopic description.

3 Transient Effects

In the preceding the current and external electric field were assume to be steady. We now consider several possibilities of transient behavior, in which the current varies, or in which the electric field varies.[6,7]

[5]The result (10) appears on pp. 347-348 of [4] and in sec. 285 of [5], but Thomson appears not to have noted the paradox that a system "at rest" has nonzero field momentum. For discussion of these early results, see [6].

[6]The example of an electric charge plus magnetic dipole is a variant on the configuration of the Aharonov-Bohm effect [18], in which an electric charge moves outside a long solenoid magnet where the external field $\mathbf{B}_{\rm solenoid}$ is negligible but the external vector potential $\mathbf{A}_{\rm solenoid}$ is nonzero. For discussion by the author of classical aspects of the Aharonov-Bohm effect, see [19].

[7]An exotic possibility, that the entire system collapses to a black hole, has been considered in [20].

3.1 The Current is Excited/De-excited

Suppose the external electric field is static, but the current in the loop is created or destroyed by a transient, external magnetic flux though the loop. The transient electric field around the loop modifies the electrical current in the loop, doing work to create/destroy the magnetic field energy of the loop (and the small kinetic energy of the circulating charges). In this process, equal and opposite amounts of mechanical and electromagnetic momentum are given/taken away from the system of loop plus external field, so no net force is exerted on that system during the transient phase.

3.2 The External Field is Due to a Single, Moving Charge

Suppose the external electric field is due to a single electric charge Q (with no magnetic moment) that has velocity $\mathbf{v}$ with $v \ll c$, while the magnetic moment $\mathbf{m}$ remains constant. This charge is subject to the Lorentz force of the magnetic dipole field of the current loop, which force does no work on the charge, such that the speed v, the relativistic mass $m_Q/\sqrt{1-v^2/c^2}$, and the kinetic energy of the charge remain constant. However, the direction of the velocity $\mathbf{v}$ changes with time, so the charge is accelerated and radiates energy and momentum. An issue raised by [20] is whether the total radiated momentum is significant compared to $P_{\rm EM} = Qm\sin\theta/cr^3$, where θ is the angle between $\mathbf{m}$ and the line joining Q and the location of the magnetic dipole $\mathbf{m}$, and r is the length of that line.

In the following we will only consider cases where $\theta = 90^\circ$; that is, the charge moves in the plane perpendicular to the constant magnetic moment $\mathbf{m}$. We also suppose that the mass of the current loop is much larger than the (rest) mass m_q of the charge, so that it is a good approximation to consider the current loop to be at rest, and at the origin.

3.2.1 The Charge Moves Radially from r_0 to ∞

Initially, the charge is at radius r_0 with respect to the magnetic moment $\mathbf{m}$, and its small velocity is radially outwards, $\mathbf{v} = v\,\hat{\mathbf{r}}$. We also suppose that v is small enough that the trajectory of the charge is nearly radial at all times.

The Lorentz force on the moving charge Q is,

$$\mathbf{F}_Q = \frac{d\mathbf{P}_Q}{dt} = Q\frac{\mathbf{v}}{c}\times\mathbf{B}_m = Q\frac{\mathbf{v}}{c}\times\left(\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}}{r^3}\right) = Q\frac{\mathbf{m}\times\mathbf{v}}{cr^3}\,, \tag{12}$$

for motion of Q in the plane perpendicular to $\mathbf{m}$. Hence, if $\mathbf{v} \approx v\,\hat{\mathbf{r}}$, then the azimuthal component of eq. (12) is,

$$\frac{dP_\phi}{dt} \approx \frac{Qmv}{cr^3}\,. \tag{13}$$

The radial component of the momentum of the charge is $P_r \approx m_Q v$, so the rate of change of the angle ϕ of the charge's trajectory with respect to the radial direction is,

$$\frac{d\phi}{dt} \approx \frac{1}{P_r}\frac{dP_\phi}{dt} \approx \frac{qm}{cm_q r^3}\,. \tag{14}$$

For nearly radial motion, $dr \approx v\,dt$, so that the total change $\Delta\phi$ in the azimuthal orientation of the trajectory of the charge as it moves from r_0 to ∞ is related by,

$$\frac{d\phi}{dr} \approx \frac{1}{v}\frac{d\phi}{dt} \approx \frac{c}{v}\frac{Qm}{m_q c^2 r^3}\,, \qquad \Delta\phi = \int_{r_0}^{\infty} \frac{d\phi}{dr}\,dr = \int_{r_0}^{\infty} \frac{c}{v}\frac{Qm}{m_Q c^2 r^3}\,dr = \frac{c}{v}\frac{Qm}{2m_Q c^2 r_0^2}\,. \tag{15}$$

Hence, the condition for quasiradial outward motion is,

$$\frac{Qm}{m_Q c^2 r_0^2} \ll \frac{v}{c} \ll 1 \qquad \text{(quasiradial outward motion).} \tag{16}$$

As discussed in sec. 2.2.6 of [21], rate of radiation of momentum by the accelerated charge is,

$$\frac{d\mathbf{P}_{\rm rad}}{dt} = \frac{dU_{\rm rad}}{dt}\frac{\mathbf{v}}{c^2} = \frac{2Q^2F_q^2}{3m_Q^2c^5}\mathbf{v} \approx \frac{2Q^4m^2v^3}{3m_Q^2c^7r^6}\hat{\mathbf{r}}, \tag{17}$$

recalling the Larmor formula for $v \ll c$ that $dU_{\rm rad}/dt = 2Q^2a^2/3c^3 = 2Q^2F_q^2/3m_Q^2c^3$. The total radiated momentum $P_{\rm rad}$ in the radial direction, follows as,

$$\frac{dP_{\rm rad}}{dr} \approx \frac{1}{v}\frac{dP_{\rm rad}}{dt} \approx \frac{2Q^4m^2v^2}{3Q_q^2c^7r^6}\,, \qquad P_{\rm rad} = \int_{r_0}^{\infty} \frac{dP_{\rm rad}}{dr}\,dr \approx \frac{2Q^4m^2v^2}{15m_Q^2c^7r_0^5}\,. \tag{18}$$

The initial electromagnetic momentum of the system follows from eq. (10) as,

$$P_{\rm EM} = \frac{Qm}{cr_0^2}\,, \tag{19}$$

so that the ratio of the radiated momentum to the initial electromagnetic momentum is,

$$\frac{P_{\rm rad}}{P_{\rm EM}} \approx \frac{Q^3mv^2}{m_Q^2c^5r_0^3} = \frac{v^2}{c^2}\frac{Qm}{m_qc^2r_0^2}\frac{Q^2/r_0}{m_qc^2} \ll \frac{v^3}{c^3}\frac{q^2/m_Qc^2}{r_0}\,. \tag{20}$$

For r_0 large compared to the classical charge radius Q^2/m_Qc^2, the ratio (20) is negligible.

In sum, as the charge moves outwards quasiradially at constant speed, the electromagnetic field momentum and the net mechanical momentum of the electric current drop to zero, remaining equal and opposite at all times in the quasistatic approximation.

3.2.2 The Charge Orbits the Magnetic Dipole

For magnetic moment $\mathbf{m} = m\,\hat{\mathbf{z}}$, a positive electric charge Q can move counterclockwise in a circular orbit of radius r with speed v given by the Lorentz force as,

$$\frac{v}{c} = \frac{Qm}{Q_qc^2r^2}\,, \tag{21}$$

when $v/c \ll 1$. We suppose that the initial radius r_0 is large enough that $v/c \ll 1$.

As noted, for example, in sec. 2.5.5 of [21], the accelerated charge radiates total power,

$$\frac{dU_{\rm rad}}{dt} = \frac{2Q^4(\mathbf{v}/c \times \mathbf{B})^2}{3m_Q^2 c^3} = \frac{2Q^4 m^2 v^2}{3m_Q^2 c^5 r^6} = \frac{2Q^6 m^4}{3m_Q^4 c^7 r^{10}}\,. \tag{22}$$

The radiation produces a back reaction on the charge which can be characterized by the radiation reaction force $\mathbf{F}_{\rm react} = -K\mathbf{v}$, such that the power delivered by this force is the negative of the radiated power,

$$\mathbf{F}_{\rm react} \cdot \mathbf{v} = -Kv^2 = -\frac{dU_{\rm rad}}{dt}, \tag{23}$$

and hence[8]

$$\mathbf{F}_{\rm react} = -K\mathbf{v} \qquad \text{with} \qquad K = \frac{2Q^4 m^2}{3m_Q^2 c^5 r^6}\,. \tag{24}$$

As a consequence, the charged particle slows down according to,

$$m_Q \frac{dv}{dt} = F_{\rm react} = -Kv, \qquad v = v_0\, e^{-Kt/m_Q}, \qquad r = \sqrt{\frac{Qm}{m_Q c v}} = r_0\, e^{Kt/2m_Q}, \tag{25}$$

and its trajectory spirals slowly out to infinite radius where the asymptotic velocity is zero. During this process the electromagnetic field momentum and the net mechanical momentum go to zero, and the initial kinetic energy of the charge is transformed into electromagnetic radiation. Momentum is radiated by the accelerating charge, but the net radiated momentum over the large number of turns in the spiral trajectory is negligible, so there is no significant "kick" imparted to the magnetic dipole.[9]

3.2.3 General Motion of the Charge

An analytic solution for the general motion of the charge is not possible, but electromagnetic radiation is generated as the charge accelerates, which reduces the kinetic energy and the speed of the charge. The charge either comes to rest at some finite distance from the magnetic moment, or moves to infinity with asymptotic speed either zero or nonzero. In the first case the final electromagnetic field momentum and final net momentum of the current are nonzero, equal and opposite, and smaller than their initials values. In the second case the final electromagnetic field momentum and final net momentum of the current are zero. In no case does anything striking take place.

3.2.4 The Charge Collides with the Magnetic Dipole

A special case is that the electric charge q has initial velocity $\mathbf{v}_0 \perp \mathbf{m}$ such that it eventually collides (totally inelastically) with the magnetic dipole $\mathbf{m}$. Just before this collision the

[8] Equation (24) also follows from the well-known form $\mathbf{F}_{\rm react} = (2Q^2/3c^3)\,\dot{\mathbf{a}} = (2Q^2/3c^3)\,\boldsymbol{\omega} \times \mathbf{a} = (2Q^2/3c^3)\,(-v\,\hat{\mathbf{z}}/r) \times (-v^2/r)\,\hat{\mathbf{r}} = -(2Q^2v^2/3c^3r^2)\,\mathbf{v} = -(2Q^4m^2/3m_Q^2c^5r^6)\,\mathbf{v}$.

[9] The radiation fields transfer net energy and momentum to the magnetic dipole (including net work done by magnetic field [22]), but this effect is negligible in the present case.

charge has velocity $\mathbf{v}_f$ ($\perp$ $\mathbf{m}$). Suppose the magnetic dipole has initial velocity $-\mathbf{v}_f\, m_Q/m_m$, such that the final state of charge + moment has zero (mechanical) velocity.

In the initial state there is electromagnetic momentum (and electromagnetic energy) associated with the separate motions of the electric charge and of the magnetic dipole. We consider that these momenta (and energies) are "renormalized" into the "mechanical" momenta (and energies) of the charge and dipole. In addition, there is a nonzero interaction field momentum in the initial state, which is well approximated for low velocities by eq. (10), $\mathbf{P}_{\rm EM,int} \approx \mathbf{E}_Q \times \mathbf{m}/c$. And to the same approximation, the initial currents of the magnetic dipole contain hidden mechanical momentum, $\mathbf{P}_{\rm hidden,mech} \approx -\mathbf{P}_{\rm EM,int} \approx -\mathbf{E}_Q \times \mathbf{m}/c$.

As the electric charge approaches the magnetic dipole the electric field experienced by the latter grows large, so the interaction field momentum and the hidden mechanical momentum grow large, and are not necessarily well approximated by the forms in the preceding paragraph. However, in the final state, where we assume that the electric charge comes to rest at the center of the current loop of the magnetic dipole, the final interaction field momentum and the final hidden mechanical momentum are zero.

As the charge moves towards the magnetic dipole they both accelerate, emitting electromagnetic energy and momentum. The electric charge radiates much more than does the magnetic dipole, assuming that the latter's mass is much larger. Then, the final state contains net radiated momentum, whose direction is roughly along the line from the initial position of the charge to the dipole. Conservation of momentum tells us that the final state of the charge + dipole is not at rest, but has constant velocity, and mechanical momentum equal and opposite to the net radiated momentum.

An observer who does not consider the radiated momentum would say that the final state of charge + dipole has an unexpected "kick". While consideration of electromagnetic radiation shows that this "kick" has nothing to do with "hidden momentum" and everything to do with radiation, the naïve observer might associate it with "hidden momentum".

4 Wave Fields *(October, 2017)*

We now consider the possible validity of the argument of sec. 2 above for the case of a wave field, of wavelength λ and angular frequency $\omega = 2\pi c/\lambda$, *i.e.*, for a wave in a medium with unit (relative) permittivity and permeability.

One premise of the argument of sec. 2 is that the electric field $\mathbf{E}$ be uniform over the loop, which implies that in the case of a wave field, the wavelength λ is large compared to the sides h and w of the loop.

In, addition, for the instantaneous "hidden" mechanical momentum to be $\mathbf{m} \times \mathbf{E}(t)/c$, it must be the period of the circulation of charges in the current loop be less than the period of the external wave field. That is, the angular velocity of the circulating charges must be larger than the angular velocity of the wave field.

We consider the implications of this requirement for the cases of current loops associated with permanent magnetism, and with diamagnetism.

4.1 Permanent Magnetism

In the case of permanent magnetism the magnetic moments are (almost entirely) associated with the intrinsic magnetic moments of electron, which have magnitude $e\hbar/2mc$. Although there is no satisfactory classical model for these intrinsic moments, we make a plausibility argument based on the model that an electron is a shell of charge of radius $\lambdabar_C = \hbar/mc$, the reduced Compton wavelength, which shell rotates with equatorial velocity c. In this model, the angular velocity of the moving charge is $c/\lambdabar_C$, so it is plausible that the relation $\mathbf{m} \times \mathbf{E}(t)/c$ describes "hidden" mechanical momentum of permanent magnetism in electromagnetic fields of (reduced) wavelength $\lambdabar$ larger than $\lambdabar_C$.

This is the same requirement that the electric field be uniform over the current distribution of the magnetic dipole.

This requirement is well satisfied by electromagnetic waves of optical frequencies.

4.2 Diamagnetism

Diamagnetism is associated with magnetic momentum to due orbital motion of atomic electrons that is induced by the external electromagnetic field.

For an order-of-magnitude estimate, we take the effective radius of the magnetic moment to be the Bohr radius $r_B = \lambdabar_C/\alpha$, where $\alpha = e^2/\hbar c$ is the fine-structure constant and e is the charge of an electron.

Then, the requirement that the external field be uniform over the magnetic moment is that the wavelength of the external field be large compared to the Bohr radius, which is well satisfied by optical fields.

The angular frequency of the induced orbital motion of electrons is less than $v_{\rm max}/r_B$, where,

$$v_{\rm max} \approx a_{\rm max} t_{\rm wave} \approx \frac{eE_0}{m}\frac{1}{\omega}\,, \tag{26}$$

where E_0 is the peak electric field of the wave. For $\mathbf{m} \times \mathbf{E}(t)/c$ to describe the "hidden" mechanical momentum associated with diamagnetism, we must also have,

$$\omega_{\rm orbital} \approx \frac{v_{\rm max}}{r_B} \approx \frac{eE_0}{m\omega r_B} = \alpha\frac{eE_0}{m\omega\lambdabar_C} \gg \omega, \qquad E_0 \gg \frac{m\omega^2\lambdabar_C}{\alpha e} = \frac{\hbar^2\omega^2}{e^3} = \frac{E_{\rm crit}}{\alpha}\left(\frac{\lambdabar_C}{\lambdabar}\right)^2, \tag{27}$$

where $E_{\rm crit} = m^2c^3/e\hbar = 1.6 \times 10^{16}$ V/cm is the so-called QED critical field strength (above which a static electric field would spontaneously produce electron-positron pairs [23]). For optical waves, this requirement is that $E_0 \gg 10^4$ V/cm, which is a reasonably strong field, but which is readily achieved in laser beams.

A Appendix: An All-Mechanical Current Loop

As suggested by J. Franklin (private communication), we can imagine the figure on p. 1 corresponds to a set of runners on a racetrack. On the left, vertical leg of the track the runners convert some of their internal energy into increased kinetic energy, ideally in a reversible manner (such that on the right, vertical leg the runners convert some of their kinetic energy back into internal energy). In this idealization the total energy $m\gamma c^2$ of each runner is constant in time; the rest mass m decreases when the kinetic energy increases, and *vice versa*. So, if the runners are distributed such that the "current" is constant around the track,

$$I = n_b v_b = n_t v_t, \tag{28}$$

then in the frame of the track the total momentum of the runners is,

$$\mathbf{P}_{\text{runners}} = (n_t l m_t \gamma_t v_t - n_b l m_b \gamma_b v_b)\,\hat{\mathbf{x}} = 0. \tag{29}$$

Hence, there is no "hidden" momentum in this all-mechanical example.

As remarked in footnote 7 of [2], it appears that "hidden" mechanical momentum exists only in examples where (moving) "matter" interacts with a "field".

B Appendix: Magnetic-Current Loop in a Magnetic Field *(October, 2017)*

If magnetic charges (monopoles) existed, we could consider the case of a loop of magnetic current in a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{y}}$, with geometry as in the figure on p. 1.

If we label a magnetic charge by q_m (with rest mass m_q), the force on it in an external magnetic field $\mathbf{B}$ is $\mathbf{F} = q_m\mathbf{B}$.[10] Then, all the discussion in sec. 2 holds with the substitutions $q \to q_m$, $E \to B$, and $I \to I_m$ (for the current), except for the last equality in eq. (5). That is, the total momentum of the charges can now be written as,

$$\mathbf{P}_{\text{charges}} = (n_t l m_q \gamma_t v_t - n_b l m_q \gamma_b v_b)\,\hat{\mathbf{x}} = \frac{l I_m \Delta U_{\text{mech}}}{c^2 q}\,\hat{\mathbf{x}} = \frac{I l_m w B}{c^2}\,\hat{\mathbf{x}}. \tag{30}$$

Here, we note that a loop of circulating magnetic charge has an electric dipole moment $\mathbf{p}_m$ given (in Gaussian units) by,

$$\mathbf{p}_m = -\int \frac{\mathbf{r} \times \mathbf{J}_m}{2c}\,d\text{Vol} \to -\frac{I_m \mathbf{Area}}{c} = \frac{I_m l w}{c}\,\hat{\mathbf{z}}, \tag{31}$$

where $\hat{\mathbf{z}}$ is out of the paper.[11] Thus, the total mechanical momentum of the moving magnetic charges is,

$$\mathbf{P}_{\text{magnetic charges}} = -\frac{\mathbf{p}_m \times \mathbf{B}}{c}\,. \tag{32}$$

B.1 B due to Magnetic Charges

For eq. (32) to be the only mechanical momentum in the example, we suppose that the external magnetic field is due to magnetic charges at rest, $\mathbf{B} = \mathbf{B}_m$, rather than electric charges in motion. Then,

$$\mathbf{P}_{\text{hidden,mech}} = \mathbf{P}_{\text{magnetic charges}} = -\frac{\mathbf{p}_m \times \mathbf{B}_m}{c}\,. \tag{33}$$

[10] For a review of classical electrodynamics with both electric and magnetic charges, see, for example, [24].

[11] For a discussion of the first equality in eq. (31), see Appendix B of [25].

As for the case of a loop of circulating electric charges (with external electric field due to other electric charges), the velocity of the center of mass of the moving magnetic charges is zero, so the momentum (32) is a "hidden" mechanical momentum, to which the electromagnetic field momentum is equal and opposite, so that the total momentum is zero in this example.

B.2 B due to Electric Currents

B.2.1 The Electric and Magnetic Currents Occupy Different Volumes

If the external field $\mathbf{B}$ were due to moving electric charges (in an electrically neutral "wire"), rather than to static magnetic charges, then there would be some "hidden" mechanical momentum associated with the electric currents. Supposing these currents to form a magnetic dipole $\mathbf{m}_e$, and that the electric field $\mathbf{E}$ due to the magnetic currents of $\mathbf{p}_m$ is sufficiently uniform over the magnetic dipole $\mathbf{m}_e$, this additional "hidden" mechanical momentum would be given by eq. (9). Then, the total "hidden" mechanical momentum would be, combining eqs. (33) and (9),

$$\begin{aligned}\mathbf{P}_{\text{hidden,mech}} &= -\frac{\mathbf{p}_m \times \mathbf{B}_{\text{at }\mathbf{p}}}{c} + \frac{\mathbf{m}_e \times \mathbf{E}_{\text{at }\mathbf{m}}}{c} \\ &= -\frac{\mathbf{p}_m}{c} \times \frac{3(\mathbf{m}_e \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_e}{r^3} + \frac{\mathbf{m}_e}{c} \times \frac{3(\mathbf{p}_m \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}_m}{r^3} \\ &= \frac{2\mathbf{p}_m \times \mathbf{m}_e}{cr^3} + \frac{3\hat{\mathbf{r}} \times [\hat{\mathbf{r}} \times (\mathbf{p}_m \times \mathbf{m}_e)]}{cr^3} = \frac{3[(\mathbf{p}_m \times \mathbf{m}_e)\cdot\hat{\mathbf{r}}]\hat{\mathbf{r}} - (\mathbf{p}_m \times \mathbf{m}_e)}{cr^3}\,,\end{aligned} \tag{34}$$

where $\mathbf{r}$ is the distance between $\mathbf{p}_m$ and $\mathbf{m}_e$.

This result (with functional form like that of the electromagnetic fields of the dipoles $\mathbf{p}_m$ and $\mathbf{m}_e$) is listed in the table appended to the link to [11], which follows from eq. (66) of that paper,

$$\mathbf{P}_{\text{EM}} = -\frac{\mathbf{p}_m \times \boldsymbol{\nabla} V_m}{c} + \boldsymbol{\nabla}\left(\mathbf{p}_m \cdot \frac{\mathbf{A}_e}{c}\right), \tag{35}$$

where V_m is the scalar potential associated with magnetic charges and $\mathbf{A}_e$ is the vector potential associated with electric currents,[12] together with the observation that for the total momentum to be zero in a static example, it must include a "hidden" mechanical momentum that obeys,[13]

$$\mathbf{P}_{\text{hidden,mech}} = -\mathbf{P}_{\text{EM}}. \tag{36}$$

In the present example, we suppose that the magnetic currents of $\mathbf{p}_m$ are magnetically neutral, such that $V_m = 0$. Then, eq. (35) implies,

$$\mathbf{P}_{\text{EM}} = \boldsymbol{\nabla}\left(\mathbf{p}_m \cdot \frac{\mathbf{A}_e}{c}\right) = \mathbf{p}_m \times \frac{\mathbf{B}_e}{c} + (\mathbf{p}_m \cdot \boldsymbol{\nabla})\frac{\mathbf{A}_e}{c}\,. \tag{37}$$

[12] $\mathbf{B}_e = \boldsymbol{\nabla} \times \mathbf{A}_e$, and for steady magnetic currents as considered in [11], $\mathbf{B}_m = -\boldsymbol{\nabla} V_m$.
[13] If $\mathbf{A}_e = 0$ but $V_m \neq 0$, we recover eq. (33) from eqs. (35)-(36).

An inference from eqs. (36)-(37) is that field quantity $-(\mathbf{p}_m \cdot \boldsymbol{\nabla})\mathbf{A}_e/c$ equals the "hidden" mechanical momentum associated with the electric currents.[14]

For example, suppose the external magnetic field is due to a loop of electric current with magnetic moment $\mathbf{m}_e$, for which its magnetic field has vector potential $\mathbf{A}_e = \mathbf{m}_e \times \mathbf{r}/r^3$, where $\mathbf{r}$ is the vector from the magnetic dipole to the observer. Then,

$$\begin{aligned}(\mathbf{p}_m \cdot \boldsymbol{\nabla})\frac{\mathbf{A}_e}{c} &= (\mathbf{p}_m \cdot \boldsymbol{\nabla})\left(\mathbf{m}_e \times \frac{\mathbf{r}}{cr^3}\right) = \mathbf{m}_e \times (\mathbf{p}_m \cdot \boldsymbol{\nabla})\frac{\mathbf{r}}{cr^3} = -\mathbf{m}_e \times -\boldsymbol{\nabla}\left(\frac{\mathbf{p}_m \cdot \mathbf{r}}{cr^3}\right) \\ &= -\frac{\mathbf{m}_e \times \mathbf{E}_m}{c}, \end{aligned} \tag{38}$$

and,

$$\begin{aligned}\mathbf{P}_{\mathrm{EM}} &= \mathbf{p}_m \times \frac{\mathbf{B}_e}{c} - \frac{\mathbf{m}_e \times \mathbf{E}_m}{c} \quad \left(= \int \frac{\mathbf{E}_m \times \mathbf{B}_e}{4\pi c}\, d\mathrm{Vol}\right) \\ &= \boldsymbol{\nabla}\left(\mathbf{p}_m \cdot \frac{\mathbf{A}_e}{c}\right) = -\frac{3[(\mathbf{p}_m \times \mathbf{m}_e)\cdot \hat{\mathbf{r}}]\hat{\mathbf{r}} - (\mathbf{p}_m \times \mathbf{m}_e)}{cr^3}. \end{aligned} \tag{39}$$

Equating the "hidden" mechanical momentum to $-\mathbf{P}_{\mathrm{EM}}$ for this static example, we have,

$$\mathbf{P}_{\mathrm{hidden,mech}} = \frac{3[(\mathbf{p}_m \times \mathbf{m}_e)\cdot \hat{\mathbf{r}}]\hat{\mathbf{r}} - (\mathbf{p}_m \times \mathbf{m}_e)}{cr^3} = -\mathbf{p}_m \times \frac{\mathbf{B}_e}{c} + \frac{\mathbf{m}_e \times \mathbf{E}_m}{c} \tag{40}$$

as expected, and which confirms the consistency of the analysis of [11] for electric and magnetic dipoles that occupy different volumes.

If we suppose that no "particles" can exist that have both an electric dipole moment due to magnetic current and a magnetic dipole due to electric currents, then the results (34) and (39) (which are bilinear in $\mathbf{p}_m$ *and* $\mathbf{m}_e$*) imply that any collection of such particles, at rest, contains hidden mechanical momentum equal and opposite to the electromagnetic field momentum of the system. However, the electrodynamics of Maxwell was based on the assumption of continuous distributions of charge, and of dipole moments, with the implication that electric and magnetic dipole moments could occupy the same volume.*

In the rest of this Appendix we explore various distributions of electric and magnetic dipole moments, at rest, and mostly find that these obey the desirable result that the total momentum of the system (at rest) is zero. However, we seem to find, sec. B.2.2 below, that continuous distributions of electric dipoles due to magnetic currents and magnetic dipoles due to electric currents, at rest, would have nonzero total momentum.

B.2.1.1 Dipoles $\mathbf{p}_m$ and $\mathbf{m}_e$ Due to Currents on Spherical Shells[15]

We consider a spherical shell of radius a, centered at the origin, that supports surface electric current density,

$$\mathbf{K}_e = c\,\mathbf{M}_e \times \hat{\mathbf{r}} = \frac{3c\,\mathbf{m}_e}{4\pi a^3} \times \hat{\mathbf{r}}, \tag{41}$$

[14] Of course, "hidden" mechanical momentum is a "mechanical" quantity, and is physically distinct from a field-related quantity like $-(\mathbf{p}_m \cdot \boldsymbol{\nabla})\mathbf{A}_e/c$.

[15] This example was suggested (and analyzed) by D. Griffiths, Apr. 16, 2018; see also sec. 5 of [29]. It is a variant on an example by Romer [30]. Other variants of such examples are considered in sec. 2.2 of [25].

where the magnetic dipole moment $\mathbf{m}_e$ is the same as that due to a volume density $\mathbf{M}_e$ of magnetization (associated with electric currents) that is uniform inside a sphere of radius a. Also, we consider a spherical shell of radius b, centered at the origin, that supports surface magnetic current density,

$$\mathbf{K}_m = c\,\mathbf{P}_m \times \hat{\mathbf{r}} = \frac{3c\,\mathbf{p}_m}{4\pi b^3} \times \hat{\mathbf{r}}, \tag{42}$$

where the electric dipole moment $\mathbf{p}_m$ is the same as that due to a volume density $\mathbf{P}_m$ of electric polarization (associated with magnetic currents) that is uniform inside a sphere of radius b. The magnetic field is uniform inside the sphere of radius a, the electric field is uniform inside the sphere of radius b, and the fields outside these spheres are those due to dipole moments $\mathbf{m}_e = 4\pi a^3\mathbf{M}_e/3$ and $\mathbf{p}_m = 4\pi b^3\mathbf{P}_m/3$,

$$\mathbf{B} = \begin{cases} \frac{2\mathbf{m}_e}{a^3} & (r < a), \\ \frac{3(\mathbf{m}_e\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}_e}{r^3} & (r > a), \end{cases} \qquad \mathbf{E} = \begin{cases} \frac{2\mathbf{p}_m}{b^3} & (r < b), \\ \frac{3(\mathbf{p}_m\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}_m}{r^3} & (r > b). \end{cases} \tag{43}$$

We now suppose that $b < a$. Then, the surface magnetic currents on sphere b are inside sphere a where the magnetic field is uniform. Hence, we can use the argument that led to eq. (32), to infer that the "hidden" mechanical momentum associated with the magnetic currents is,

$$\mathbf{P}_{\mathrm{hidden,mmech},m} = -\frac{\mathbf{p}_m \times \mathbf{B}(r < a)}{c} = -\frac{2\mathbf{p}_m \times \mathbf{m}_e}{c\,a^3}\,. \tag{44}$$

However, the surface electric currents on sphere a experience the dipole electric field $3(\mathbf{p}_m \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_m/a^3$, which varies with position on the shell $r = a$. So we cannot immediately use the argument of sec. 2 above, but must note that the change ΔU_{mech} in the mechanical energy of an electric charge q as it moves between two points on its current loop is $q\Delta V$, rather than that of eq. (3), where V is the electric scalar potential associated with the electric field due to the surface magnetic currents $\mathbf{K}_m$ on sphere b. On sphere a, which is outside sphere b, the electric scalar potential has the dipole form $V = \mathbf{p}_m \cdot \hat{\mathbf{r}}/a^2$.

We consider a rectangular coordinate system (x, y, z) with $\hat{\mathbf{z}}$ parallel to $\mathbf{m}_e$, and $\mathbf{p}_m$ in the x-z plane at angle θ_p to the z-axis. Then, in a corresponding spherical coordinate system (r, θ, ϕ), the electric scalar potential on the surface of sphere a is,

$$\begin{aligned} V(a,\theta,\phi) &= \frac{p_m}{a^2}(\hat{\mathbf{x}}\sin\theta_p + \hat{\mathbf{z}}\cos\theta_p)\cdot(\hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta) \\ &= \frac{p_m}{a^2}(\sin\theta_p\sin\theta\cos\phi + \cos\theta_p\cos\theta), \end{aligned} \tag{45}$$

Equation (3) now has the form,

$$\Delta U_{\mathrm{mech}}(\theta,\phi) = [\gamma(\theta,\phi) - \gamma(\theta,0)]\,m\,c^2 = q[V(a,\theta,\phi) - V(a,\theta,0)] = \frac{q\,p_m}{a^2}\sin\theta_p\sin\theta(\cos\phi - 1). \tag{46}$$

For a loop on sphere a of extent $d\theta$ at angle θ, the (azimuthal) electric current is,

$$d\mathbf{I}(\theta) = \mathbf{K}_e(\theta)\,a\,d\theta = \boldsymbol{\phi}\,\frac{3c\,m_e\sin\theta}{4\pi a^2}\,d\theta = q\,dn(\theta,\phi)\,\mathbf{v}(\theta,\phi), \tag{47}$$

where the moving charges at (a, θ, ϕ) have velocity $\mathbf{v}$ and density (per unit length around the loop) dn. The "hidden" mechanical momentum of the moving electric charges in the loop (of circumference $2\pi a \sin\theta$) is,

$$
\begin{aligned}
d\mathbf{P}_{\text{hidden,mech},e}(\theta) &= \int_0^{2\pi} a \sin\theta \, d\phi \; \gamma(\theta,\phi)\, m\, \mathbf{v}(\theta,\phi)\, dn(\theta,\phi) \\
&= \int_0^{2\pi} a \sin\theta \, d\phi \left[\gamma(\theta,0) + \frac{q\, p_m}{a^2\, m\, c^2} \sin\theta_p \sin\theta (\cos\phi - 1)\right] m\,(-\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi)\, \frac{3c\, m_e \sin\theta}{4\pi a^2\, q}\, d\theta \\
&= \hat{\mathbf{y}}\, \frac{3 p_m m_e}{4 a^3 c} \sin\theta_p \sin^3\theta \, d\theta.
\end{aligned}
\tag{48}
$$

The total "hidden" mechanical momentum of the moving electric charges is,

$$
\mathbf{P}_{\text{hidden},e} = \int d\mathbf{P}_{\text{hidden},e}(\theta) = \int_0^{\pi} \hat{\mathbf{y}}\, \frac{3 p_m m_e}{4 a^3 c} \sin\theta_p \sin^3\theta \, d\theta = \hat{\mathbf{y}}\, \frac{p_m m_e}{c\, a^3} \sin\theta_p = \frac{\mathbf{p}_m \times \mathbf{m}_e}{c\, a^3}\,. \tag{49}
$$

The total "hidden" mechanical momentum of both the electric and magnetic charges is, recalling eq. (44),

$$
\mathbf{P}_{\text{hidden,mech}} = \mathbf{P}_{\text{hidden,mech},e} + \mathbf{P}_{\text{hidden,mech},m} = -\frac{\mathbf{p}_m \times \mathbf{m}_e}{c\, a^3}\,. \tag{50}
$$

This "hidden" momentum is independent of radius b, so the result (49) also holds when $b = a$.

The electromagnetic field momentum for a sphere of radius a containing uniform volume densities $\mathbf{M}_e$ and $\mathbf{P}_m$ has been evaluated in Appendix A.1.3 of [25], and found to be,

$$
\mathbf{P}_{\text{EM}} = \frac{\mathbf{p}_m \times \mathbf{m}_e}{c\, a^3}\,. \tag{51}
$$

The electromagnetic fields of the present example, due to surface currents (41)-(42), are the same as those due to volume densities $\mathbf{M}_e$ and $\mathbf{P}_m$, so the electromagnetic field momentum in the present example is also given by eq. (50). Hence, for the present example with surface currents,

$$
\mathbf{P}_{\text{hidden,mech}} + \mathbf{P}_{\text{EM}} = 0, \tag{52}
$$

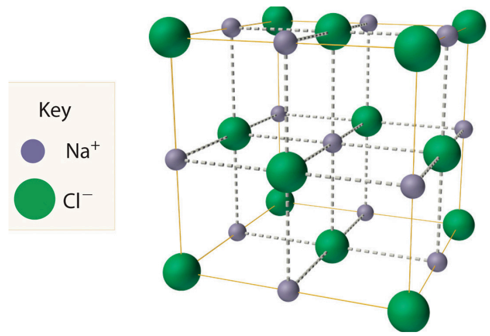
as is agreeable for a system "at rest".

However, it is not evident that the "hidden" mechanical momentum is the same in the two cases (surface currents, volume densities), although the electromagnetic fields are the same. The above calculation of the "hidden" mechanical momentum of the surface currents used eq. (46) for the change in mechanical energy of electric charges as they flowed around loops on the surface of sphere a. While the volume density $\mathbf{M}_e$ can be associated with an "equivalent" surface current density $\mathbf{K}_e$ according to eq. (41), the physical charges of density $\mathbf{M}_e$ do not flow in the large loops considered in eqs. (47)-(48), so these equations do not necessarily represent the "hidden" mechanical momentum of the volume densities.

We return to this issue in Appendix B.2.2 below.

B.2.1.2 Dipoles $\mathbf{p}_m$ and $\mathbf{m}_e$ on a Cubic Lattice[16]

As a special case of when the dipoles $\mathbf{p}_m$ and $\mathbf{m}_e$ occupy different volumes, suppose they are arrayed at alternating sites on a cubic lattice, as are Na and Cl ions in salt, and the lattice fills a sphere of radius a which is much larger than the lattice spacing.



The total "hidden" mechanical momentum can be obtained by summing either form of eq. (40) over all pairs of dipoles $(\mathbf{p}_m, \mathbf{m}_e)$.

While it is tempting to suppose that the arrays of dipoles correspond to continuous macroscopic polarization densities $\mathbf{P}_m$ and $\mathbf{M}_e$, such that the double summation over the dipoles $(\mathbf{p}_m, \mathbf{m}_e)$ is equivalent to a double integral, it turns out that the double volume integral of the first form of eq. (40) over a sphere vanishes, as discussed below in sec. B.2.2.2.

So, we instead sum the second form of eq. (40), supposing that the fields inside the sphere of radius a due to the current-based dipoles are uniform, and given by,

$$\mathbf{E}_m(r < a) = \frac{2\mathbf{p}_{m,\text{total}}}{a^3}\,, \qquad \mathbf{B}_e(r < a) = \frac{2\mathbf{m}_{e,\text{total}}}{a^3}\,, \tag{53}$$

to find,

$$\begin{aligned}\mathbf{P}_{\text{hidden,mech}} &= -\sum_p \mathbf{p}_m \times \frac{\mathbf{B}_e}{c} + \sum_m \frac{\mathbf{m}_e \times \mathbf{E}_m}{c} \\ &= -\mathbf{p}_{m,\text{total}} \times \frac{2\mathbf{m}_{e,\text{total}}}{a^3 c} + \mathbf{m}_{e,\text{total}} \times \frac{2\mathbf{p}_{m,\text{total}}}{a^3 c} = -4\frac{\mathbf{p}_{m,\text{total}} \times \mathbf{m}_{e,\text{total}}}{a^3 c}\,. \end{aligned} \tag{54}$$

This result is disconcerting in that the total field momentum for this example, assuming that the fields inside the sphere are given by eq. (53) (and that the fields outside the sphere are the fields due to total dipole moments $\mathbf{p}_{m,\text{total}}$ and $\mathbf{m}_{e,\text{total}}$), is,

$$\mathbf{P}_{\text{EM}} = \frac{\mathbf{p}_{m,\text{total}} \times \mathbf{m}_{e,\text{total}}}{a^3 c}\,, \tag{55}$$

as deduced, for example, in Appendix A.1.3 of [25].

However, the assumption of uniform fields for dipoles on a cubic lattice is not warranted, but rather the fields of eq. (53) should be regarded as the macroscopic average fields, while the microscopic fields vary considerably across a unit cell of the lattice. In particular the electric field due to dipoles $\mathbf{p}_m$ at the centers of dipoles $\mathbf{m}_e$ is zero, and similarly the magnetic

[16] This example was suggested by P. Saldanha, Oct. 30, 2017.

field due to dipoles $\mathbf{m}_e$ at the centers of dipoles $\mathbf{p}_m$ is zero, as discussed in sec. 2.6 of [26].[17] That is, the fields are not uniform over the various dipoles, the average values of the fields on the two types of dipoles are less that the macroscopic averages given in eq. (53), and the estimate (54) of the "hidden" mechanical momentum of the current-based dipoles is too large in magnitude. If the effective average fields on the dipoles are 1/4 of the macroscopic averages (53), then we would have an accounting that the total momentum is zero, as expected for a system "at rest".

We continue this theme in the next section.

B.2.2 The Electric and Magnetic Currents Occupy the Same Volume

Another special case is that the two dipoles $\mathbf{p}_m$ and $\mathbf{m}_e$ occupy the same volume, instead of being external to one another as assumed is sec. B.2.1.

As an example, we consider the case of dipoles due to uniform densities $\mathbf{P}_m$ and $\mathbf{M}_e$ of electric polarization due to magnetic currents and magnetic polarization due to electric currents, all inside a sphere of radius a.[18]

Again, we might expect that the total "hidden" mechanical momentum would be, combining eqs. (33) and (9),

$$\mathbf{P}_{\text{hidden,mech}} = -\frac{\mathbf{p}_m \times \mathbf{B}_{\text{at }\mathbf{p}}}{c} + \frac{\mathbf{m}_e \times \mathbf{E}_{\text{at }\mathbf{m}}}{c} \tag{56}$$

Then, since the internal fields of these current-based dipoles are,

$$\mathbf{E}_m(r < a) = \frac{2\mathbf{p}_m}{a^3}, \qquad \mathbf{B}_e(r < a) = \frac{2\mathbf{m}_e}{a^3}, \tag{57}$$

we would have,

$$\mathbf{P}_{\text{hidden,mech}} = -4\frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3 c}. \tag{58}$$

We recall that in static examples like the present, the "hidden" mechanical momentum should be equal and opposite to the electromagnetic field momentum (so that the total momentum of the system is zero). A computation of the field momentum for the present example is given in Appendix A.1.3 of [25], with the result,

$$\mathbf{P}_{\text{EM}} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\text{Vol} = \frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3 c}, \tag{59}$$

so we infer that the "hidden" mechanical momentum is,

$$\mathbf{P}_{\text{hidden,mech}} = -\mathbf{P}_{\text{EM}} = -\frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3 c}, \tag{60}$$

---

[17] A closely related result is that if one of the polarized entities were removed from the lattice, the field due to the remaining polarized entities would be zero at the center of the vacant site, as deduced by Lorentz on p. 306 (Note 55) of [27]. See also sec. 4.5 of [28].

[18] This example is an extension to the case of magnetic charges and currents of that discussed by Romer in [30]. See also Appendix A.1 of [25].

which differs from the result of eq. (58) by a factor of 4.

B.2.2.1 Digression: Field Momentum According to Third Form of Eq. (37)

As a check, we compute the field mechanical momentum according to the relation (37), noting that $V_m = 0$ in the present example,

$$\mathbf{P}_{\mathrm{EM}} = \boldsymbol{\nabla}\left(\mathbf{p}_m \cdot \frac{\mathbf{A}_e}{c}\right) = \mathbf{p}_m \times \frac{\mathbf{B}_e}{c} + (\mathbf{p}_m \cdot \boldsymbol{\nabla})\frac{\mathbf{A}_e}{c}. \tag{61}$$

The vector potential of the uniform magnetic field inside the sphere has only an azimuthal component, $A_{e,\phi}(r < a) = rB_e/2$, with respect to the direction of $\mathbf{m}_e$ and $\mathbf{B}_e$. A delicacy in the argument is to suppose for the time being that $\mathbf{p}_m$ is just an element of the polarization density $\mathbf{P}_m$, at distance r from the center of the sphere. Then, at the location of the element $\mathbf{p}_m$, we have,

$$(\mathbf{p}_m \cdot \boldsymbol{\nabla})\mathbf{A}_e = (\mathbf{p}_m \cdot \hat{\mathbf{r}})\frac{B_e}{2}\,\hat{\boldsymbol{\phi}}, \tag{62}$$

since $\mathbf{A}_e$ has only a ϕ component, which depends only on r. This result is independent of r, so we can sum up the contributions from all elements $\mathbf{p}_m$, and again consider the symbol $\mathbf{p}_m$ to represent the total electric dipole moment. Now,

$$(\mathbf{p}_m \cdot \hat{\mathbf{r}})\frac{B_e}{2}\,\hat{\boldsymbol{\phi}} = \frac{\mathbf{B}_e \times \mathbf{p}_m}{2} = -\frac{\mathbf{p}_m \times \mathbf{B}_e}{2}, \tag{63}$$

so altogether, recalling eq. (57),

$$\mathbf{P}_{\mathrm{EM}} = \boldsymbol{\nabla}\left(\mathbf{p}_m \cdot \frac{\mathbf{A}_e}{c}\right) = \frac{\mathbf{p}_m \times \mathbf{B}_e}{c} - \frac{\mathbf{p}_m \times \mathbf{B}_e}{2c} = \frac{\mathbf{p}_m \times \mathbf{B}_e}{2c} = \frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3 c}, \tag{64}$$

in agreement with eq. (59).

B.2.2.2 Digression: Field Momentum According to the Last Form of Eq. (39)

We also give an analysis (based on comments by D, Griffiths and V. Hnizdo) that will show a connection to the last form of eq. (39), which is also a re-expression of $\mathbf{P}_{\mathrm{EM}}$.

Again starting from eq. (37), the field mechanical momentum of the sphere of radius a can be expressed as a volume integral over elements of electric polarization, $d\mathbf{p}_m = \mathbf{P}_m\, d\mathrm{Vol}_p$,

$$\begin{aligned}
\mathbf{P}_{\mathrm{EM}} &= \boldsymbol{\nabla}\left(\mathbf{p}_m \cdot \frac{\mathbf{A}_e}{c}\right) = \int d\mathrm{Vol}_p\, \boldsymbol{\nabla}_p\left(\frac{\mathbf{P}_m \cdot \mathbf{A}_e}{c}\right) \\
&= \frac{1}{c}\int d\mathrm{Vol}_p\, \{\mathbf{P}_m \times [\boldsymbol{\nabla}_p \times \mathbf{A}_e(\mathbf{r}_p)] + [\mathbf{P}_m \cdot \boldsymbol{\nabla}_p]\mathbf{A}_{\mathbf{e}}(\mathbf{r}_p)\},
\end{aligned} \tag{65}$$

where $\boldsymbol{\nabla}_p = \partial/\partial\mathbf{r}_p$. The vector potential $\mathbf{A}_e$ is due to the magnetic polarization density $\mathbf{M}_e$, and can be expressed as a volume integral over elements $d\mathbf{m}_e = \mathbf{M}_e\, d\mathrm{Vol}_m$,

$$\mathbf{A}_{\mathbf{e}}(\mathbf{r}_p) = \int d\mathrm{Vol}_m\, \mathbf{M}_e \times \frac{\mathbf{r}_p - \mathbf{r}_m}{c\,|\mathbf{r}_p - \mathbf{r}_m|^3}. \tag{66}$$

Using this in eq. (65), we have, with $\mathbf{B}_e = \boldsymbol{\nabla} \times \mathbf{A}_e$, and recalling eq. (38),

$$\mathbf{P}_{\rm EM} = \frac{1}{c} \int d{\rm Vol}_p \, \mathbf{P}_m \times \mathbf{B}_e(\mathbf{r}_p) + \frac{1}{c} \int d{\rm Vol}_p \int d{\rm Vol}_m \, \mathbf{M}_e \times \boldsymbol{\nabla}_p \left(\mathbf{P}_m \cdot \frac{\mathbf{r}_p - \mathbf{r}_m}{|\mathbf{r}_p - \mathbf{r}_m|^3} \right). \quad (67)$$

If we ignored the special cases when the volume elements $d{\rm Vol}_p$ and $d{\rm Vol}_m$ are the same, the arguments of sec. B.2.1 above would lead to the form,

$$\begin{aligned} \mathbf{P}_{\rm EM,1} &= \frac{1}{c} \int d{\rm Vol}_p \int d{\rm Vol}_m \frac{3[(\mathbf{P}_m \times \mathbf{M}_e) \cdot \hat{\mathbf{r}}]\hat{\mathbf{r}} - (\mathbf{P}_m \times \mathbf{M}_e)}{r^3} \\ &= \frac{2\,|\mathbf{P}_m \times \mathbf{M}_e|}{c} \sqrt{\frac{4\pi}{5}} \int d{\rm Vol}_p \int d{\rm Vol}_m \frac{Y_{20}(\theta)}{r^3} \,, \end{aligned} \quad (68)$$

where $\mathbf{r} = \mathbf{r}_p - \mathbf{r_m}$, Y_{20} is a spherical harmonic, and angle θ is with respect to the direction of $\mathbf{P}_m \times \mathbf{M}_e$. This integral is zero for the present example where $\mathbf{P}_m$ and $\mathbf{M}_e$ occupy the same volume!

This possibly surprising result is a special case of an analysis of related double-volume integrals given in [32], using a so-called Fourier-Bessel expansion. Our eq. (68) has the form of eq. (1) of [32], but with the vector $\mathbf{r}$ in that equation set to zero. Then, the main result, eq. (21) of [32] (which involves the product of 8 series expansions, 3 multipole analyses, and 4 Wigner 3-j symbols [33]), simplifies in that the spherical-Bessel-function factor $j_\lambda(q_n^{(l)} r)$ reduces to $j_\lambda(0)$, which is zero unless index $\lambda = 0$.

The next steps involve consideration of the Wigner 3-j symbols,

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_2 \end{pmatrix}, \quad (69)$$

for which in a classical context the j_i are non-negative integers that obey the "triangle relations" (familiar from the quantum addition of angular momenta, as in $\mathbf{j}_1 + \mathbf{j}_2 = \mathbf{j}_3$),

$$|j_1 - j_2| \le j_3 \le j_1 + j_2, \ \ etc., \quad (70)$$

and the integers k_i obey $k_1 + k_2 + k_3 = 0$.

The first and second 3-j symbols of eq. (21) of [32] have upper indices $j_1, j_2, j_3 = l_1, l_2, \lambda'$, which must satisfy the triangle inequality,

$$|l_1 - l_2| \le \lambda' \le l_1 + l_2, \quad (71)$$

for the 3-j symbols to be nonzero. The indices l_1 and l_2 are determined by the multipole (spherical-harmonic) expansion of the scalar "densities" $\rho_1(\mathbf{r}_1)$ and $\rho_2(\mathbf{r}_2)$ that characterize the volumes of integration over $\mathbf{r}_1$ and $\mathbf{r}_2$. For spherically symmetric "densities", as in the present example where $\rho_1(r_1 < 1) = 1 = \rho_2(r_2 < a)$ and are zero otherwise, $l_1 = 0 = l_2$, so the triangle inequality (71) takes the form,

$$0 \le \lambda' \le 0, \quad (72)$$

which can only be satisfied for $\lambda' = 0$.

The third and fourth 3-j symbols in eq. (21) of [32] have upper indices $j_1, j_2, j_3 = \lambda', l, \lambda$, where the index $l = 2$ is that of the spherical harmonic $Y_{lm} = Y_{20}$ which is a factor in the integrand of eq. (68), and $\lambda = 0$ as previously argued. For these 3-j symbols to be nonzero, the triangle inequality,

$$|\lambda' - l| \le \lambda \le \lambda' + l, \qquad i.e., \qquad |\lambda' - 2| \le 0 \le \lambda' + 2, \tag{73}$$

must be satisfied, which is possible only for $\lambda' = 2$

In the present example, at least two of the four 3-j symbols of eq. (21) of [32] vanish for all indices λ', according to the "triangle rule".

As such, eq. (21) of [32], and the integral of eq. (68) above, vanish for the present example.[19]

Hence, the "hidden" mechanical momentum can be computed via eq. (67) with only those terms related to the behavior when the electric- and magnetic-dipole elements $d\mathbf{p}_m$ and $d\mathbf{m}_e$ occupy the same volume.

Since both polarization densities $\mathbf{P}_m$ and $\mathbf{M}_e$ are current based, the electric and magnetic fields of elements $d\mathbf{p}_m$ and $d\mathbf{m}_e$ centered on $\mathbf{r}_p$ and $\mathbf{r}_m$ can be written as,

$$\mathbf{E}_m(\mathbf{r}) = \frac{3[d\mathbf{p}_m \cdot (\mathbf{r} - \mathbf{r}_p)](\mathbf{r} - \mathbf{r}_p)}{|\mathbf{r} - \mathbf{r}_p|^5} - \frac{d\mathbf{p}_m}{|\mathbf{r} - \mathbf{r}_p|^3} + \frac{8\pi\, d\mathbf{p}_m}{3}\delta^3(\mathbf{r} - \mathbf{r}_p), \tag{74}$$

$$\mathbf{B}_e(\mathbf{r}) = \frac{3[d\mathbf{m}_e \cdot (\mathbf{r} - \mathbf{r}_m)](\mathbf{r} - \mathbf{r}_m)}{|\mathbf{r} - \mathbf{r}_m|^5} - \frac{d\mathbf{m}_e}{|\mathbf{r} - \mathbf{r}_m|^3} + \frac{8\pi\, d\mathbf{m}_e}{3}\delta^3(\mathbf{r} - \mathbf{r}_m). \tag{75}$$

The difference between equations (67) and (68) is that the internal fields of elements $d\mathbf{p}_m$ and $d\mathbf{m}_e$, represented by the delta-functions terms in eqs. (74)-(75), have not been included in eq. (68). We infer that the first term of eq. (67) has an additional contribution to the "hidden" mechanical momentum of,

$$\mathbf{P}_{\mathrm{EM},2} = \frac{1}{c}\int d\mathrm{Vol}_p\, \mathbf{P}_m \times \int d\mathrm{Vol}_m\, \frac{8\pi\, \mathbf{M}_e}{3}\delta^3(\mathbf{r}_p - \mathbf{r}_m) = 2\frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3 c}, \tag{76}$$

recalling that $\mathbf{p}_m = 4\pi a^3 \mathbf{P}_m/3$ and $\mathbf{m}_e = 4\pi a^3 \mathbf{M}_e/3$.

The second term of eq. (67) also has an additional contribution, associated with the field $-\boldsymbol{\nabla}_p[\mathbf{P}_m \cdot (\mathbf{r}_p - \mathbf{r}_m)/\left|\mathbf{r}_p - \mathbf{r}_m\right|^3]$ at points $\mathbf{r}_p = \mathbf{r}_m$. Although the electric polarization density $\mathbf{P}_m$ is current based, this term has the form of the field of an electric dipole formed from a pair of electric charges, for which the delta-function term is $-(4\pi/3)\mathbf{P}_m\delta^3(\mathbf{r} - \mathbf{r}_m)$. Hence, the additional contribution to the "hidden" mechanical momentum associated with the second term in eq. (67) is,

$$\mathbf{P}_{\mathrm{EM},3} = -\frac{1}{c}\int d\mathrm{Vol}_p \int d\mathrm{Vol}_m\, \mathbf{M}_e \times \frac{4\pi}{3}\mathbf{P}_m\delta^3(\mathbf{r} - \mathbf{r}_m) = -\frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3 c}. \tag{77}$$

[19] For volumes of integration that are not spherically symmetric, indices l_1 and l_2 can be nonzero, and the integral of eq. (68) need not vanish. For example, if $l_1 = 1 = l_2$, then both triangle inequalities (71) and (73) are satisfied for $\lambda' = 2$ (and $\lambda = 0$).

The total electromagnetic field momentum in the present example is,

$$\mathbf{P}_{\rm EM} = \mathbf{P}_{\rm EM,1} + \mathbf{P}_{\rm EM,2} + \mathbf{P}_{\rm EM,3} = \frac{\mathbf{p}_m \times \mathbf{m}_e}{a^3 c}\,, \tag{78}$$

which is also in agreement with eq. (59).

According to eq. (78), the field momentum can be attributed entirely to the delta-function terms associated with elements $d\mathbf{p}_m$ and $d\mathbf{m}_e$ that occupy the same volume.[20]

B.2.2.3 Integrals of "Hidden"-Momentum Density for $\mathbf{P}_m$ and $\mathbf{M}_e$

As remarked in [34], for cases like the present example that contain volume densities of electric and magnetic polarization, it is more correct to write eq. (56) as,

$$\mathbf{P}_{\rm hidden} = -\int \frac{\mathbf{P}_m \times \mathbf{B}}{c}\, d{\rm Vol}_p + \int \frac{\mathbf{M}_e \times \mathbf{E}}{c}\, d{\rm Vol}_m = -\frac{\mathbf{P}_m}{c} \times \int \mathbf{B}\, d{\rm Vol}_p + \frac{\mathbf{M}_e}{c} \times \int \mathbf{E}\, d{\rm Vol}_m . \tag{79}$$

The electric and magnetic fields can be regarded as the integrals of contributions from dipole elements throughout the sphere of radius a of the present example,

$$\mathbf{E}(\mathbf{r}_m) = \int d{\rm Vol}_p\; d\mathbf{E}(\mathbf{r}_p), \qquad \mathbf{B}(\mathbf{r}_p) = \int d{\rm Vol}_m\; d\mathbf{B}(\mathbf{r}_m). \tag{80}$$

where the fields due to the various current-based dipole elements $d\mathbf{p}_m$ and $d\mathbf{m}_e$ can be written as,

$$d\mathbf{E}(\mathbf{r}) = \frac{3[d\mathbf{p}_m \cdot (\mathbf{r} - \mathbf{r}_p)](\mathbf{r} - \mathbf{r}_p)}{|\mathbf{r} - \mathbf{r}_p|^5} - \frac{d\mathbf{p}_m}{|\mathbf{r} - \mathbf{r}_p|^3} + \frac{8\pi\, d\mathbf{p}_m}{3}\delta^3(\mathbf{r} - \mathbf{r}_p), \tag{81}$$

$$d\mathbf{B}(\mathbf{r}) = \frac{3[d\mathbf{m}_e \cdot (\mathbf{r} - \mathbf{r}_m)](\mathbf{r} - \mathbf{r}_m)}{|\mathbf{r} - \mathbf{r}_m|^5} - \frac{d\mathbf{m}_e}{|\mathbf{r} - \mathbf{r}_m|^3} + \frac{8\pi\, d\mathbf{m}_e}{3}\delta^3(\mathbf{r} - \mathbf{r}_m). \tag{82}$$

Then,

$$\begin{aligned} &\int \mathbf{B}\, d{\rm Vol}_p \qquad (83) \\ &= \int\int \left(\frac{3[d\mathbf{m}_e \cdot (\mathbf{r}_p - \mathbf{r}_m)](\mathbf{r}_p - \mathbf{r}_m)}{|\mathbf{r}_p - \mathbf{r}_m|^5} - \frac{d\mathbf{m}_e}{|\mathbf{r}_p - \mathbf{r}_m|^3} + \frac{8\pi\, d\mathbf{m}_e}{3}\delta^3(\mathbf{r}_p - \mathbf{r}_m) \right) d{\rm Vol}_p\, d{\rm Vol}_m \\ &= \frac{8\pi \mathbf{m}_e}{3} + \int\int \left(\frac{3[d\mathbf{m}_e \cdot (\mathbf{r}_p - \mathbf{r}_m)](\mathbf{r}_p - \mathbf{r}_m)}{|\mathbf{r}_p - \mathbf{r}_m|^5} - \frac{d\mathbf{m}_e}{|\mathbf{r}_p - \mathbf{r}_m|^3} \right) d{\rm Vol}_p\, d{\rm Vol}_m = \frac{8\pi \mathbf{m}_e}{3}\,, \end{aligned}$$

where we recall the lengthy argument in the preceding digression that showed how the integral of eq. (68) vanishes. Similarly, $\int \mathbf{E}\, d{\rm Vol}_m = 8\pi \mathbf{p}_m/3$, and eq. (79) becomes,

$$\mathbf{P}_{\rm hidden} = -\mathbf{P}_m \times \frac{8\pi \mathbf{m}_e}{3c} + \mathbf{M}_e \times \frac{8\pi \mathbf{p}_m}{3c} = -2\mathbf{p}_m \times \frac{\mathbf{m}_e}{a^3 c} + 2\mathbf{m}_e \times \frac{\mathbf{p}_m}{a^3 c} = -4\mathbf{p}_m \times \frac{\mathbf{m}_e}{a^3 c}\,, \tag{84}$$

[20] The result (78) is very satisfactory, but it is perhaps surprising that it was obtained using the delta-function term for a magnetic field based on (magnetic) currents and for an electric field based on (electric) charges, while in the present example all polarizations are due to currents. We note that we could have based the derivation on the (dual) form $\mathbf{P}_{\rm EM} = \int d{\rm Vol}_m\, \boldsymbol{\nabla}(\mathbf{M}_e \cdot \mathbf{A}_m/c)$, which follows from eq. (70) of [11]. In this case, $\mathbf{P}_{\rm EM,2}$ would use the delta-function term for an electric field based on (electric) currents while and $\mathbf{P}_{\rm EM,3}$ would use the delta-function term for a magnetic field based on (magnetic) charges.

in agreement with eq. (58), but not with eq. (60).

B.2.2.4 Integrals of "Hidden"-Momentum Density for $(\mathbf{P}_e, \mathbf{M}_e)$ and $(\mathbf{P}_m, \mathbf{M}_m)$

We can also apply the argument of this subsection to the case of a sphere with uniform densities of polarizations $\mathbf{P}_e$ and $\mathbf{M}_e$ (or of $\mathbf{P}_m$ and $\mathbf{M}_m$), for which the field momentum has been computed in Appendix A.1.1 of [25] to be $\mathbf{P}_{\rm EM} = \mathbf{m}_e \times \mathbf{p}_e/a^3 c = -\mathbf{m}_e \times \mathbf{E}/c$ (or $\mathbf{m}_e \times \mathbf{p}_e/a^3 c = \mathbf{p}_m \times \mathbf{B}/c$). According to eq. (9), we expect the "hidden" mechanical momentum of the electric current to be,

$$\mathbf{P}_{\rm hidden,mech} = \int \frac{\mathbf{M}_e \times \mathbf{E}}{c}\, d{\rm Vol}_m = \frac{\mathbf{M}_e}{c} \times \int \mathbf{E}\, d{\rm Vol}_m. \tag{85}$$

In this example, the electric field on the current-based density $\mathbf{M}_e$ is due to the density $\mathbf{P}_e$ which is based on electric charges, so,

$$d\mathbf{E}(\mathbf{r}) \;=\; \frac{3[d\mathbf{p}_e \cdot (\mathbf{r} - \mathbf{r}_p)](\mathbf{r} - \mathbf{r}_p)}{|\mathbf{r} - \mathbf{r}_p|^5} - \frac{d\mathbf{p}_e}{|\mathbf{r} - \mathbf{r}_p|^3} - \frac{4\pi\, d\mathbf{p}_e}{3}\delta^3(\mathbf{r} - \mathbf{r}_p), \tag{86}$$

$$\begin{aligned}
&\int \mathbf{E}\, d{\rm Vol}_m \qquad (87)\\
&= \int\!\!\int \left(\frac{3[d\mathbf{p}_e \cdot (\mathbf{r}_p - \mathbf{r}_m)](\mathbf{r}_p - \mathbf{r}_m)}{|\mathbf{r}_p - \mathbf{r}_m|^5} - \frac{d\mathbf{p}_e}{|\mathbf{r}_p - \mathbf{r}_m|^3} - \frac{4\pi\, d\mathbf{p}_e}{3}\delta^3(\mathbf{r}_p - \mathbf{r}_m)\right) d{\rm Vol}_p\, d{\rm Vol}_m\\
&= -\frac{4\pi \mathbf{p}_e}{3} + \int\!\!\int \left(\frac{3[d\mathbf{p}_e \cdot (\mathbf{r}_p - \mathbf{r}_m)](\mathbf{r}_p - \mathbf{r}_m)}{|\mathbf{r}_p - \mathbf{r}_m|^5} - \frac{d\mathbf{p}_e}{|\mathbf{r}_p - \mathbf{r}_m|^3}\right) d{\rm Vol}_p\, d{\rm Vol}_m = -\frac{4\pi \mathbf{p}_e}{3},
\end{aligned}$$

where we recall the lengthy argument in the preceding digression that showed how the integral of eq. (68) vanishes. The "hidden" mechanical momentum eq. (85) becomes,

$$\mathbf{P}_{\rm hidden,mech} = \mathbf{M}_e \times -\frac{4\pi \mathbf{p}_e}{3c} = -\mathbf{m}_e \times \frac{\mathbf{p}_e}{a^3 c} = -\mathbf{P}_{\rm EM}, \tag{88}$$

as expected.

For the case of uniform polarization densities $\mathbf{P}_m$ and $\mathbf{M}_m$,

$$\begin{aligned}
\mathbf{P}_{\rm hidden,mech} &= -\int \frac{\mathbf{P}_m \times \mathbf{B}}{c}\, d{\rm Vol}_p = -\frac{\mathbf{P}_m}{c} \times \int \mathbf{B}\, d{\rm Vol}_p = -\frac{\mathbf{P}_m}{c} \times -\frac{4\pi \mathbf{m}_m}{3} = \frac{\mathbf{p}_m \times \mathbf{m}_m}{a^3 c}\\
&= -\mathbf{P}_{\rm EM}.
\end{aligned} \tag{89}$$

B.2.2.5 Computations Using Equivalent Currents

This section is based on comments by Vladimir Hnizdo and Pablo Saldanha.

The polarized spheres can be thought of as supporting equivalent current densities $\mathbf{J} = c\boldsymbol{\nabla} \times \mathbf{M}$ and $c\boldsymbol{\nabla} \times \mathbf{P}$, which are zero for uniform polarization, as well as surface current densities $\mathbf{K} = c\,\mathbf{M} \times \hat{\mathbf{r}}$ and $c\,\mathbf{P} \times \hat{\mathbf{r}}$.

For the case of polarization densities $\mathbf{P}_e$ and $\mathbf{M}_e$, the electric field can be related to a scalar potential, $\mathbf{E}_e = -\boldsymbol{\nabla} V_e$ where $V_e = \mathbf{p}_e \cdot \mathbf{r}/R^3(r)$ with $R(r > a) = r$ and $R(r < a) = a$, and $\mathbf{p}_e = 4\pi a^2 \mathbf{P}_e/3$ = total electric dipole moment of the sphere, Then, the "hidden" mechanical momentum associated with the electric currents of $\mathbf{M}_e$ can be written as,[21]

$$
\begin{aligned}
\mathbf{P}_{\rm hidden,mech} &= \int \frac{\mathbf{M}_e \times \mathbf{E_e}}{c}\, d\mathrm{Vol}_p = -\int \frac{\mathbf{M}_e \times \boldsymbol{\nabla} V_e}{c}\, d\mathrm{Vol}_p \\
&= -\int \frac{V\mathbf{M}_e \times \hat{\mathbf{r}}}{c}\, d\mathrm{Area}_p - \int \frac{V\boldsymbol{\nabla} \times \mathbf{M}_e}{c}\, d\mathrm{Vol}_p = -\int \frac{V\mathbf{K}_e}{c^2}\, d\mathrm{Area}_p - \int \frac{V\mathbf{J}_e}{c^2}\, d\mathrm{Vol}_p \\
&= -\frac{1}{a^2 c}\int (\mathbf{p}_e \cdot \hat{\mathbf{r}})\, \mathbf{M}_e \times \hat{\mathbf{r}}\, d\mathrm{Area}_p = \frac{4\pi \mathbf{p}_e \times \mathbf{M}_e}{3c} = \frac{\mathbf{p}_e \times \mathbf{m}_e}{a^3 c} = -\mathbf{P}_{\rm EM}, \qquad (90)
\end{aligned}
$$

as previously found in eq. (88).[22] This is agreeable in that the field momentum in a static system can be computed as $\int V\mathbf{J}\, d\mathrm{Vol}/c^2$, eq. (10), as reviewed in [35].

Turning to the case of uniform polarization densities $\mathbf{P}_m$ and $\mathbf{M}_e$, the "hidden" mechanical momentum can be computed via the two integrals in eq. (79). Considering the second of these integrals, we note that the electric field due to the polarization density $\mathbf{P}_m$ is not related to a scalar potential, but can be deduced from a vector potential, $\mathbf{E}_m = \boldsymbol{\nabla} \times \mathbf{A}_m$, where $\mathbf{A}_m = \mathbf{p}_m \times \mathbf{r}/R^3(r)$ with $R(r > a) = r$ and $R(r < a) = a$. Then, we integrate by parts, noting that the result volume integral is zero for uniform $\mathbf{P}_m$,

$$
\begin{aligned}
\int \frac{\mathbf{M}_e \times \mathbf{E}_m}{c}\, d\mathrm{Vol}_m &= \int \frac{\mathbf{M}_e \times (\hat{\mathbf{r}} \times \mathbf{A}_m)}{a^3 c}\, d\mathrm{Area}_m \\
&= \int \frac{\hat{\mathbf{r}} \times (\mathbf{M}_e \times \mathbf{A}_m) - \mathbf{A}_m \times (\mathbf{M}_e \times \hat{\mathbf{r}})}{a^3 c}\, d\mathrm{Area}_m \\
&= \int \frac{\hat{\mathbf{r}} \times (\mathbf{M}_e \times \mathbf{A}_m)}{a^3 c}\, d\mathrm{Area}_m - \int \frac{\mathbf{A}_m \times \mathbf{K}_e}{a^3 c^2}\, d\mathrm{Area}_m \qquad (91)
\end{aligned}
$$

With some effort, each of the two integrals in the last line of eq. (91) can be shown to be equal to $\mathbf{p}_m \times \mathbf{m}_e/a^3 c$ (and by extention to the first integral of eq. (90), the total "hidden" mechanical momentum is again $4\mathbf{p}_m \times \mathbf{m}_e/a^3 c$). Thus, for the case of $\mathbf{P}_m$ and $\mathbf{M}_e$, the "hidden" mechanical momentum cannot be computed using only equivalent currents (and vector potentials).

B.2.2.6 Comments

For the examples where uniform electric and magnetic polarization densities occupy the same volumes within a sphere, the use of uniform (macroscopic) fields inside the sphere leads to an understanding that the sum of the field momentum and the "hidden" mechanical momentum is zero when the fields on the current-based dipoles are due to charged-based dipoles, but is nonzero when the fields on the current-based dipoles are due to current-based dipoles.

If we take a microscopic view, that the polarization densities are due to a collection, such as a cubic lattice, of entities that posses both an electric and a magnetic moment, then (as seen in the example in Appendix B.2.1.2 above), we expect that the fields at the center of these entities, due to the other entities, are zero. That is, the effective field on a dipole could

[21] Nominally, the integration by parts of $-\int \mathbf{M}_e \times \boldsymbol{\nabla} V_e\, d\mathrm{Vol}_p$ would lead to a term $\int V\mathbf{M}_e \times \boldsymbol{\nabla}\, d\mathrm{Vol}_p$, but in the usual convention that the $\boldsymbol{\nabla}$ acts to the right, we write this term as $-\int V\boldsymbol{\nabla} \times \mathbf{M}_e\, d\mathrm{Vol}_p$.

[22] The result (90) is consistent with sec. IV of [14], particularly eq. (35).

well be different from the macroscopic average field. It appears that when the fields on the current-based dipoles (that possess "hidden" mechanical momentum) are due to charged-based dipoles, the use of a macroscopic analysis is (perhaps surprisingly) satisfactory, but when the fields on the current-based dipoles of one type are due to current-based dipoles of the other type, the macroscopic analysis overestimates the "hidden" mechanical momentum.

While the discrepancies between eqs. (54) and (55), and between eqs. (58) and (60), are not fully resolved, it is very plausible that if the field momentum and the "hidden" mechanical momentum could be correctly computed their sum would be zero, as expected on general principle for these examples of systems "at rest".

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