Special Relativity and the Precession of the Perihelion

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(October 25, 2022; updated April 4, 2023)

1 Problem

In 1915, Einstein computed the precession of the perihelion of Mercury [1] as the first application of his new theory of general relativity, finding that advance in angle θ of the perihelion with respect to the Sun (assumed to be spherical and not rotating) to be,

$$\Delta\theta = \frac{6\pi GM}{ac^2(1-\epsilon^2)} = \frac{24\pi^3 a^2}{T^2c^2(1-\epsilon^2)}\,, \tag{1}$$

per revolution, where G is Newton's gravitational constant, M is the (rest) mass of the Sun, the orbit has semimajor axis a and eccentricity ϵ, c is the speed of light in vacuum, and $T^2 = 4\pi^2 a^3/GM$ relates the period T of the orbit to a via Kepler's third law. The result (1) is accurate to order v^2/c^2, where v is the velocity of Mercury with respect to the Sun,[1,2] and the mass m of Mercury has been neglected in comparison to mass M of the Sun.[3,4]

[1]Einstein's approximate result can be called the first example of the use of a "post-Newtonian approximation" based on general relativity. See, for example, [2].

[2]For a review of methods of computing the precession of the perihelion via general relativity, see [3].

[3]In 1967, Dicke [4, 5] noted that the precession of the perihelion of Mercury could be explained by the possible oblateness of the Sun. Evidence for this is marginal, as reviewed in [6].

[4]The famous data on the precession of the perihelion of Mercury were compiled by Le Verrier (1859) [7, 8]. For a review as of 1903, see secs. 23-24 of [9].

Among the many attempts around 1890 to explain the precession of the perihelion of Mercury, Lévy [10] proposed that gravity is deducible from a scalar potential,

$$V = \frac{GM}{r}\left(1 - \frac{\dot r^2}{v_g^2}\right), \tag{2}$$

where r is the present distance from the source to the observer, $\dot r$ is the speed of the source, and v_g is the speed of gravity. Apparently Lévy was inspired by Weber's electrodynamics, and hoped that $v_g = c$ would explain the data; however it did not quite. Earlier attempts based on Weber's electrodynamics include [11]-[13].

In 1898, Gerber [14, 15] gave a model of gravity based on the scalar potential,

$$V = \frac{GM}{r(1-\dot r/v_g)^2} \approx \frac{GM}{r}\left(1 + \frac{2\dot r}{v_g} + \frac{3\dot r^2}{v_g^2}\right), \tag{3}$$

This potential is an approximate form of an approximate retarded potential, as discussed in [16]. From this potential he computed the precession of the perihelion of Mercury per revolution, finding,

$$\Delta\theta = \frac{24\pi^3 a^2}{T^2 v_g^2(1-\epsilon^2)}\,. \tag{4}$$

Based on the data, Gerber inferred that $v_g = c$ to good accuracy. (continued on p. 2)

Einstein's result (1) agrees well with data.[5]

The question arises as to what is the prediction from special relativity. The literature on this is rather erratic.

Early work (1906-1911) by Poincare [25, 26], Lorentz [27], de Sitter [28] and others (for a survey, see p. 158 of [22]) inferred that the result from special relativity for the precession of the perihelion of Mercury is only 1/6 that of the observed value. An effort by Nordström in 1912 [29] predicted precession -1/6(!) of the observed value.

In 1917, Lodge [30] claimed to be inspired by special relativity to consider velocity-dependent corrections to the precession of the perihelion, but actually reverted to Newton's analysis of precession in case of a force law $1/r^n$ for n different than 2 [31], as extended by [33, 34]. A debate followed between Eddington and Lodge [35]-[39].

In 1929, Kennedy [40] gave two analyses of Newtonian precession of the perihelion, with corrections for retardation and for special relativity, claiming negligible effects in both cases.

It was stated by Goldstein in Exercise 6, Chap. 13 of [41] (1950)[6] that the result from special relativity is 1/6 that of general relativity, our eq. (1).

A review of "relativistic" theories of gravity was given by Whitrow and Morduch in 1965 [42], including Nordström-like theories.

In 1984, Phipps [43] claimed that the result of special relativity is the same as that of general relativity.

In 1986, Peters [44] noted that Phipps made a computational error, and claimed the correct result of Phipps' model is 1/2 that of general relativity. Peters then endorsed the claim of Goldstein [41] as the "standard" view of special relativity.

In 1987, Biswas [45] claimed that the result of (his interpretation of) special relativity is the same as that of general relativity.

In 1988, Frisch [46] discussed "post-Newtonian" approximations, claiming that use of "relativistic momentum" but Newtonian gravity gives the result of Goldstein [41], 1/6 of the observed precession of the perihelion of Mercury, while including the gravitation due to gravitational field energy doubles the result, to 1/3 of the observed precession of the perihelion of Mercury.

In 1989, Peters [47] argued that Biswas' calculation was in error.

In 2006, Jefimenko proposed a theory of "cogravitation", and claime it predicted 1/3 of the observed precession of the perihelion of Mercury

In 2008, Biswas [49] published another version of his 1998 paper [45], again claiming that his model, based on special relativity, explains the full precession of the perihelion of Mercury.

In retrospect, this is less surprising in that eq. (4) is identical to the result (1) of Einstein [1], computed via his theory of general relativity with $v_g = c$. This "coincidence" led to accusations that Einstein plagiarized Gerber's result [17, 18], which reverberate to this day. See [19] for extensive comments on various approximations to gravity in the solar system.

Gerber's effort is often dismissed as naïve, but is perhaps better appreciated as an attempt at an effective theory of gravity beyond that of Newton (as discussed in sec. 6.3 of [20]; see also [21]).

Other efforts in the 1800's to include a finite speed of propagation of gravity are reviewed in Chap. 6 of [22], and in [23].

[5] An illustration of how an *ad hoc* modification to Newton's "orbit equation" can reproduce the result of general relativity for the precession of the perihelion is given, for example, in Sec. 8.9, p. 312 of [24].

[6] Exercise 26, Chap. 7 of the 3rd ed. (2002)

In 2015, Wayne [50] claimed that special relativity can explain the precession of the perihelion of Mercury.

In 2016, Lemmon and Mondragon [51] argued that special relativity predicts 1/3 of the rate of the precession of the perihelion according to general relativity.

In 2020, Corda [52] claimed that Newtonian gravity completely explains the precession of the perihelion of Mercury (without consideration of relativity), but not that of other planets. Then, he argued that general relativity also explains the precession, but only if one includes the effect of "rotational time dilation".

In 2022, D'Abramo [53] claimed that Corda [52] was wrong.

What is going on here?

2 Solution

2.1 Goldstein

Goldstein [41] deduced the form of a single-particle Lagrangian in special relativity as,

$$\mathcal{L} = -mc^2\sqrt{1 - v^2/c2} - V, \tag{5}$$

where m is the rest mass of the particle, whose velocity is $\mathbf{v}$, and V is the potential energy of the particle. The low-velocity limit of eq. (5) is the usual nonrelativistic form $\mathcal{L} = T - V$ with kinetic energy $T = mv^2/2$.

Goldstein argued that for a particle of electric charge q in an electromagnetic field, one can use $V = q\phi - \mathbf{v} \cdot \mathbf{A}$ (in SI units), where $(\phi, \mathbf{A})$ are the electromagnetic potentials of the field in some gauge. However, Goldstein did not mention what he considered the gravitational potential to be in special relativity. One infers that he assumed it to be just the nonrelativistic form $V = -GMm/r$ for (spherical) masses M and m distance r apart.

2.2 Phipps

Phipps [43] considered the Lagrangian (5), but supposed the gravitational potential in special relativity, for (spherical) mass m with velocity $\mathbf{v}$ with respect to (large, spherical) mass M is,

$$V = -\frac{GMm}{r\sqrt{1 - v^2/c^2}} = -\frac{\gamma GMm}{r}, \qquad \text{with} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{6}$$

That is, he supposed that the "relativistic mass" γm is also the "gravitational mass" of m.[7,8]

[7] In 1911, Einstein [54] considered that "gravitational" mass equals (relativistic) inertial mass, although this notion does not appear in the later theory of general relativity, where only rest mass is emphasized.

[8] In sec. III of [43], Phipps considered the motion of mass m along a radius with respect to (fixed) mass M, and speculated that this might involve some kind of "antigravity" effect. He seemed unaware of the literature on this topic, which is reviewed in [55].

2.3 Peters

This section follows the analysis by Peters in [44]. He used the Lagrangian (5) and Phipps' form (6) of the gravitational potential energy, and expanded these in powers of $1/c$, writing,

$$\mathcal{L} \approx -mc^2 + \frac{mv^2}{2} + \frac{mv^4}{8c^2} + \frac{GMm}{r}\left(1 + \frac{\alpha v^2}{2c^2}\right), \tag{7}$$

where $\alpha = 1$ for the potential (6), while $\alpha = 0$ for the nonrelativistic potential used by Goldstein [41].

Rather than proceeding to deduce Lagrange's equations of motion, Peters considered the Hamiltonian (anticipating that $dH/d\theta$ will be an "orbit equation"),

$$H = \sum_{i=1,3} p_i v_i - \mathcal{L}, \qquad p_i = \frac{\partial \mathcal{L}}{\partial v_i} \approx m v_i \left(1 + \frac{v^2}{2c^2} + \frac{\alpha GM}{rc^2}\right), \tag{8}$$

$$\begin{aligned} H &\approx mv^2\left(1 + \frac{v^2}{2c^2} + \frac{\alpha GM}{rc^2}\right) + mc^2 - \frac{mv^2}{2} - \frac{mv^4}{8c^2} - \frac{GMm}{r}\left(1 + \frac{\alpha v^2}{2c^2}\right) \\ &= mc^2 + \frac{mv^2}{2} + \frac{3mv^4}{8c^2} - \frac{GMm}{r}\left(1 - \frac{\alpha v^2}{2c^2}\right). \end{aligned} \tag{9}$$

The Hamiltonian (9) is independent of time, and so is a constant of the motion, but it is not the total energy $E = T + V$, from which it differs by the sign of the term in α. For $\alpha = 0$, Goldstein's assumption, the Hamiltonian is the energy.

This has the implication that the total energy is not conserved in Phipps' model, which suggests that Phipps' model is not physically plausible, and perhaps should not be considered further.[9,10]

[9]Instead of using the approximate Lagrangian (14), we can consider form (5) with the potential energy (6). Then, instead of eqs. (8)-(9) we have,

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{\gamma GMm}{r}, \tag{10}$$

$$H = \sum_{i=1,3} p_i v_i - \mathcal{L}, \qquad p_i = \frac{\partial \mathcal{L}}{\partial v_i} = \gamma m v_i + \frac{\gamma^2 GMm v_i}{rc^2}, \tag{11}$$

$$H = \gamma mv^2 + \frac{\gamma^2 GMmv^2}{rc^2} + \frac{mc^2}{\gamma} - \frac{\gamma GMm}{r} = \gamma mc^2 + (\gamma^2 - \gamma - 1)\frac{GMm}{r}. \tag{12}$$

If we ignore gravity, setting $G = 0$, we just have a free particle of mass m, with total energy γmc^2, which equals the Hamiltonian of eq. (12) in this case.

For Goldstein's assumption that Newtonian gravity should hold in special relativity, we set $\gamma = 1$ in the terms involving G, and the Hamiltonian (12) is the total energy $\gamma mc^2 + V$ (for the Newtonian potential $V = -GMm/r$).

But, for Phipps' model, total energy is not conserved, while the Hamiltonian of (12) is a conserved quantity.

[10]Conservation of energy is a complicated issue in general relativity. See, for example, [56]. But, if we ignore the nonlinear effect of the curvature of spacetime by the moving object on that object itself, and ignore the gravitational radiation of the moving object,one can consider that energy in conserved in the general-relativistic description of the orbit of the moving object about the much larger "fixed" mass M.

2.3.1 Continuation of Peters' Analysis

For completeness, I transcribe the rest of Peters analysis in [44].

The (planar) orbit can be described by coordinates r and θ, in which the velocity can be expressed as,

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2. \tag{13}$$

The Lagrangian (7) does not depend on θ, so there is a conserved canonical momentum,

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \approx mr^2\dot{\theta}\left(1 + \frac{v^2}{2c^2} + \frac{\alpha GM}{rc^2}\right), \tag{14}$$

which we identify as the orbital angular momentum. With this, we can express the angular velocity as,

$$\dot{\theta} \approx \frac{L}{mr^2}\left(1 - \frac{v^2}{2c^2} - \frac{\alpha GM}{rc^2}\right). \tag{15}$$

Next, Peters followed a method of Newton to replace the radius r by its reciprocal $u = 1/r$. For this, we note that,

$$\begin{aligned} v^2 = \dot{r}^2 + r^2\dot{\theta}^2 &= \left(\frac{d}{dt}\frac{1}{u}\right)^2 + \frac{\dot{\theta}^2}{u^2} = \left(\frac{d\theta}{dt}\frac{d}{d\theta}\frac{1}{u}\right)^2 + \frac{\dot{\theta}^2}{u^2} = \left(-\dot{\theta}\frac{u'}{u^2}\right)^2 + \frac{\dot{\theta}^2}{u^2} = \frac{\dot{\theta}^2 u'^2}{u^4} + \frac{\dot{\theta}^2}{u^2} \\ &= (u'^2 + u^2)\frac{\dot{\theta}^2}{u^4} = (u'^2 + u^2)r^4\dot{\theta}^2, \end{aligned} \tag{16}$$

where $u' = du/d\theta$. From eq. (15), we have,

$$r^4\dot{\theta}^2 \approx \frac{L^2}{m^2}\left(1 - \frac{v^2}{c^2} - \frac{2\alpha GM}{rc^2}\right). \tag{17}$$

Using this in eq. (16), we find,

$$v^2 \approx \frac{L^2}{m^2}(u'^2 + u^2)\left(1 - \frac{v^2}{c^2} - \frac{2\alpha GM}{rc^2}\right). \tag{18}$$

Peters noted that in the first approximation, $v^2 \approx (L^2/m^2)(u'^2 + u^2)$, to rewrite eq. (18) as,

$$v^2 \approx \frac{L^2}{m^2}(u'^2 + u^2) - \frac{v^4}{c^2} - \frac{2\alpha GMv^2}{rc^2}. \tag{19}$$

With this, the Hamiltonian (9) becomes,

$$H \approx mc^2 + \frac{L^2}{2m}(u'^2 + u^2) - \frac{mv^4}{8c^2} - GMmu\left(1 + \frac{\alpha v^2}{2c^2}\right). \tag{20}$$

For a Newtonian orbit, the total energy E is related to the semimajor axis a by,

$$E = \frac{mv^2}{2} - \frac{GMm}{r} = -\frac{GMm}{2a}, \qquad v^2 = \frac{2GM}{r} - \frac{GM}{a} = GM\left(2u - \frac{1}{a}\right). \tag{21}$$

We follow Peters in using the Newtonian relation (21) for v^2 in the terms of order $1/c^2$ of the special-relativistic Hamiltonian (20) to find,

$$
\begin{aligned}
H &\approx mc^2 + \frac{L^2}{2m}(u'^2 + u^2) - \frac{G^2M^2m}{8c^2}\left(4u^2 - \frac{u}{a} + \frac{1}{a^2}\right) - GMmu\left(1 + \frac{\alpha GM}{2c^2}\left(2u - \frac{1}{a}\right)\right) \\
&= mc^2 + \frac{L^2}{2m}(u'^2 + u^2) - GMmu - \frac{G^2M^2m}{2c^2}\left((1+2\alpha)u^2 - \frac{(1+\alpha)u}{a} + \frac{1}{4a^2}\right). \qquad (22)
\end{aligned}
$$

Then, we can find an "orbit equation" by taking the derivative of the (constant) Hamiltonian with respect to θ,

$$
H' = \frac{dH}{d\theta} \approx \frac{L^2}{m}u'(u'' + u) - GMmu' - \frac{G^2M^2m}{2c^2}\left(2(1+2\alpha)uu' - \frac{(1+\alpha)u'}{a}\right) = 0, \qquad (23)
$$

$$
u'' + u\left(1 - \frac{(1+2\alpha)G^2M^2m^2}{c^2L^2}\right) \approx \frac{GMm^2}{L^2} - \frac{G^2M^2m^2}{2c^2L^2}\frac{(1+\alpha)}{a} = \text{const.} \qquad (24)
$$

To order $1/c^2$, it suffices to use the value $L^2 = GMm^2a(1-\epsilon^2)$ of the Newtonian orbit,[11] so the special-relativistic orbit equation is,

$$
u'' + (1-k)u = \text{const}, \qquad \text{where} \qquad k = \frac{(1+2\alpha)GM}{ac^2(1-\epsilon^2)} \ll 1. \qquad (25)
$$

This equation has solutions of the form,

$$
u(\theta) = u_0 + u_1\cos(\sqrt{1-k}\,\theta), \qquad (26)
$$

whose period T in angle θ is, for small k,

$$
T = \frac{2\pi}{\sqrt{1-k}} \approx 2\pi\left(1 + \frac{k}{2}\right) = 2\pi + \pi k. \qquad (27)
$$

That is, the perihelion of the orbit advances/precesses by angle,

$$
\Delta\theta = \pi k = (1+2\alpha)\frac{\pi GM}{ac^2(1-\epsilon^2)}\,, \qquad (28)
$$

per revolution. For $\alpha = 0$ (Newtonian gravity) this is 1/6 of the result (1) of general relativity, as claimed in [41]. For the model that the relativistic mass γm is also the gravitational mass, as in eq. (5), $\alpha = 1$ and the precession of the perihelion is 1/2 that of general relativity.

The result (1) of general relativity would be predicted by a version of Phipps' model with $\alpha = 5/2$, but this model would not conserve energy, and so cannot be considered a reasonable explanation.

2.4 Biswas

Biswas' model [45] seems to be essentially the same as that of Phipps, but with an additional dimensionless parameter, κ of his eq. (19b), which he sets to 2 to obtain agreement with the observed precession of the perihelion of Mercury.

[11] See, for example, eq. (3.63) of [41].

2.5 Frisch

Frisch's second "post-Newtonian approximation" [46] included an effect of "time dilation" *vs.* distance r from the mass M (in the rest frame of that mass), which is a kind of "curved time" that goes beyond special relativity, the nominal theme of the present note. He claimed this led to 1/3 of the observed precession of the perihelion of Mercury.

I consider that "post-Newtonian approximations" start from knowledge of general relativity and "'work backwards" to formalism that resembles Newtonian mechanics, (without metric tensors and notions of curved spacetime). The precession rate (1) is also found in the "first post-Newtonian approximation", as reviewed, for example, in [58, 59] and more briefly in [60]. In contrast, this note concerns additions to special relativity to include gravity that is related to a single scalar potential for a 2-body system, perhaps via the "weak equivalence principle, that inertial mass is the same as "gravitational mass".

The full post-Newtonian approximation reproduces the prediction of general relativity for the precession of the perihelion of Mercury. See sec. 4.2, p. 46 of [2].

2.6 Jefimenko

Jefimenko's gravitational theory [48] is a kind of step towards a post-Newtonian approximation, without using general relativity. It apparently leads to the same result for the precession of the perihelion of Mercury as does Frisch's approximation [46].

2.7 Wayne

Wayne [50] seemed to argue that the "relativistic mass" of a moving object with rest mass m should be $m(1 + v^2/c^2)$ at order $1/c^2$ rather then the usual approximation $m(1 + v^2/2c^2)$, and then claimed that "special relativity" predicts the observed precession of the perihelion of Mercury.

2.8 Lemmon and Mondragon

Lemmon and Mondragon [51] first reviewed that model of Goldstein [41], using "relativistic momentum" but Newtonian gravity, confirming that this model predicts 1/6 of the observed precession of the perihelion of Mercury. In their Sec. IV, they appeared to consider the model of Phipps [43], which assumes that the "gravitational mass" of moving object m is its "relativistic mass" γm. But, they proceeded in a slightly different manner than the analysis of Peters [44] reviewed in our sec. 2.3 above, arriving at the potential energy given in their eq. (37). From this, they inferred that the precession of the perihelion of Mercury is 1/3 that predicted by general relativity (which happens to agree with the "post-Newtonian" result of Frisch, which is a different model).

I am skeptical of the approach of Lemmon and Mondragon, and consider that the analysis of Peters, given above, is the more correct version of Phipps' model.

2.9 Corda

Corda first claimed, in sec. 2 of [52], that according to Newtonian gravity, the nonzero mass of a planet would make its orbit about the center of mass of the Sun-planet system "precess",[12] in that the period of the orbit is slightly different than if the planetary mass is negligible. As remarked by D'Abramo [53], it is a coincidence that this "explanation" works for Mercury.

In sec. 3 of [52], Corda misrepresented the discussion in [57] about the effect of the outer planets on the precession of the perihelion of Mercury, supposing that even when omitting consideration of the outer planets, the orbit of Mercury could be nonplanar, claiming that this effect also fully explains the precession of the perihelion.

In sec. 6 of [52], Corda gave a nonstandard analysis via general relativity, not finding the usual prediction (1). Then, in secs. 7-8 Corda claimed that an effect of the "rotational time dilation" of clocks on Mercury, when added to his previous general-relativity analysis, fully explains the precession of its perihelion.

Corda concluded (sec. 9) that he had given three "explanations" of the precession of the perihelion of Mercury, which he somehow considered to be equivalent rather than distinct.[13]

A "standard" view is that all three of Corda's explanations are bogus.

Thanks to Derek Abbott and Germano D'Abramo for e-discussions of the problem.

References

[1] A. Einstein, *Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie*, Sitz. König. Preuss. Akad. Wiss. Berlin, 831 (1915),
http://kirkmcd.princeton.edu/examples/GR/einstein_skpaw_831_15.pdf
http://kirkmcd.princeton.edu/examples/GR/einstein_skpaw_831_15_english.pdf

[2] C.M. Will, *The Confrontation between General Relativity and Experiment*, Living Rev. Rel. **17**, 4 (2014), http://kirkmcd.princeton.edu/examples/GR/will_lrr_17_4_14.pdf

[3] J.M. Pons, *Expanding the range of validity of the simplest computation of the perihelion precession in Schwarzschild spacetime*, Am. J. Phys. **92**, 23 (2024),
http://kirkmcd.princeton.edu/examples/GR/pons_ajp_92_23_24.pdf

[4] R.H. Dicke and H.M. Goldenberg, *Solar Oblateness and General Relativity*, Phys. Rev. Lett. **18**, 313 (1967), http://kirkmcd.princeton.edu/examples/GR/dicke_prl_18_313_67.pdf

[12] Although the concept of the perihelion is only defined for elliptical orbits (with nonzero eccentricity) eq. (1) still has a value for zero eccentricity. This leaves room for definitions of the precession of a circular orbit to be almost anything, including Corda's view that it should mean the difference in the angle swept out in one turn of a circular orbit of radius r_0 and period $T_0 = 2\pi r_0^{3/2}/\sqrt{GM}$ of a zero mass particle about mass M (just 2π radians) and the angle swept out by a particle of nonzero mass m in a circular orbit of the same radius during time T_0 (rather than during its orbital period $T = 2\pi r_0^{3/2}/\sqrt{G(M+m)}$).

[13] Corda noted that Newton's 2$^{\rm nd}$ law, $mv^2/r_0 = GMm/r_0^2$ (aka Kepler's 3$^{\rm rd}$ law) for a planet of mass m with velocity v in a circular orbit of radius r_0 about mass M can be rewritten as $2r_0v^2/c^2 = 2GM/c^2 =$ the Schwarzschild radius of mass M according to general relativity.

Since the time dilation of special relativity involves the factor $1/\sqrt{1-v^2/c^2}$, Corda appeared to infer that Newtonian gravity, special relativity and general relativity are essentially the same theory.

[5] R.H. Dicke, *The Oblateness of the Sun and Relativity*, Science **184**, 419 (1974), http://kirkmcd.princeton.edu/examples/GR/dicke_science_184_419_74.pdf

[6] C. Damiani *et al.*, *A brief history of the solar oblateness. A review*, J. Atmos. Solar-Terr. Phys. **73**, 241 (2011), http://kirkmcd.princeton.edu/examples/GR/damiani_jastp_73_241_11.pdf

[7] U.J. LeVerrier, *Sur la théorie de Mercure et sur le mouvement de le périhélie de cette planète*, Compt. Rend. Acad. Sci. **49**, 379 (1859), http://kirkmcd.princeton.edu/examples/GR/leverrier_cras_49_379_59.pdf See p. 25 for a summary of the data.

[8] U.J. LeVerrier, *Théorie du Mouvement de Mercure*, Ann. Obs. Imp. Paris **5**, 1 (1859), http://kirkmcd.princeton.edu/examples/GR/leverrier_aoip_5_1_59.pdf

[9] J. Zenneck, *Gravitation*, Ency. Math. Wissens. 25 (1903), http://kirkmcd.princeton.edu/examples/GR/zenneck_emw_25_03.pdf

[10] M. Lévy, *Sur l'application des lois électrodynamique au mouvement des planètes*, Compt. Rend. Acad. Sci. **110**, 545 (1890), http://kirkmcd.princeton.edu/examples/GR/levy_cras_110_545_90.pdf

[11] G. Holzmüller, *Ueber die Anwendung der Jacobi-Hamilton'schen Methode auf den Fall der Anziehung nach dem elektrodynamishen Gesetze von Weber*, Z. Math. Phys. **15**, 69 (1870), http://kirkmcd.princeton.edu/examples/GR/holzmuller_zmp_15_69_70.pdf

[12] F. Tisserand, *Sur le mouvement des planètes autour du Soleil, d'après la loi électrodynamique de Weber*, Comptes Rend. Acad. Sci. **75**, 760 (1872), http://kirkmcd.princeton.edu/examples/GR/tisserand_cras_75_760_72.pdf

[13] F. Tisserand, *Sur le mouvement des planètes, en supposant l'attraction représentée par l'une des lois électrodynamiques de Gauss ou de Weber*, Comptes Rend. Acad. Sci. **110**, 313 (1890), http://kirkmcd.princeton.edu/examples/GR/tisserand_cras_110_313_90.pdf

[14] P. Gerber, *Die räumliche und zeitliche Ausbreitung der Gravitation*, Z. Math. Phys. **43**, 93 (1898), http://kirkmcd.princeton.edu/examples/GR/gerber_zmp_43_93_98.pdf http://kirkmcd.princeton.edu/examples/GR/gerber_zmp_43_93_98_english.pdf

[15] P. Gerber, *Die Fortpflanzungsgeschwindigkeit der Gravitation*, Ann. d. Phys. **52**, 415 (1917), http://kirkmcd.princeton.edu/examples/GR/gerber_ap_52_415_17.pdf

[16] J. Giné, *On the origin of the anomalous precession of Mercury's perihelion*, Chaos, Solitons Fractals **35**, 1004 (2008), http://kirkmcd.princeton.edu/examples/GR/gine_csf_35_1004_08.pdf

[17] E. Gehrcke, *Zur Kritik umd Geschichte der neueren Gravitationstheorien*, Ann. d. Phys. **51**, 119 (1916), http://kirkmcd.princeton.edu/examples/GR/gehrcke_ap_51_119_16.pdf Several subsequent discussion are appended to this file.

[18] A. Einstein, *Ueber die antirelativiätstheoretische*, Berliner Tageblatt (Aug. 27, 1920), http://kirkmcd.princeton.edu/examples/GR/einstein_bt_200827.pdf
See the appended notes in English.

[19] A.L. Kholodenko, *Newtonian limit of Einsteinian gravity: from dynamics of Solar system to dynamics of stars in spiral galaxies* (May 7, 2015), http://arxiv.org/abs/1006.4650

[20] J.D. Wells, *When effective theories predict: the inevitability of Mercury's anomalous perihelion precession* (June 8, 2011), http://arxiv.org/abs/1106.1568

[21] K. Brown, *Gerber's Gravity*, http://www.mathpages.com/home/kmath527/kmath527.htm

[22] N.T. Roseveare, *Mercury's perihelion, from Le Verrier to Einstein* (Clarendon Press, 1982). http://kirkmcd.princeton.edu/examples/GR/roseveare_82.pdf
https://archive.org/details/mercurysperiheli0000rose/page/n5/mode/2up

[23] H.-J. Treder, *The Post-Newtonian Effect of Gravitation and the Retardation of the Gravitational Potential in Classical and Relativistic Theories of Gravitation*, Ann d. Phys. **32**, 338 (1975). http://kirkmcd.princeton.edu/examples/GR/treder_ap_32_338_75.pdf

[24] S.T. Thornton and J.B. Marion, *Classical Dynamics of Particles and Systems*, 5$^{\text{th}}$ ed. (Cengage Learning, 2003), http://kirkmcd.princeton.edu/examples/mechanics/thornton_dynamics_03.pdf

[25] H. Poincaré, *Limites de la loi de Newton*, Bull. Astro. **17**, 121 (1953). Lectures from 1906. See pp. 236-239. http://kirkmcd.princeton.edu/examples/GR/poincare_ba_17_121_53.pdf

[26] H. Poincaré, *La Dynamique de l'Électron*, Rev. Gen. sci. Pure Appl. **19**, 386 (1908). See pp. 389-401. http://kirkmcd.princeton.edu/examples/EM/poincare_rgspa_19_386_08.pdf

[27] H.A. Lorentz, *Alte und Neue Fragen der Physik*, Phys. Z. **11**, 1234 (1910). See p. 1240.
http://kirkmcd.princeton.edu/examples/EM/lorentz_pz_11_1234_10.pdf
http://kirkmcd.princeton.edu/examples/EM/lorentz_pz_11_1234_10_english.pdf

[28] W. De Sitter, *On the bearing of the Principle of Relativity on Gravitational Astronomy*, Mon. Not. Roy. Astro. Soc. **71**, 388 (1911).
http://kirkmcd.princeton.edu/examples/GR/desitter_mnras_71_388_11.pdf

[29] G. Nordström, *Relativitätsprinzip und Gravitation*, Phys. Z. **13**, 1126 (1912).
http://kirkmcd.princeton.edu/examples/GR/nordstrom_pz_13_1126_12.pdf

[30] O. Lodge, *Astronomical Consequences of the Electrical Theory of Matter*, Phil. Mag. **34**, 81 (1917). http://kirkmcd.princeton.edu/examples/GR/lodge_pm_34_81_17.pdf

[31] I. Newton, *Philosophiæ Naturalis Principia Mathematica* (1686), Prop 45, Book 1, p. 177. kirkmcd.princeton.edu/examples/mechanics/newton_principia.pdf See also [32].

[32] S.R. Valluri, c. Wilson C and W.L. Harper, J. Hist. Astron. **28**, 13 (1997).
http://kirkmcd.princeton.edu/examples/GR/valluri_jha_28_13_97.pdf

[33] J. Bertrand, *Théorème relatif au mouvement d'un point attiré vers un centre fixe*, Compt. Rendus Acad. Sci. **77**, 749 (1873).
http://kirkmcd.princeton.edu/examples/mechanics/bertrand_cras_77_849_73.pdf
http://kirkmcd.princeton.edu/examples/mechanics/bertrand_cras_77_849_73_english.pdf

[34] A. Hall, *A Suggestion in the Theory of Mercury*, Astron. J. **14**, 49 (1894).
http://kirkmcd.princeton.edu/examples/GR/hall_aj_14_49_94.pdf

[35] A.S. Eddington, *Astronomical Consequences of the Electrical Theory of Matter. A Note on Sir Oliver Lodge's Suggestions*, Phil. Mag. **34**, 163 (1917).
http://kirkmcd.princeton.edu/examples/GR/eddington_pm_34_163_17.pdf

[36] A.S. Eddington, *Astronomical Consequences of the Electrical Theory of Matter. A Note on Sir Oliver Lodge's Suggestions, II*, Phil. Mag. **34**, 321 (1917).
http://kirkmcd.princeton.edu/examples/GR/eddington_pm_34_321_17.pdf

[37] O. Lodge, *Astronomical Consequences of the Electrical Theory of Matter. Supplementary Note*, Phil. Mag. **34**, 517 (1917).
http://kirkmcd.princeton.edu/examples/GR/lodge_pm_34_517_17.pdf

[38] O. Lodge, *Continued Discussion of the Astronomical and Gravitational Bearings of the Electrical Theory of Matter*, Phil. Mag. **35**, 141 (1918).
http://kirkmcd.princeton.edu/examples/GR/lodge_pm_35_141_18.pdf

[39] A.S. Eddington, *Electrical Theories of Matter and their Astronomical Consequences with special reference to the Principle of Relativity*, Phil. Mag. **35**, 481 (1918).
http://kirkmcd.princeton.edu/examples/GR/eddington_pm_35_481_18.pdf

[40] R.J. Kennedy, *Planetary Motion in a Retarded Newtonian Potential Field*, Proc. Nat. Acad. Sci. **1**, 744 (1929).
http://kirkmcd.princeton.edu/examples/GR/kennedy_pnas_15_744_29.pdf

[41] H. Goldstein, *Classical Mechanics* (Addison-Wesley, 1950). H. Goldstein, C.P. Poole and J. Safko, *Classical Mechanics*, 3rd ed. (Addison-Wesley, 2002), Ex. 26, Chap. 7.
http://kirkmcd.princeton.edu/examples/mechanics/goldstein_3ed.pdf

[42] G.J Whitrow and G.E. Morduch, *Relativistic Theories of Gravitation*, Vistas Astro. **6**, 1 (19652). http://kirkmcd.princeton.edu/examples/GR/whitrow_va_6_1_65.pdf

[43] T.E. Phipps Jr, *Mercury's precession according to special relativity*, Am. J. Phys. **54**, 245 (1986), http://kirkmcd.princeton.edu/examples/GR/phipps_ajp_54_245_86.pdf

[44] P.C. Peters, *Comment on "Mercury's precession according to special relativity"*, Am. J. Phys. **55**, 757 (1987), http://kirkmcd.princeton.edu/examples/GR/peters_ajp_55_757_87.pdf

[45] T. Biswas, *Minimally relativistic Newtonian gravity*, Am. J. Phys. **56**, 1032 (1988), http://kirkmcd.princeton.edu/examples/GR/biswas_ajp_56_1032_88.pdf

[46] D.H. Frisch, *Simple aspects of post-Newtonian gravitation*, Am. J. Phys. **58**, 332 (1990), http://kirkmcd.princeton.edu/examples/GR/frisch_ajp_58_332_90.pdf

[47] P.C. Peters, *Comment on "Minimally relativistic Newtonian gravity"*, Am. J. Phys. **58**, 188 (1990), http://kirkmcd.princeton.edu/examples/GR/peters_ajp_58_188_90.pdf

[48] O.D. Jefimenko, *Gravitation and Cogravitation* (Electret Scientific Company, Star City, 2006), sec. 20-2, p. 333, http://kirkmcd.princeton.edu/examples/GR/jefimenko_06.pdf

[49] A. Biswas and K.R.S. Mani, *Relativistic perihelion precession of orbits of Venus and the Earth*, Cent. Eur. J. Phys. **6**, 754 (2008), http://kirkmcd.princeton.edu/examples/GR/biswas_cejp_6_754_08.pdf

[50] R. Wayne, *Explanation of the Perihelion Motion of Mercury in Terms of a Velocity-Dependent Correction to Newton's Law of Gravitation*, Afr. Rev. Phys. **10**, 26 (2015), http://kirkmcd.princeton.edu/examples/GR/wayne_arp_10_26_15.pdf

[51] T.J. Lemmon and A.R. Mondragon, *Kepler's Orbits and Special Relativity in Introductory Classical Mechanics* (Sept. 10, 2020), https://arxiv.org/abs/1012.5438

[52] C. Corda, *The Advance of Planets' Perihelion in Newtonian Theory Plus Gravitational and Rotational Time Dilation* (Sept. 10, 2020), http://kirkmcd.princeton.edu/examples/GR/corda_preprints202009.0235.pdf
The secret of planets' perihelion between Newton and Einstein, Phys. Dark Universe **32**, 100834 (2021), http://kirkmcd.princeton.edu/examples/GR/corda_pdu_32_100834_21.pdf

[53] G. D'Abramo, *Comment on "The secret of planets perihelion between Newton and Einstein"*, Phys. Dark Universe **37**, 101076 (2022), http://kirkmcd.princeton.edu/examples/GR/dabramo_pdu_37_101076_22.pdf

[54] A. Einstein, *Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes*, Ann. d. Phys. **35**m 898 (1911), http://kirkmcd.princeton.edu/examples/GR/einstein_ap_35_898_11.pdf
http://kirkmcd.princeton.edu/examples/GR/einstein_ap_35_898_11_english2.pdf

[55] K.T. McDonald, *Gravitational Acceleration of a Moving Object at the Earth's Surface* (April 4, 2016), http://kirkmcd.princeton.edu/examples/gravity_moving.pdf

[56] M. Weiss, *Is Energy Conserved in General Relativity?* (2017), https://math.ucr.edu/home/baez/physics/Relativity/GR/energy_gr.html

[57] M.P. Price and W.F. Rush, *Nonrelativistic contribution to Mercury's perihelion precession*, Am. J. Phys. **47**, 531 (1979), http://kirkmcd.princeton.edu/examples/GR/peters_ajp_47_531_79.pdf

[58] T. Damour and N. Deruelle, *General relativistic celestial mechanics of binary systems I. The post-Newtonian motion*, Ann. Inst. H. Poincaré **43**, 107 (1985), sec. 4. http://kirkmcd.princeton.edu/examples/GR/damour_aihpa_43_107_85.pdf

[59] J. Forlano, *Coming out of the woodwork: Post-Newtonian approximations and applications* (Mar. 25, 2015). http://kirkmcd.princeton.edu/examples/GR/forlano_150325.pdf

[60] H.P. Robertson, *Relativity and Cosmology*, In *Space Age Astronomy* (Academic Press, 1962), p. 228. http://kirkmcd.princeton.edu/examples/GR/robertson_saa_228_61.pdf