

Magnetic Force on a Permeable Wire

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1 Problem

What is the force per unit length on a wire of radius a and (relative) permeability μ' when it carries uniform current density $\mathbf{J} = I \hat{\mathbf{z}}/\pi a^2$ and is placed along the z axis in a magnetic field whose form is $\mathbf{B}_i = B_0 \hat{\mathbf{x}} + B_1[(x/a) \hat{\mathbf{x}} - (y/a) \hat{\mathbf{y}}]$ before the wire is placed in that field? The medium surrounding the wire is a nonconducting liquid with relative permeability $\mu \neq 1$, and the wire is held at rest by “mechanical” forces.

The form of the initial magnetic field has been chosen so that there will be both a $\mathbf{J} \times \mathbf{B}$ force associated with the uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{x}}$, as well as a force due to the interaction of the induced magnetization with the nonuniform field $\mathbf{B}_1 = B_1[(x/a) \hat{\mathbf{x}} - (y/a) \hat{\mathbf{y}}]$.

2 Solution

This old problem [1]¹ was recently reconsidered by Casperson [3],² who reported an experimental result that appears to disagree with the theory he presented. The impression was given that no straightforward theory exists for this problem, which this note hopes to correct by presenting three “standard” solutions that are in agreement. We also explore use of the somewhat hybrid methods for calculating forces on magnetic media advocated by the so-called Coulomb Committee [9, 10, 11, 12] and obtain success only if those methods are revised in an important way. An overall perspective on these issues is given in [13].

A variant on this problem has practical import to high-energy physicists such as the present author, who consider it to be experimentally confirmed (and continually reconfirmed) for over 50 years [14] that high-energy particles of charge q and velocity v obey the Lorentz force law of the form,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}/c \times \mathbf{B}), \quad (1)$$

(in Gaussian units) even when in a permeable medium where the magnetic fields are related by $\mathbf{B} = \mu \mathbf{H} \gg \mathbf{H}$. The problem concerns macroscopic version of the above microscopic force law.

If magnetic charges (monopoles) existed, the case of a magnetic-current-carrying “wire” embedded in a dielectric medium would be the dual of the present example. One could then ask whether the force density on a magnetic current density \mathbf{J}_m in the wire is $-\mathbf{J}_m \times \mathbf{D}$ (the canonical expression) or $-\mathbf{J}_m \times \mathbf{E}$? See, [15, 16].

¹A closely related problem is discussed at the end of chap. 7 of [2].

²In 1908-10 Einstein [4, 5, 6, 7] advocated that the force on free electrical currents is $\mathbf{J}_{\text{free}} \times \mathbf{H}$, rather than $\mathbf{J}_{\text{free}} \times \mathbf{B}$, which may have prolonged confusion on this issue. See [8] for additional discussion.

A suggestive principle is that no object with steady motion in vacuum can exert a net force on itself (Newton’s first law).³ In particular, the fields that are set up or modified when the wire, with a steady current, is added to the problem cannot result in a net force on the wire. Hence, the force on a permeable wire in vacuum can be calculated via the interaction of the current and magnetization in the wire with only the fields present before the wire was added to the problem, as emphasized in sec. 35 of [18] and also in [19].⁴ This important result is explicitly contained in the original formulation of the Biot-Savart force law (for media of unit permeability), that the magnetic force on circuit a is due to effects caused by some other circuit b ,⁵

$$\mathbf{F}_a = \frac{I_a I_b}{c^2} \oint_a d\mathbf{l}_a \times \oint_b \frac{d\mathbf{l}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2} = \frac{I_a}{c} \oint_a d\mathbf{l}_a \times \mathbf{B}_b, \quad \text{where} \quad \mathbf{B}_b(a) = \frac{I_b}{c} \oint_b \frac{d\mathbf{l}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2}. \quad (2)$$

Thus, we expect the simple result that the force per unit length on the wire due to the initial uniform field $\mathbf{B}_i = \mathbf{B}_0$ (in vacuum) is,

$$\mathbf{F} = \frac{\mathbf{I}}{c} \times \mathbf{B}_0, \quad (3)$$

where $\mathbf{I} = \pi a^2 \mathbf{J}$, which is the macroscopic equivalent of the Lorentz force law (1).⁶

³Accelerated charges can be subject to the so-called radiation-reaction force, which is a self force (first noted by Lorentz [17], and so should be considered as part of the “Lorentz force law”). Not all accelerated charges are subject to the radiation reaction force, since interference of the fields of the various charges may cancel the total radiation, as for steady current loops. Also, a uniformly accelerated charge (which is a kind of steady motion) famously experiences no self/radiation-reaction force.

⁴If the wire is not in vacuum this prescription does not hold in general. For example, a current-carrying wire is repelled from a perfectly conducting plane although that plane has no magnetic field in the absence of the wire. See also the problem in sec. 35 of [18], which is discussed further in [20].

⁵Biot and Savart [21, 22] had no concept of a magnetic field \mathbf{B} due to an electric current I , and discussed only the force on a magnetic pole p as $p \oint \mathbf{I} d\mathbf{l} \times \hat{\mathbf{r}}/cr^2$, although not, of course, in vector form. The form (2) can be traced to Grassmann (1845) [24], still not in vector form. The vector relation (3) appears without attribution as eq. (11) of Art. 603 of Maxwell’s *Treatise* [25], while Einstein may have been the first to call this the Biot-Savart law, in sec. 2 of [26].

⁶The transverse force (3) acts most directly on the conduction electrons, but in the steady state there cannot be a transverse component to the conduction current. However, it might be that the current density is displaced transversely, which would manifest itself as a change in the electrical resistance of the wire. This possibility was investigated by Hall in 1879 [27], who, on finding no change in the resistance, had the important insight that a charge separation arises when the wire first experiences the external magnetic, which field leads to a transverse electric field \mathbf{E} inside the wire such that the combined electric and magnetic force on the conduction electrons is zero. However, this electric field then acts on the positive charges of the lattice of the wire, exerting an equal and opposite force on them, which therefore equals the original force (3) on the conduction electrons. If the lattice of the wire is to remain at rest (or in a state of uniform motion), there must be a “mechanical” force on the wire equal and opposite to the force on the lattice, *i.e.*, to the force of eq. (3). See [28] for further discussion.

If the wire of radius a is free to move through the surrounding liquid it reaches a transverse terminal velocity \mathbf{v} , and experiences a drag force per unit length in a liquid with viscosity η given by,

$$\mathbf{F}_{\text{drag}} = -K(a, b)\eta\mathbf{v}, \quad (4)$$

where $K(a, b) \approx 2$ when the fluid lies between walls with separation b large (but not too large) compared to a . No simple expression for the drag force, *i.e.*, for the dimensionless function $K(a, b)$, exists, as anticipated

The initial magnetic field \mathbf{B}_i also induces magnetization \mathbf{M} in the wire, and an additional force results if the magnetic field is nonuniform at the position of the wire. According to the preceding argument, it should be possible to calculate this force as an interaction between the initial magnetic field and some representation of the induced magnetization in terms of bound currents or effective magnetic poles.

Nonetheless, it is also desirable to have a method for calculating the magnetic force that uses the total fields in the problem, including those generated by the wire. Two successful approaches are to use the Maxwell stress tensor (sec. 2.2), and the bulk force density of Helmholtz (sec. 2.3), both of which confirm the result (3). A third popular method based on the concept of virtual work [32] is not reviewed here.

In sec. 3 we calculate the force on a permeable, current-carrying wire by combining the Biot-Savart force law force the volume force density \mathbf{f} ,

$$\mathbf{f} = \frac{\mathbf{J}_{\text{cond}}}{c} \times \mathbf{B}_i, \quad (6)$$

with terms due to either magnetization currents or effective magnetic pole densities. As expected from the preceding argument, this approach is successful if only the initial magnetic fields are used in the force law. However, it turns out that when using the method of magnetization currents, the initial magnetic field to be used in the Biot-Savart law is \mathbf{H}_i rather than \mathbf{B}_i . For these calculations, the total magnetic fields \mathbf{B} and \mathbf{H} and the induced magnetization density \mathbf{M} in the permeable media will first be deduced in sec. 2.1.

It is important to note that one cannot, in general, compute magnetic forces correctly using the form,

$$\mathbf{f} = \frac{\mathbf{J}_{\text{total}}}{c} \times \mathbf{B}_{\text{total}}, \quad (7)$$

as discussed in sec. 4. This is unfortunate, as many text recommend use of this form.

2.1 The Fields \mathbf{B} , \mathbf{H} and \mathbf{M}

We adopt a coordinate system in which the axis of the wire is the z axis with the conduction current density being,

$$\mathbf{J}_{\text{cond}} = \frac{I}{\pi a^2} \hat{\mathbf{z}} \quad (8)$$

inside the wire of radius a .

Because we are dealing with magnetic media with nonzero magnetization \mathbf{M} , both the magnetic fields \mathbf{H} and $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M} = \mu\mathbf{H}$ are of utility. The initial external field is,

$$\mathbf{H}_i = H_0 \hat{\mathbf{x}} + H_1 \left(\frac{x}{a} \hat{\mathbf{x}} - \frac{y}{a} \hat{\mathbf{y}} \right), \quad \mathbf{B}_i = \mu\mathbf{H}_i, \quad B_0 = \mu H_0, \quad B_1 = \mu H_1, \quad (9)$$

by Stokes [29]. An approximate theory and numerical results are presented, for example, in Fig. 4 of [30]. This drag force is balanced by the force (3) on the wire, and that force does work per unit length at the rate,

$$P = \mathbf{F} \cdot \mathbf{v} = \frac{F^2}{K(a, b)\eta} \approx \frac{F^2}{2\eta}. \quad (5)$$

This is another example in which the magnetic force on a macroscopic system does work [31].

where μ is the permeability of the medium surrounding the wire. When the wire is placed into this medium, we expect a force in the $+y$ direction according to the Biot-Savart law (6), and a magnetization force in the $+x$ direction due to the nonuniform field \mathbf{H}_i that increases with x .

In addition to the rectangular coordinate system (x, y, z) , we will work in a cylindrical coordinate system (r, θ, z) . The usual transformation of the units vectors between these two coordinate systems are,

$$\hat{\mathbf{x}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}}, \quad \hat{\mathbf{y}} = \sin \theta \hat{\mathbf{r}} + \cos \theta \hat{\boldsymbol{\theta}}, \quad (10)$$

and,

$$\hat{\mathbf{r}} = \cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}}, \quad \hat{\boldsymbol{\theta}} = -\sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{y}}. \quad (11)$$

The current density (8) causes magnetic field \mathbf{H}_{cond} according to Ampère's law, $\nabla \times \mathbf{H} = (4\pi/c) \mathbf{J}_{\text{cond}}$,

$$\mathbf{H}_{\text{cond}} = \frac{2I}{c} \hat{\boldsymbol{\theta}} \begin{cases} \frac{r}{a^2} & (r < a), \\ \frac{1}{r} & (r > a). \end{cases} \quad (12)$$

The part of the field \mathbf{H} not due to \mathbf{J}_{cond} we label as \mathbf{H}_{ind} (for induced), which then obeys $\nabla \times \mathbf{H}_{\text{ind}} = 0$. Hence, we may deduce this part of the field as $\mathbf{H}_{\text{ind}} = -\nabla \Phi_{\text{ind}}$ from a scalar potential Φ_{ind} that obeys Laplace's equation, $\nabla^2 \Phi_{\text{ind}} = 0$.

The external field (9) can be regarded as due to the scalar potential,

$$\Phi_i = -H_0 x - \frac{H_1}{2} \frac{x^2 - y^2}{a} = -H_0 r \cos \theta - \frac{H_1}{2} \frac{r^2}{a} \cos 2\theta. \quad (13)$$

The external field induces additional terms in the scalar potential that also vary as $\cos \theta$ or $\cos 2\theta$, since these are two of the set of orthogonal functions in which the scalar potential $\Phi_{\text{ind}}(r, \theta)$ can be expanded. In particular, we can write,

$$\Phi_{\text{ind}} = \begin{cases} -H_0 r \cos \theta - \frac{H_1}{2} \frac{r^2}{a} \cos 2\theta + A_0 \frac{r}{a} \cos \theta + \frac{A_1}{2} \frac{r^2}{a^2} \cos 2\theta & (r < a), \\ -H_0 r \cos \theta - \frac{H_1}{2} \frac{r^2}{a} \cos 2\theta + A_0 \frac{a}{r} \cos \theta + \frac{A_1}{2} \frac{a^2}{r^2} \cos 2\theta & (r > a), \end{cases} \quad (14)$$

which is continuous at $r = a$. The induced fields obey the additional matching condition that the radial component $B_r = \mu H_r$ of the magnetic field is continuous at $r = a$ (since $\nabla \cdot \mathbf{B} = 0$). As we have different permeabilities μ' for $r < a$ and μ for $r > a$, the condition is that,

$$\mu \frac{\partial \Phi_{\text{ind}}(r = a^+)}{\partial r} = \mu' \frac{\partial \Phi_{\text{ind}}(r = a^-)}{\partial r}, \quad (15)$$

and hence,

$$\begin{aligned} & \mu \left(-H_0 \cos \theta - H_1 \cos 2\theta - \frac{A_0}{a} \cos \theta - \frac{A_1}{a} \cos 2\theta \right) \\ &= \mu' \left(-H_0 \cos \theta - H_1 \cos 2\theta + \frac{A_0}{a} \cos \theta + \frac{A_1}{a} \cos 2\theta \right). \end{aligned} \quad (16)$$

The equality holds separately for the coefficients of the orthogonal functions $\cos \theta$ and $\cos 2\theta$, so that,

$$A_{0,1} = \frac{\mu' - \mu}{\mu' + \mu} a H_{0,1}, \quad (17)$$

$$\Phi_{\text{ind}} = \begin{cases} -\frac{2\mu}{\mu'+\mu} \left(H_0 r \cos \theta + \frac{H_1 r^2}{2a} \cos 2\theta \right) & (r < a), \\ -H_0 \left(r - \frac{\mu'-\mu}{\mu'+\mu} \frac{a^2}{r} \right) \cos \theta - \frac{H_1}{2} \left(\frac{r^2}{a} - \frac{\mu'-\mu}{\mu'+\mu} \frac{a^3}{r^2} \right) \cos 2\theta & (r > a), \end{cases} \quad (18)$$

and,

$$\begin{aligned} \mathbf{H}_{\text{ind}} &= -\frac{\partial \Phi_{\text{ind}}}{\partial r} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial \Phi_{\text{ind}}}{\partial \theta} \hat{\boldsymbol{\theta}} \\ &= \begin{cases} \frac{2\mu}{\mu'+\mu} (H_0 \cos \theta + H_1 \frac{r}{a} \cos 2\theta) \hat{\mathbf{r}} - \frac{2\mu}{\mu'+\mu} (H_0 \sin \theta + H_1 \frac{r}{a} \sin 2\theta) \hat{\boldsymbol{\theta}} & (r < a), \\ \left[H_0 \left(1 + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^2}{r^2} \right) \cos \theta + H_1 \left(\frac{r}{a} + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^3}{r^3} \right) \cos 2\theta \right] \hat{\mathbf{r}} \\ - \left[H_0 \left(1 - \frac{\mu'-\mu}{\mu'+\mu} \frac{a^2}{r^2} \right) \sin \theta + H_1 \left(\frac{r}{a} - \frac{\mu'-\mu}{\mu'+\mu} \frac{a^3}{r^3} \right) \sin 2\theta \right] \hat{\boldsymbol{\theta}} & (r > a), \end{cases} \\ &= \begin{cases} \frac{2\mu}{\mu'+\mu} (H_0 + H_1 \frac{x}{a}) \hat{\mathbf{x}} - \frac{2\mu}{\mu'+\mu} H_1 \frac{y}{a} \hat{\mathbf{y}} & (r < a), \\ \left[H_0 \left(1 + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^2}{r^2} \cos 2\theta \right) + H_1 \left(\frac{x}{a} + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^3}{r^3} \cos 3\theta \right) \right] \hat{\mathbf{x}} \\ + \left[\frac{\mu'-\mu}{\mu'+\mu} H_0 \frac{a^2}{r^2} \sin 2\theta + H_1 \left(-\frac{y}{a} + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^3}{r^3} \sin 3\theta \right) \right] \hat{\mathbf{y}} & (r > a). \end{cases} \quad (19) \end{aligned}$$

The total magnetic field is the sum of eqs. (12) and (19),

$$\mathbf{H} = \begin{cases} \frac{2\mu}{\mu'+\mu} (H_0 \cos \theta + H_1 \frac{r}{a} \cos 2\theta) \hat{\mathbf{r}} + \left[\frac{2Ir}{ca^2} - \frac{2\mu}{\mu'+\mu} (H_0 \sin \theta + H_1 \frac{r}{a} \sin 2\theta) \right] \hat{\boldsymbol{\theta}} & (r < a), \\ \left[H_0 \left(1 + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^2}{r^2} \right) \cos \theta + H_1 \left(\frac{r}{a} + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^3}{r^3} \right) \cos 2\theta \right] \hat{\mathbf{r}} \\ + \left[\frac{2I}{cr} - H_0 \left(1 - \frac{\mu'-\mu}{\mu'+\mu} \frac{a^2}{r^2} \right) \sin \theta - H_1 \left(\frac{r}{a} - \frac{\mu'-\mu}{\mu'+\mu} \frac{a^3}{r^3} \right) \sin 2\theta \right] \hat{\boldsymbol{\theta}} & (r > a), \end{cases} \quad (20)$$

$$= \begin{cases} \left[\frac{2\mu}{\mu'+\mu} (H_0 + H_1 \frac{r}{a} \cos \theta) - \frac{2Ir}{ca^2} \sin \theta \right] \hat{\mathbf{x}} + \left(\frac{2Ir}{ca^2} \cos \theta - \frac{2\mu}{\mu'+\mu} H_1 \frac{r}{a} \sin \theta \right) \hat{\mathbf{y}} & (r < a), \\ \left[H_0 \left(1 + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^2}{r^2} \cos 2\theta \right) + H_1 \left(\frac{r}{a} \cos \theta + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^3}{r^3} \cos 3\theta \right) - \frac{2I}{cr} \sin \theta \right] \hat{\mathbf{x}} \\ + \left[\frac{\mu'-\mu}{\mu'+\mu} H_0 \frac{a^2}{r^2} \sin 2\theta + H_1 \left(-\frac{r}{a} \sin \theta + \frac{\mu'-\mu}{\mu'+\mu} \frac{a^3}{r^3} \sin 3\theta \right) + \frac{2I}{cr} \cos \theta \right] \hat{\mathbf{y}} & (r > a). \end{cases} \quad (21)$$

Of course,

$$\mathbf{B} = \begin{cases} \mu' \mathbf{H} & (r < a), \\ \mu \mathbf{H} & (r > a). \end{cases} \quad (22)$$

These forms obey the matching conditions that B_r and H_θ are continuous at the boundary $r = a$. Similarly, the magnetization is given by,

$$\mathbf{M} = \begin{cases} \frac{\mu'-1}{4\pi} \mathbf{H} & (r < a), \\ \frac{\mu-1}{4\pi} \mathbf{H} & (r > a). \end{cases} \quad (23)$$

2.2 Calculation of the Force via the Maxwell Stress Tensor

We calculate the force on unit length of the wire by integrating the Maxwell stress tensor over a cylindrical surface of radius $r > a$, so that any effects at the surface $r = a$ (both the outer surface of the wire at $r = a^-$ and the inner surface of the liquid at $r = a^+$) are included. The surface element at radius r is,

$$d\mathbf{S} = r d\theta \hat{\mathbf{r}} = r d\theta (\cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}}). \quad (24)$$

In rectangular coordinates, and for $r > a$ where the permeability is μ , the Maxwell stress tensor for the magnetic fields (ignoring magnetostriction) is,⁷

$$T_{ij} = \frac{1}{4\pi} \left(B_i H_j - \frac{\delta_{ij}}{2} \mathbf{B} \cdot \mathbf{H} \right) = \frac{\mu}{4\pi} \left(H_i H_j - \frac{\delta_{ij}}{2} H^2 \right) \quad (25)$$

We first calculate the x component of the force which is not expected to depend on the current I , so we drop terms in I^2 and IH that would eventually integrate to zero. Then,

$$\begin{aligned} F_x &= \int (T_{xx} dS_x + T_{xy} dS_y) = \frac{\mu r}{8\pi} \int_0^{2\pi} (H_x^2 - H_y^2) \cos \theta d\theta + \frac{\mu r}{4\pi} \int_0^{2\pi} H_x H_y \sin \theta d\theta \\ &= \frac{\mu r}{8\pi} \int_0^{2\pi} \left\{ \left[H_0 \left(1 + \frac{\mu' - \mu a^2}{\mu' + \mu r^2} \cos 2\theta \right) + H_1 \left(\frac{r}{a} \cos \theta + \frac{\mu' - \mu a^3}{\mu' + \mu r^3} \cos 3\theta \right) \right]^2 \right. \\ &\quad \left. - \left[\frac{\mu' - \mu}{\mu' + \mu} H_0 \frac{a^2}{r^2} \sin 2\theta + H_1 \left(-\frac{r}{a} \sin \theta + \frac{\mu' - \mu a^3}{\mu' + \mu r^3} \sin 3\theta \right) \right]^2 \right\} \cos \theta d\theta \\ &\quad + \frac{\mu r}{4\pi} \int_0^{2\pi} \left[H_0 \left(1 + \frac{\mu' - \mu a^2}{\mu' + \mu r^2} \cos 2\theta \right) + H_1 \left(\frac{r}{a} \cos \theta + \frac{\mu' - \mu a^3}{\mu' + \mu r^3} \cos 3\theta \right) \right] \\ &\quad \left[\frac{\mu' - \mu}{\mu' + \mu} H_0 \frac{a^2}{r^2} \sin 2\theta + H_1 \left(-\frac{r}{a} \sin \theta + \frac{\mu' - \mu a^3}{\mu' + \mu r^3} \sin 3\theta \right) \right] \sin \theta d\theta \\ &= \frac{\mu r}{8\pi} \int_0^{2\pi} \left[H_0^2 + 2H_0 H_1 \frac{r}{a} \cos \theta + H_1^2 \frac{r^2}{a^2} \cos^2 \theta \right. \\ &\quad \left. + 2 \frac{\mu' - \mu}{\mu' + \mu} \left(H_0 H_1 \frac{a}{r} \cos \theta + (H_0^2 + H_1^2) \frac{a^2}{r^2} \cos 2\theta + H_0 H_1 \frac{a^3}{r^3} \cos 3\theta \right) \right. \\ &\quad \left. + \left(\frac{\mu' - \mu}{\mu' + \mu} \right)^2 \left(H_0^2 \frac{a^4}{r^4} \cos 4\theta + 2H_0 H_1 \frac{a^5}{r^5} \cos 5\theta + H_1^2 \frac{a^6}{r^6} \cos 6\theta \right) \right] \cos \theta d\theta \\ &\quad + \frac{\mu r}{8\pi} \int_0^{2\pi} \left[-2H_0 H_1 \frac{r}{a} \sin \theta - H_1^2 \frac{r^2}{a^2} \sin^2 \theta \right. \\ &\quad \left. + 2 \frac{\mu' - \mu}{\mu' + \mu} \left(H_0 H_1 \frac{a}{r} \sin \theta + (H_0^2 + H_1^2) \frac{a^2}{r^2} \sin 2\theta + H_0 H_1 \frac{a^3}{r^3} \sin 3\theta \right) \right. \\ &\quad \left. + \left(\frac{\mu' - \mu}{\mu' + \mu} \right)^2 \left(H_0^2 \frac{a^4}{r^4} \sin 4\theta + 2H_0 H_1 \frac{a^5}{r^5} \sin 5\theta + H_1^2 \frac{a^6}{r^6} \sin 6\theta \right) \right] \sin \theta d\theta \end{aligned}$$

⁷This form differs slightly from that discussed by Maxwell in secs. 639-459 of his *Treatise* [25], and corresponds to the stress tensor for linear magnetic media, as apparently first deduced by Lorentz [33], starting from the (Lorentz) force density $\mathbf{f} = \rho \mathbf{E} + \mathbf{J}/c \times \mathbf{B}$.

$$= \frac{\mu' - \mu}{\mu' + \mu} \frac{\mu H_0 H_1 a}{4\pi} \int_0^{2\pi} d\theta + \frac{\mu H_0 H_1 r^2}{4\pi a} \int_0^{2\pi} \cos 2\theta d\theta = \frac{\mu' - \mu}{\mu' + \mu} \frac{a B_0 H_1}{2}. \quad (26)$$

We now calculate the y component of the force, which is expected to exhibit the $\mathbf{J} \times \mathbf{B}$ force (3) proportional to $I H_0$, so we drop terms in I^2 and H^2 that eventually would integrate to zero. Then,

$$\begin{aligned} F_y &= \int (T_{yx} dS_x + T_{yy} dS_y) = \frac{\mu r}{4\pi} \int_0^{2\pi} H_x H_y \cos \theta d\theta + \frac{\mu r}{8\pi} \int_0^{2\pi} (H_y^2 - H_x^2) \sin \theta d\theta \\ &= \frac{\mu r}{4\pi} \int_0^{2\pi} \left[H_0 \left(1 + \frac{\mu' - \mu}{\mu' + \mu} \frac{a^2}{r^2} \cos 2\theta \right) + H_1 \left(\frac{r}{a} \cos \theta + \frac{\mu' - \mu}{\mu' + \mu} \frac{a^3}{r^3} \cos 3\theta \right) - \frac{2I}{cr} \sin \theta \right] \\ &\quad \left[\frac{\mu' - \mu}{\mu' + \mu} H_0 \frac{a^2}{r^2} \sin 2\theta + H_1 \left(-\frac{r}{a} \sin \theta + \frac{\mu' - \mu}{\mu' + \mu} \frac{a^3}{r^3} \sin 3\theta \right) + \frac{2I}{cr} \cos \theta \right] \cos \theta d\theta \\ &+ \frac{\mu r}{8\pi} \int_0^{2\pi} \left\{ \left[\frac{\mu' - \mu}{\mu' + \mu} H_0 \frac{a^2}{r^2} \sin 2\theta + H_1 \left(-\frac{r}{a} \sin \theta + \frac{\mu' - \mu}{\mu' + \mu} \frac{a^3}{r^3} \sin 3\theta \right) + \frac{2I}{cr} \cos \theta \right]^2 \right. \\ &\quad \left. - \left[H_0 \left(1 + \frac{\mu' - \mu}{\mu' + \mu} \frac{a^2}{r^2} \cos 2\theta \right) + H_1 \left(\frac{r}{a} \cos \theta + \frac{\mu' - \mu}{\mu' + \mu} \frac{a^3}{r^3} \cos 3\theta \right) - \frac{2I}{cr} \sin \theta \right]^2 \right\} \sin \theta d\theta \\ &= \frac{\mu I}{2\pi c} \int_0^{2\pi} \left[H_0 \left(1 + \frac{\mu' - \mu}{\mu' + \mu} \frac{a^2}{r^2} \cos 2\theta \right) \cos^2 \theta + \frac{\mu' - \mu}{\mu' + \mu} H_1 \frac{a^3}{r^3} \cos 3\theta \cos^2 \theta \right. \\ &\quad \left. - \frac{H_0}{2} \frac{\mu' - \mu}{\mu' + \mu} \frac{a^2}{r^2} \sin^2 2\theta - \frac{\mu' - \mu}{\mu' + \mu} H_1 \frac{a^3}{r^3} \sin 3\theta \sin \theta \cos \theta \right] d\theta \\ &+ \frac{\mu I}{2\pi c} \int_0^{2\pi} \left[\frac{H_0}{2} \frac{\mu' - \mu}{\mu' + \mu} \frac{a^2}{r^2} \sin^2 2\theta + \frac{\mu' - \mu}{\mu' + \mu} H_1 \frac{a^3}{r^3} \sin 3\theta \sin \theta \cos \theta \right. \\ &\quad \left. + H_0 \left(1 + \frac{\mu' - \mu}{\mu' + \mu} \frac{a^2}{r^2} \cos 2\theta \right) \sin^2 \theta - \frac{\mu' - \mu}{\mu' + \mu} H_1 \frac{a^3}{r^3} \cos 3\theta \cos^2 \theta \right] d\theta \\ &= \frac{\mu I H_0}{2\pi c} \int_0^{2\pi} \left(1 + \frac{\mu' - \mu}{\mu' + \mu} \frac{a^2}{r^2} \cos 2\theta \right) d\theta = \frac{\mu I H_0}{c} = \frac{I B_0}{c}. \quad (27) \end{aligned}$$

The force is independent of the choice of the radius r , so long as $r > a$, is independent of the permeability μ' of the wire, and agrees with the simple expectation (3).

For the record, if we had integrated the stress tensor over a cylinder of radius $r < a$ the result would be $\mathbf{F} = 2\mu' I B_0 r^2 \hat{\mathbf{y}}/a^2(\mu' + \mu)$. Since the limit of this as $r \rightarrow a$ does not equal the result for $r > a$, we infer that there are important effects at the interface $r = a$. The permeable liquid is presumably contained in a tank of some characteristic radial scale $b \gg a$, at whose surface additional magnetization forces will arise. We consider these forces as distinct from those at the interface $r = a$, and that only the latter are part of the forces on the wire.

Equation (27) was deduced in a similar manner in ref. [1].

2.3 Calculation Using the Bulk Force Density

An expression for a bulk force density \mathbf{f} in magnetic media can be obtained by transformation of the surface integral of the stress tensor into a volume integral. See, for example, secs. 15

and 35 of [18]. The result, again ignoring magnetostriction, is,

$$\mathbf{f} = \frac{\mathbf{J}_{\text{cond}}}{c} \times \mathbf{B} - \frac{H^2}{8\pi} \nabla \mu, \quad (28)$$

which is due to Helmholtz [34].

In the present problem, $\nabla \mu = 0$ except across the surface $r = a$ that separates the wire of permeability μ' from the surrounding medium of permeability μ . Hence, the $\nabla \mu$ term of the volume integral of eq. (28) becomes a surface integral on the cylinder $r = a$. However, care is required in this procedure when H^2 is not continuous across this surface. Recalling that the tangential component H_t and the normal component $B_n = \mu H_n$ of the magnetic fields are continuous across a boundary, it is preferable to write $H^2 = H_t^2 + H_n^2 = H_t^2 + B_n^2/\mu^2$. Then,⁸

$$\int H^2 \nabla \mu \, d\text{Vol} = \int \left(H_t^2 + \frac{B_n^2}{\mu^2} \right) \frac{\partial \mu}{\partial n} \hat{\mathbf{n}} \, dS = \int \left(H_t^2 \frac{\partial \mu}{\partial n} - B_n^2 \frac{\partial(1/\mu)}{\partial n} \right) \hat{\mathbf{n}} \, dS, \quad (29)$$

and,

$$\mathbf{F} = \int \mathbf{f} \, d\text{Vol} = \int \frac{\mathbf{J}_{\text{cond}}}{c} \times \mathbf{B} \, d\text{Vol} - \frac{\mu - \mu'}{8\pi} \int H_t^2 \hat{\mathbf{n}} \, dS + \frac{1}{8\pi} \left(\frac{1}{\mu} - \frac{1}{\mu'} \right) \int B_n^2 \hat{\mathbf{n}} \, dS. \quad (30)$$

In contrast to the simple prescription given at the beginning of sec. 2, in this integral the magnetic field \mathbf{B} is the field on the current element $\mathbf{J}_{\text{cond}} \, d\text{Vol}$ from all sources, including those in element $d\text{Vol}$. For the second and third terms of eq. (30) where $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, we recall eq. (11) that for the x and y components we need only the parts of H_θ^2 and B_r^2 that vary as $\cos \theta$ and $\sin \theta$, respectively. From eq. (21) we find,

$$H_\theta^2 = 4B_0 H_1 \frac{\mu}{(\mu' + \mu)^2} \cos \theta - \frac{8}{\mu' + \mu} \frac{IB_0}{ca} \sin \theta + \dots, \quad (31)$$

$$B_r^2 = 4B_0 H_1 \frac{\mu \mu'^2}{(\mu' + \mu)^2} \cos \theta + \dots \quad (32)$$

Thus,

$$\begin{aligned} \mathbf{F} = & \frac{1}{c} \int_0^a r \, dr \int_0^{2\pi} d\theta \frac{I}{\pi a^2} \hat{\mathbf{z}} \times \\ & \mu' \left\{ \left[\frac{2}{\mu' + \mu} \left(B_0 + B_1 \frac{r}{a} \cos \theta \right) - \frac{2Ir}{ca^2} \sin \theta \right] \hat{\mathbf{x}} + \left(\frac{2Ir}{ca^2} \cos \theta - \frac{2}{\mu' + \mu} B_1 \frac{r}{a} \sin \theta \right) \hat{\mathbf{y}} \right\} \\ & - \frac{\mu - \mu'}{8\pi} \int_0^{2\pi} a \, d\theta \left(4B_0 H_1 \frac{\mu}{(\mu' + \mu)^2} \cos \theta \right) \cos \theta \hat{\mathbf{x}} \\ & - \frac{\mu - \mu'}{8\pi} \int_0^{2\pi} a \, d\theta \left(-\frac{8}{\mu' + \mu} \frac{IB_0}{ca} \sin \theta \right) \sin \theta \hat{\mathbf{y}} \\ & + \frac{\mu' - \mu}{8\pi \mu \mu'} \int_0^{2\pi} a \, d\theta \left(4B_0 H_1 \frac{\mu \mu'^2}{(\mu' + \mu)^2} \cos \theta \right) \cos \theta \hat{\mathbf{x}} \end{aligned}$$

⁸Thanks to J. Castro for pointing out this trick. It appears in eq. (7-21) of [35] when evaluating the electric integral $\int E^2 \nabla \epsilon \, d\text{Vol}$, and in Ex. 13.5 of [36] for the magnetic case.

$$\begin{aligned}
&= \frac{2\mu'}{\mu' + \mu} \frac{IB_0}{c} \hat{\mathbf{y}} - (\mu - \mu') \frac{aB_0H_1}{2} \frac{\mu}{(\mu' + \mu)^2} \hat{\mathbf{x}} + \frac{\mu - \mu'}{\mu' + \mu} \frac{IB_0}{c} \hat{\mathbf{y}} - (\mu - \mu') \frac{aB_0H_1}{2} \frac{\mu'}{(\mu' + \mu)^2} \hat{\mathbf{x}} \\
&= \frac{\mu' - \mu}{\mu' + \mu} \frac{aB_0H_1}{2} \hat{\mathbf{x}} + \frac{IB_0}{c} \hat{\mathbf{y}}.
\end{aligned} \tag{33}$$

This agrees with the calculation of the previous section via the stress tensor.

3 Use of Magnetization Currents or Effective Magnetic Poles

We now take a different approach to the calculation of the force on the wire, whereby the effects due to magnetization are included via either the magnetization current density $\mathbf{J}_M = c\nabla \times \mathbf{M}$ (and the associated surface current $K_M = c \mathbf{M} \times \hat{\mathbf{n}}$) or via the effective pole density $\rho_{M,\text{eff}} = -\nabla \cdot \mathbf{M}$ (and the associated surface pole density $\sigma_{M,\text{eff}} = \mathbf{M} \cdot \hat{\mathbf{n}}$).⁹ The wire is considered to consist of filaments along the z axis, and the total force is calculated as an integral over the force on the filaments in the spirit of the Biot-Savart force law.

The hope is that this approach would provide more intuitive explanations for the term $-(H^2/8\pi)\nabla\mu$ in the bulk force expression (28). However, we achieve success only with prescriptions that differ in an important way from those advocated by Brown [11] on behalf of the Coulomb Committee.

3.1 Calculation Using Magnetization Currents

In the first version of this calculation, we include the so-called bound current density due to the bulk magnetization \mathbf{M} ,

$$\mathbf{J}_M = c\nabla \times \mathbf{M} = \frac{\mu' - 1}{4\pi/c} \nabla \times \mathbf{H} = (\mu' - 1)\mathbf{J}_{\text{cond}}, \tag{34}$$

since the magnetic field \mathbf{H} is related to the conduction current density \mathbf{J}_{cond} by Ampère's law, $\nabla \times \mathbf{H} = 4\pi\mathbf{J}_{\text{cond}}/c$ in magnetostatics. Thus, the total current in the interior of the wire is,

$$\mathbf{J}_{\text{total}} = \mathbf{J}_M + \mathbf{J}_{\text{cond}} = \mu'\mathbf{J}_{\text{cond}}. \tag{35}$$

The current density $\mathbf{J}_{\text{total}}$ is the one that should be used in the microscopic version of Ampère's law $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}_{\text{total}}$, which leads to $\mathbf{B}(I) = 2\mu'Ir \hat{\boldsymbol{\theta}}/ca^2 = \mu'\mathbf{H}(I)$ inside the wire. Since eq. (35) by itself also implies that $\mathbf{B}(I) = 2\mu'I \hat{\boldsymbol{\theta}}/cr$ outside the wire, we see that in addition to the volume magnetization current density \mathbf{J}_M , there must be surface currents at $r = a$. On the outer surface of the wire the current density is given by,

$$\begin{aligned}
\mathbf{K}_M(r = a^-) = c \mathbf{M}(r = a^-) \times \hat{\mathbf{r}} &= c \frac{\mu' - 1}{4\pi} H_\theta(r = a^-) \hat{\boldsymbol{\theta}} \times \hat{\mathbf{r}} \\
&= -\frac{\mu' - 1}{4\pi} \left[\frac{2I}{a} - \frac{2\mu}{\mu' + \mu} c(H_0 \sin \theta + H_1 \sin 2\theta) \right] \hat{\mathbf{z}},
\end{aligned} \tag{36}$$

⁹The effective magnetic poles are a representation of effects of Ampèrian currents, and are distinct from possible "true" (Gilbertian) magnetic poles that apparently do not exist in Nature.

3.2 Calculation Using Effective Magnetic Poles

The forces on the magnetization of the media might also be considered as due to a density of effective magnetic poles, rather than being due to currents \mathbf{J}_M and \mathbf{K}_M . Some care is required to use this approach, since a true magnetic pole density ρ_M would imply $\nabla \cdot \mathbf{B} = 4\pi\rho_M$, and the bulk force density on these poles would be $\mathbf{F} = \rho_M\mathbf{H}$.¹⁰ However, in reality $0 = \nabla \cdot \mathbf{B} = \nabla \cdot (\mathbf{H} + 4\pi\mathbf{M})$, so we can write,

$$\nabla \cdot \mathbf{H} = -4\pi\nabla \cdot \mathbf{M} = 4\pi\rho_{M,\text{eff}}, \quad (41)$$

and we identify $\rho_{M,\text{eff}} = -\nabla \cdot \mathbf{M}$ as the volume density of effective magnetic poles. Inside linear magnetic media, such as those considered here, $\mathbf{B} = \mu'\mathbf{H}$ and $\nabla \cdot \mathbf{B} = 0$ together imply that $\rho_{M,\text{eff}} = 0$. However, a surface density $\sigma_{M,\text{eff}}$ of effective poles can exist on an interface between two media, and we see that Gauss' law for the field \mathbf{H} implies that,

$$\sigma_{M,\text{eff}} = \frac{(\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{n}}}{4\pi}, \quad (42)$$

where unit normal $\hat{\mathbf{n}}$ points across the interface from medium 1 to medium 2. The surface pole density can also be written in terms of the magnetization $\mathbf{M} = (\mathbf{B} - \mathbf{H})/4\pi$ as,

$$\sigma_{M,\text{eff}} = (\mathbf{M}_1 - \mathbf{M}_2) \cdot \hat{\mathbf{n}}, \quad (43)$$

since $\nabla \cdot \mathbf{B} = 0$ insures that the normal component of \mathbf{B} is continuous at the interface.

In the present problem, the density of effective magnetic poles on the surface $r = a$ is given by,

$$\sigma_{M,\text{eff}} = \frac{H_r(r = a^+) - H_r(r = a^-)}{4\pi} = \frac{1}{2\pi} \frac{\mu' - \mu}{\mu' + \mu} (H_0 \cos \theta + H_1 \cos 2\theta). \quad (44)$$

The force on the surface density of effective magnetic poles is,

$$\mathbf{F} = \sigma_{M,\text{eff}}\mathbf{B}(r = a), \quad (45)$$

noting that the effective (ampèrian) poles couple to \mathbf{B} rather than to \mathbf{H} .^{11,12}

¹⁰See [16] for additional discussion of true and effective magnetic charges.

¹¹Poisson [38] worked exclusively with the magnetic field \mathbf{H} , but realized that the effective force on a true (Gilbertian) magnetic pole p is not necessarily $\mathbf{F} = p\mathbf{H}$ if the pole is at rest inside a bulk medium, which results in an altered force on the pole depending on the assumed shape of the surrounding cavity. W. Thomson (Lord Kelvin) noted in 1850 [39] that for a pole in a disk-shaped cavity with axis parallel to the magnetization \mathbf{M} of the medium, the force would be $\mathbf{F} = p(\mathbf{H} + 4\pi\mathbf{M})$, and therefore he introduced the magnetic field $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ "according to the electromagnetic definition" (in Gaussian units). In sec. 400 of his *Treatise* [25], Maxwell follows Thomson in stating that the effective force on a true magnetic pole is usefully considered to be $\mathbf{F} = p\mathbf{B}$ (Gaussian units). This convention for the effective force on a true (Gilbertian) magnetic pole is the same as the "true" force on an effective (Ampèrian) magnetic pole, which latter is the topic of the sec. 3.2 of this note.

¹²We note that Brown in his eq. (1.3-3) of [11] recommends that the initial field \mathbf{H}_i be used rather than \mathbf{B}_i when using the method of effective magnetic poles. However, this would imply a force $1/\mu$ times that of eq. (45).

The total force on the medium in this view is the sum of the force on the conduction current plus the force on the effective surface poles, where to avoid calculating a spurious force of the wire on itself we use the initial magnetic field \mathbf{B}_i ,

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}_{\text{cond}} \times \mathbf{B}_i \, d\text{Vol} + \int \sigma_{M,\text{eff}} \mathbf{B}_i(r=a) \, dS. \quad (46)$$

Then,

$$\begin{aligned} \mathbf{F} &= \frac{1}{c} \int_0^a r \, dr \int_0^{2\pi} d\theta \frac{I}{\pi a^2} \hat{\mathbf{z}} \times B_0 \hat{\mathbf{x}} \\ &\quad + \int_0^{2\pi} a \, d\theta \frac{1}{2\pi} \frac{\mu' - \mu}{\mu' + \mu} (H_0 \cos \theta + H_1 \cos 2\theta) \cdot [(B_0 + B_1 \cos \theta) \hat{\mathbf{x}} - B_1 \sin \theta \hat{\mathbf{y}}] \\ &= \frac{\mu' - \mu}{\mu' + \mu} \frac{a H_0 B_1}{2} \hat{\mathbf{x}} + \frac{I B_0}{c} \hat{\mathbf{y}}, \end{aligned} \quad (47)$$

in agreement with eqs. (26) and (27), since $H_0 B_1 = B_0 H_1$.

3.3 The Biot-Savart Force Law in a Permeable Medium

The results of secs. 3.1 and 3.2 show that care is needed when using the Biot-Savart force law in permeable media. We review this issue by starting with the simpler case that the wire and the surrounding liquid both have the same permeability $\mu \neq 1$. Then, there is neither a surface current nor an effective pole density at the interface between the wire and the liquid. However, there remains the volume current density $\mathbf{J}_M = (\mu - 1)\mathbf{J}_{\text{cond}}$, so that the total current is still $\mathbf{J}_{\text{total}} = \mu\mathbf{J}_{\text{cond}}$. Since the force on the wire is correctly calculated via the force law,

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}_{\text{cond}} \times \mathbf{B}_i \, d\text{Vol}, \quad (48)$$

using the conduction current, we see that if we wish to use the total current we must write,

$$\mathbf{F} = \frac{1}{c} \int \frac{\mathbf{J}_{\text{total}}}{\mu} \times \mathbf{B}_i \, d\text{Vol} = \frac{1}{c} \int \mathbf{J}_{\text{total}} \times \mathbf{H}_i \, d\text{Vol}. \quad (49)$$

The other aspect of the analysis of Biot and Savart is the calculation of the magnetic field from the current density. The microscopic version of Ampère's law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}_{\text{total}}, \quad (50)$$

corresponds to the prescription that,

$$\mathbf{B} = \frac{1}{c} \int \frac{\mathbf{J}_{\text{total}} \times \hat{\mathbf{r}}}{r^2} \, d\text{Vol} = \frac{\mu}{c} \int \frac{\mathbf{J}_{\text{cond}} \times \hat{\mathbf{r}}}{r^2} \, d\text{Vol} = \mu \mathbf{H}. \quad (51)$$

Hence, the macroscopic version of Ampère's law,

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{\text{cond}}, \quad (52)$$

corresponds to the prescription that,

$$\mathbf{H} = \frac{1}{c} \int \frac{\mathbf{J}_{\text{cond}} \times \hat{\mathbf{r}}}{r^2} d\text{Vol}, \quad (53)$$

independent of the permeability.

The form of the eq. (2) for the force on circuit a due to circuit b supposing the wires and the surrounding media all have permeability μ is therefore,

$$\mathbf{F}_a = \frac{I_a}{c} \oint_a d\mathbf{l}_a \times \mathbf{B}_b = \mu \frac{I_a I_b}{c^2} \oint_a d\mathbf{l}_a \times \oint_b \frac{d\mathbf{l}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2}, \quad (54)$$

where I_a and I_b are the conduction currents in the circuits. We also see that eq. (54) holds even if the wires have permeabilities μ_a and μ_b that differ from the permeability μ of the surrounding medium, since the magnetic field due to wire b at the position of wire a before wire a was introduced is given by $\mathbf{B}_b = \mu \mathbf{H}_b$, which depends on neither μ_a nor μ_b .

An extensive bibliography on conceptual issues in magnetism is at [40].

4 There is No General Macroscopic Version of the Lorentz Force Law Using Total Fields

The Lorentz Force law for a moving electric charge q in microscopic electrodynamics has the well-known form,¹³

$$\mathbf{F} = q \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \quad (\text{microscopic}). \quad (55)$$

This suggests that in macroscopic electrodynamics the force density on charge and currents densities ρ and \mathbf{J} might be written as,

$$\mathbf{F} \stackrel{?}{=} \rho \mathbf{E} + \frac{\mathbf{J}}{c} \times \mathbf{B} \quad (\text{macroscopic}), \quad (56)$$

where here the total fields \mathbf{E} and \mathbf{B} are the macroscopic averages of the corresponding microscopic fields, and the charge and current densities might be either the “free” or “total”,

$$\rho_{\text{total}} = \rho_{\text{free}} + \rho_{\text{bound}} = \rho_{\text{free}} - \nabla \cdot \mathbf{P}, \quad \mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{bound}} = \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{P}}{\partial t} + c \nabla \times \mathbf{M}. \quad (57)$$

However, the present example shows that the supposed macroscopic form (56) is not generally valid if the total fields \mathbf{E} and \mathbf{B} are used.¹⁴

¹³It is generally considered that Heaviside first gave the Lorentz force law (28) for electric charges in [42], but the key insight is already visible for the electric case in [41] (and for the magnetic case in [43]). Lorentz himself seems to have advocated the form $q\mathbf{v} \times \mu_0 \mathbf{H}$ in eq. (V), sec. 12 of [33]. See also eq. (23) of [44]. Maxwell gave a version of the Lorentz force law in eq. (10), Art. 599 of [25], where he wrote $\mathbf{F}/q = \mathbf{v} \times \mathbf{B} - \nabla \phi - \partial \mathbf{A}/\partial t$, but he made little use of this elsewhere.

¹⁴Another example of the delicacy in the use of a macroscopic Lorentz force law concerns the torque on a magnetized object that is immersed in a permeable liquid, all in a magnetic field [45].

To see this, it suffices to consider only the case where the external field, in the medium with relative permeability μ , has the form,

$$\mathbf{B}_i = B_0 \hat{\mathbf{x}} = \mu H_0 \hat{\mathbf{x}}. \quad (58)$$

Then, from eq. (20) with $B_0 = \mu H_0$ and $H_1 = 0$ the total magnetic field inside the wire (of relative permeability μ') is,

$$\begin{aligned} \mathbf{B}_{\text{total}}(r < a) &= \mu' \mathbf{H}_{\text{total}}(r < a) = \frac{2\mu'}{\mu' + \mu} B_0 \cos \theta \hat{\mathbf{r}} + \left(\frac{2\mu' I r}{ca^2} - \frac{2\mu'}{\mu' + \mu} B_0 \sin \theta \right) \hat{\boldsymbol{\theta}} \\ &= \left(\frac{2\mu'}{\mu' + \mu} B_0 - \frac{2\mu' I y}{ca^2} \right) \hat{\mathbf{x}} + \frac{2\mu' I x}{ca^2} \hat{\mathbf{y}} = \frac{2\mu' I r}{ca^2} \hat{\boldsymbol{\theta}} + \frac{2\mu'}{\mu' + \mu} B_0 \hat{\mathbf{x}}. \end{aligned} \quad (59)$$

Thus,

$$\int_{r < a} \frac{\mathbf{J}_{\text{free}}}{c} \times \mathbf{B}_{\text{total}} d\text{Vol} = \frac{I}{\pi a^2 c} \int \hat{\mathbf{z}} \times \mathbf{B}_{\text{total}} d\text{Vol} = \frac{2\mu'}{\mu' + \mu} \frac{I B_0}{c} \hat{\mathbf{c}} = \frac{2\mu'}{\mu' + \mu} \mathbf{F}_{\text{wire}}. \quad (60)$$

Hence, the candidate macroscopic force law,

$$\mathbf{F} \stackrel{?}{=} \rho_{\text{free}} \mathbf{E}_{\text{total}} + \frac{\mathbf{J}_{\text{free}}}{c} \times \mathbf{B}_{\text{total}} \quad (\text{macroscopic}), \quad (61)$$

is not valid for the present example if either μ or μ' differ from unity or from each other.

To consider the candidate macroscopic force law,

$$\mathbf{F} \stackrel{?}{=} \rho_{\text{total}} \mathbf{E}_{\text{total}} + \frac{\mathbf{J}_{\text{total}}}{c} \times \mathbf{B}_{\text{total}} \quad (\text{macroscopic}), \quad (62)$$

we note that the magnetization inside the wire is given by,

$$\begin{aligned} \mathbf{M}_{\text{wire}}(r < a) &= \frac{\mu' - 1}{4\pi} \mathbf{H}_{\text{total}}(r < a) = \frac{\mu' - 1}{4\pi\mu'} \mathbf{B}_{\text{total}}(r < a) \\ &= \frac{\mu' - 1}{2\pi} \left[\frac{B_0 \cos \theta}{\mu' + \mu} \hat{\mathbf{r}} + \left(\frac{I r}{ca^2} - \frac{B_0 \sin \theta}{\mu' + \mu} \right) \hat{\boldsymbol{\theta}} \right], \end{aligned} \quad (63)$$

and since $\nabla \times \mathbf{B}_i = 0$ inside the wire the bound current density associated with the wire is,

$$\mathbf{J}_{\text{bound}}(r < a) = c \nabla \times \mathbf{M} = \frac{\mu' - 1}{\pi a^2} I \hat{\mathbf{z}}, \quad \text{so} \quad \mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{bound}} = \frac{\mu' I}{\pi a^2} \hat{\mathbf{z}} = \mu' \mathbf{J}_{\text{free}}. \quad (64)$$

In addition, there is a bound current density on the surface of the wire,¹⁵ as in eq. (36) with $H_1 = 0$,

$$\mathbf{K}_{\text{bound}}(r = a^-) = c \mathbf{M}(r = a^-) \times \hat{\mathbf{r}} = \frac{\mu' - 1}{2\pi} \left(\frac{c B_0 \sin \theta}{\mu' + \mu} - \frac{I}{a} \right) \hat{\mathbf{z}}. \quad (65)$$

When evaluating the force on the surface current $\mathbf{K}_{\text{bound}}$ we should not use the magnetic field $\mathbf{B}_{\text{total}}(r = a^-)$, since in the microscopic view the magnetic field varies continuously

¹⁵There is also a bound surface current density on the inner surface of the medium at $r > a$.

(and approximately linearly) with r across the surface current layer, which has a small but finite thickness. Considering a loop in a plane of constant z , with azimuthal length dl and infinitesimal thickness in r from $r = a^-$ to $r = a$, which surrounds an element of the surface current on the wire, we have that,

$$\begin{aligned} \frac{4\pi}{c} \int \mathbf{K}_{\text{bound}}(r = a^-) \cdot d\mathbf{Area} &= \frac{4\pi}{c} K_{\text{bound}}(r = a^-) dl \\ &= \oint \mathbf{B}_{\text{total}} \cdot d\mathbf{l} = [B_{\theta, \text{total}}(r = a) - B_{\theta, \text{total}}(r = a^-)] dl, \end{aligned} \quad (66)$$

such that,

$$B_{\theta, \text{total}}(r = a) = B_{\theta, \text{total}}(r = a^-) + \frac{4\pi}{c} K_{\text{bound}}(r = a^-). \quad (67)$$

Of course, $B_{r, \text{total}}(r = a) = B_{r, \text{total}}(r = a^-)$, so the effective magnetic field which acts on the surface current is,

$$\begin{aligned} \mathbf{B}_{\text{total,eff}}(r = a^-) &= \frac{\mathbf{B}_{\text{total}}(r = a^-) + \mathbf{B}_{\text{total}}(r = a)}{2} = \mathbf{B}_{\text{total}}(r = a^-) + \frac{2\pi}{c} K_{\text{bound}}(r = a^-) \boldsymbol{\theta} \\ &= \frac{2\mu' B_0}{\mu' + \mu} \hat{\mathbf{x}} + \left[\frac{2\mu' I}{ca} + (\mu' - 1) \left(\frac{B_0 \sin \theta}{\mu' + \mu} - \frac{I}{ca} \right) \right] \boldsymbol{\theta} \\ &= \frac{2\mu' B_0}{\mu' + \mu} \hat{\mathbf{x}} + \left(\frac{(\mu' + 1)I}{ca} + \frac{(\mu' - 1)B_0 \sin \theta}{\mu' + \mu} \right) \boldsymbol{\theta}. \end{aligned} \quad (68)$$

Then, the candidate force on the total currents at $r < a$ is,

$$\begin{aligned} &\int_{r < a} \frac{\mathbf{J}_{\text{total}}}{c} \times \mathbf{B}_{\text{total}} d\text{Vol} + \int_{r = a^-} \frac{\mathbf{K}_{\text{total}}}{c} \times \mathbf{B}_{\text{total,eff}} d\text{Area} \\ &= \int_{r < a} \frac{\mu' \mathbf{J}_{\text{free}}}{c} \times \mathbf{B}_{\text{total}} d\text{Vol} + \int_{r = a^-} \frac{\mathbf{K}_{\text{bound}}}{c} \times \mathbf{B}_{\text{total,eff}} d\text{Area} \\ &= \frac{2\mu'^2}{\mu' + \mu} \frac{IB_0 \hat{\mathbf{y}}}{c} - \frac{2\mu'^2 - \mu' - 1}{\mu' + \mu} \frac{IB_0 \hat{\mathbf{y}}}{c} = \frac{\mu' + 1}{\mu' + \mu} \frac{IB_0 \hat{\mathbf{y}}}{c}, \end{aligned} \quad (69)$$

recalling eq. (60).

It is also reasonable to suppose that the total current should include the surface current at $r = a^+$ on the inner surface of the permeable liquid that surrounds the wire. From eq. (37) with $H_1 = 0$,

$$\mathbf{K}_{\text{bound}}(r = a^+) = c \mathbf{M}(r = a^+) \times (-\hat{\mathbf{r}}) = \frac{\mu - 1}{2\pi} \left[\frac{I}{a} - \frac{cB_0 \sin \theta}{\mu' + \mu} \right] \hat{\mathbf{z}}. \quad (70)$$

An argument similar to that which led to eq. (67) indicates that,

$$B_{\theta, \text{total}}(r = a) = B_{\theta, \text{total}}(r = a^+) - \frac{4\pi}{c} K_{\text{bound}}(r = a^+). \quad (71)$$

Of course, $B_{r,\text{total}}(r = a) = B_{r,\text{total}}(r = a^-)$, so the effective magnetic field which acts on the surface current is,

$$\begin{aligned}\mathbf{B}_{\text{total,eff}}(r = a^+) &= \frac{\mathbf{B}_{\text{total}}(r = a^+) + \mathbf{B}_{\text{total}}(r = a)}{2} \\ &= \mathbf{B}_{\text{total}}(r = a^+) - \frac{2\pi}{c} K_{\text{bound}}(r = a^+) \boldsymbol{\theta}, \\ B_{\text{total,eff},x}(r = a^+) &= B_0 \left(1 + \frac{\mu' - \mu}{\mu' + \mu} \cos 2\theta \right) - \left[\frac{(\mu + 1)I}{ca} + \frac{(\mu - 1)B_0 \sin \theta}{\mu' + \mu} \right] \sin \theta.\end{aligned}\tag{72}$$

The force on the surface current $\mathbf{K}_{\text{bound}}(r = a^+)$ has only a y -component, given by,

$$\begin{aligned}\int_{r=a^+} \frac{\mathbf{K}_{\text{total}}}{c} \times \mathbf{B}_{\text{total,eff}} d\text{Area} \Big|_y &= a \int_{r=a^+} \frac{K_{\text{bound}}(r = a^+)}{c} B_{\text{total,eff},x}(r = a^+) d\theta \\ &= (\mu - 1) \left(1 + \frac{1}{\mu' + \mu} \right) \frac{IB_0}{c}.\end{aligned}\tag{73}$$

Adding this to the result of eq. (69), the candidate force on the total current is,

$$\frac{\mu IB_0}{c} \hat{\mathbf{y}}.\tag{74}$$

Hence, the candidate form (62) is also not valid for the present example if μ differs from unity.^{16,17}

We have seen in previous sections that the magnetic force on a wire of relative permeability μ' can be written as,

$$\mathbf{F} = \int \frac{\mathbf{J}_{\text{free}}}{c} \times \mathbf{B}_i d\text{Vol} = \int \frac{\mathbf{J}_{\text{total}}}{c} \times \mathbf{H}_i d\text{Vol},\tag{75}$$

where $\mathbf{B}_i = \mu' \mathbf{H}_i$ are the fields that would exist at the location of the wire in its absence. It is clear from the results in this section that the force on such a wire cannot be written as,

$$\begin{aligned}\int \frac{\mathbf{J}_{\text{free}}}{c} \times \mathbf{H}_{\text{total}} d\text{Vol}, & \quad \text{nor as} \quad \int \frac{\mathbf{J}_{\text{total}}}{c} \times \mathbf{H}_{\text{total}} d\text{Vol}, \quad \text{nor as} \quad \int \frac{\mathbf{J}_{\text{free}}}{c} \times \mathbf{H}_i d\text{Vol}, \\ \text{nor as} \quad \int \frac{\mathbf{J}_{\text{free}}}{c} \times \mathbf{B}_{\text{total}} d\text{Vol}, & \quad \text{nor as} \quad \int \frac{\mathbf{J}_{\text{total}}}{c} \times \mathbf{B}_i d\text{Vol}.\end{aligned}\tag{76}$$

¹⁶If the medium surrounding the wire has unit relative permeability, $\mu = 1$, then the form (62) is still valid. This has given validation to that form for the unwary, as in [5] and prob. 13.14 of [36]. This reinforces the theme of [3] that one must consider examples with two different, nonunit relative permeabilities to confront the full subtlety of magnetic force calculations.

¹⁷The Einstein-Laub force density [5] appears to be derived from the force law (62) according to Appendix B of [46], and hence is not valid in general.

4.1 Radial Gap between the Wire and the Surrounding Permeable Medium

To illustrate further the possible applicability of the candidate force density (62) when permeable media are present, we consider the case of a wire of relative permeability μ' and radius a along the z -axis, surrounded by a medium of relative permeability μ in the region $r > b$. The region $a < r < b$ is vacuum. The initial/external magnetic field \mathbf{B}_i , in the absence of the wire is $B_0 \hat{\mathbf{x}}$ for $r < b$, so we might again expect the force on the wire to be given by eq. (3) when it carries current \mathbf{I} .

At large r the initial/external field has the form,

$$\mathbf{B}_i = B_\infty \hat{\mathbf{x}}, \quad \mathbf{H}_i = \frac{B_\infty}{\mu} \hat{\mathbf{x}} = -\nabla \left(-\frac{B_\infty}{\mu} x \right) = -\nabla \left(-\frac{B_\infty}{\mu} r \cos \theta \right), \quad (77)$$

where the relation between B_∞ and B_0 is to be determined such that as $b \rightarrow a$ the fields become those of eq. (59).

To use the candidate force density (62) we need to know $\mathbf{B}_{\text{total}}$ when the wire is present and carrying current I . We begin by computing the magnetic fields \mathbf{B}_j and \mathbf{H}_j when the wire is present but with zero current.

When the free current in the wire is zero, $\nabla \times \mathbf{H}_j = 0$ (except at the distant location of the sources of the initial field), so we can write $\mathbf{H}_j = -\nabla \Phi_M$, where Φ_M is a (continuous) magnetic scalar potential of the form,

$$\Phi_M(r < a) = A \frac{r}{a} \cos \theta, \quad (78)$$

$$\Phi_M(a < r < b) = B \frac{r}{a} \cos \theta + C \frac{a}{r} \cos \theta, \quad (79)$$

$$\Phi_M(r > b) = D \frac{a}{r} \cos \theta - \frac{B_\infty}{\mu} r \cos \theta. \quad (80)$$

Continuity of Φ_M at $r = a$ and b implies that,

$$A = B + C, \quad \text{and} \quad D = \frac{b^2}{a^2} B + C + \frac{b^2 B_\infty}{\mu a}. \quad (81)$$

Continuity of the radial component of \mathbf{B}_j at $r = a$ and b implies that,

$$\mu' \frac{\partial \Phi_M(r = a^-)}{\partial r} = \frac{\partial \Phi_M(r = a^+)}{\partial r}, \quad \text{and} \quad \frac{\partial \Phi_M(r = b^-)}{\partial r} = \mu \frac{\partial \Phi_M(r = b^+)}{\partial r}, \quad (82)$$

and hence,

$$\mu' A = B - C, \quad \text{and} \quad B - \frac{a^2}{b^2} C = -\mu \frac{a^2}{b^2} D - a B_\infty. \quad (83)$$

After some algebra,

$$B = \frac{\mu' + 1}{2} A, \quad C = -\frac{\mu' - 1}{2} A, \quad D = \frac{A}{2a^2} [\mu'(b^2 - a^2) + b^2 + a^2],$$

$$\frac{A}{a} = -\frac{4b^2 B_\infty}{(\mu + \mu')(b^2 + a^2) + (1 + \mu\mu')(b^2 - a^2)} = -\frac{4b^2 B_\infty}{(\mu' + 1)(\mu + 1)b^2 - (\mu' - 1)(\mu - 1)a^2}. \quad (84)$$

As $b \rightarrow a$ the magnetic field at $r < a$ with the wire present but with zero current is,

$$\mathbf{B}_j(r < a) = \mu' \mathbf{H}_j(r < a) = -\frac{\mu' A(b \rightarrow a)}{a} \hat{\mathbf{x}} = \frac{2\mu' B_\infty}{\mu' + \mu} \hat{\mathbf{x}}. \quad (85)$$

For this limit to agree with eq. (59) we simply take $B_\infty = B_0$, and the magnetic field for $b > a$ with the wire present but with zero current is,¹⁸

$$\mathbf{B}_j(r < a) = \mu' \mathbf{H}_j(r < a) = -\frac{\mu' A}{a} \hat{\mathbf{x}} = \frac{4\mu' B_0}{(\mu' + 1)(\mu + 1) - (\mu' - 1)(\mu - 1)a^2/b^2} \hat{\mathbf{x}}. \quad (86)$$

The expression for the magnetic field due to the current I in the wire is the same for $r < a$ in the present case and in eq. (59), so formally the total field \mathbf{B} for $r < a$ in the present case differs from the previous case only by an additional factor (which goes to 1 as $b \rightarrow a$),

$$\frac{2(\mu' + \mu)}{(\mu' + 1)(\mu + 1) - (\mu' - 1)(\mu - 1)a^2/b^2}, \quad (87)$$

that multiplies terms in B_0 (in \mathbf{B} and in $\mathbf{K}_{\text{bound}}$). As such, we can immediately transcribe the result (69) for the force per unit length on the wire (but not on the medium outside the wire) computed with the candidate force density (62) as,

$$\begin{aligned} \mathbf{F}_{\text{wire}} &= \frac{\mu' + 1}{\mu' + \mu} \frac{IB_0 \hat{\mathbf{y}}}{c} \frac{2(\mu' + \mu)}{(\mu' + 1)(\mu + 1) - (\mu' - 1)(\mu - 1)a^2/b^2} \\ &= \frac{2(\mu' + 1)}{(\mu' + 1)(\mu + 1) - (\mu' - 1)(\mu - 1)a^2/b^2} \frac{IB_0 \hat{\mathbf{y}}}{c}. \end{aligned} \quad (88)$$

For confirmation of eq. (88), we consider the force per unit length on the wire as deduced from the Maxwell stress tensor just outside the surface of the wire, where $r = a^+$. The total field here can be obtained from the total field just inside the wire, at $r = a^-$, by recalling that the radial component of \mathbf{B} and the tangential component of \mathbf{H} are continuous across the surface of the wire. Hence,

$$\begin{aligned} \mathbf{B}(r = a^+) &= \mathbf{H}(r = a^+) = -\frac{\mu' A}{a} \hat{\mathbf{r}} \cos \theta + \frac{A}{a} \hat{\boldsymbol{\theta}} \sin \theta + \frac{2I}{ca} \hat{\boldsymbol{\theta}} \\ &= -\left(\frac{A}{2a} [\mu' + 1 + (\mu' - 1) \cos 2\theta] + \frac{2I}{ca} \sin \theta \right) \hat{\mathbf{x}} + \left(-(\mu' - 1) \frac{A}{2a} \sin 2\theta + \frac{2I}{ca} \cos \theta \right) \hat{\mathbf{y}}. \end{aligned} \quad (89)$$

We expect only a nonzero y -component to the force on the wire, which is computed at $r = a^+$ via the Maxwell stress tensor as in eq. (27),

$$\begin{aligned} F_y &= \int (T_{yx} dS_x + T_{yy} dS_y) = \frac{a}{4\pi} \int_0^{2\pi} H_x H_y \cos \theta d\theta + \frac{a}{8\pi} \int_0^{2\pi} (H_y^2 - H_x^2) \sin \theta d\theta \\ &= \frac{a}{4\pi} \int_0^{2\pi} \left[\frac{A}{2a} [\mu' + 1 + (\mu' - 1) \cos 2\theta] + \frac{2I}{ca} \sin \theta \right] \end{aligned}$$

¹⁸If $\mu' = 1$ then eq. (86) becomes the initial field in the absence of the wire, $\mathbf{B}_i(r < a) = 2B_0 \hat{\mathbf{x}}/(\mu + 1)$, whose strength differs from the initial field B_0 when the permeability is μ in the absence of the wire.

$$\begin{aligned}
& \left[-(\mu' - 1) \frac{A}{2a} \sin 2\theta + \frac{2I}{ca} \cos \theta \right] \cos \theta d\theta \\
+ & \frac{a}{8\pi} \int_0^{2\pi} \left\{ \left[-(\mu' - 1) \frac{A}{2a} \sin 2\theta + \frac{2I}{ca} \cos \theta \right]^2 \right. \\
& \left. - \left[\frac{A}{2a} [\mu' + 1 + (\mu' - 1) \cos 2\theta] + \frac{2I}{ca} \sin \theta \right]^2 \right\} \sin \theta d\theta \tag{90} \\
= & \frac{AI(\mu' + 1)}{2\pi ca} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{AI(\mu' + 1)}{2ca} = \frac{2(\mu' + 1)}{(\mu' + 1)(\mu + 1) - (\mu' - 1)(\mu - 1)a^2/b^2} \frac{IB_0}{c},
\end{aligned}$$

which agrees with the candidate expression (88) for the force on the wire.¹⁹

However, the force (88) and (90) is not of the form $\mathbf{I} \times \mathbf{B}_i/c$, as the initial field in the absence of the wire is $\mathbf{B}_i(r < a) = 2B_0 \hat{\mathbf{x}}/(\mu + 1)$. Thus it appears that the prescription (3) is not true in general, as anticipated in footnote 3.

While the macroscopic Lorentz force “law” (62) does not hold in general, the present considerations do not exclude that this form is valid for computation of the force on charges and currents that are surrounded by vacuum.

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