5   The Physics of Rotating Bodies

Introduction

Newton's law $\vec{F} = m\vec{a}$ describes the motion of the center of mass of an object. But in general the motion of a rigid body consists of a rotation about the center of mass as well as movement of the c.m. In this Lab you will explore the physics of rotation in four situations:

1. Measurement of the moment of inertia of a massive pulley.

2. Measurement of energy balance in a block-and-tackle system.

3. Verification of the vector character of the laws of angular momentum.

4. Hands-on experience with physics 'toys' in which you are part of the rotating system.

Because this lab offers the opportunity of rapid verification of the impressive phenomena of rotational motion, we will be somewhat more ambitious than the course text in parts 3 and 4 of the Lab. An Appendix gives vector derivations of the laws of rotating objects for those who enjoy such details.

For parts 1 and 2 of the Lab we need only those simplifications of the theory that hold for rotary motion about a symmetry axis (axle) with fixed direction.

5.1   Moment of Inertia of a Massive Pulley

In this part you should verify that the moment of inertia of a uniform disk is

$$I_{calc} = \frac{1}{2}MR^2 = \frac{\pi}{2}\rho R^4 H \tag{1}$$

where R is the radius, H is the thickness and ρ is the density.

The 'disk' to be studied is a pulley that actually consists of an aluminum disk and a small aluminum cylinder bolted together so as to rotate about their common axis. The total moment of inertia of the pulley is the sum of the moments of its two components

Calculate the moment of inertia of the pulley by measuring the radii and thicknesses of the disk and cylinder, and note that density $\rho = 2.70$ g/cm^3 for aluminum. Negligible error will occur in the calculation if you ignore the various holes and grooves in the disks.

An error analysis of eq. (1) tells us that

$$\frac{\sigma_{I,calc}}{I} \approx \max\left\{4\frac{\sigma_R}{R}, \ \frac{\sigma_H}{H}\right\}. \tag{2}$$

Hence is it especially important to make a good measurement of the radius of the larger disk. The goal of this part is 1% accuracy.

To measure the moment of inertia of the pulley you will use a variation of Atwood's machine, sketched in Fig. 1. A mass m is suspended from a string that wraps around the smaller disk of the pulley at radius r. The tension T in the string exerts a torque of magnitude

$$\tau = Tr \tag{3}$$

about the axle of the pulley. This torque leads to an angular acceleration α of the pulley related by

$$\tau = I\alpha \tag{4}$$

From a measurement of the angular acceleration you could then deduce the moment of inertia as

$$I = \frac{Tr}{\alpha}, \tag{5}$$
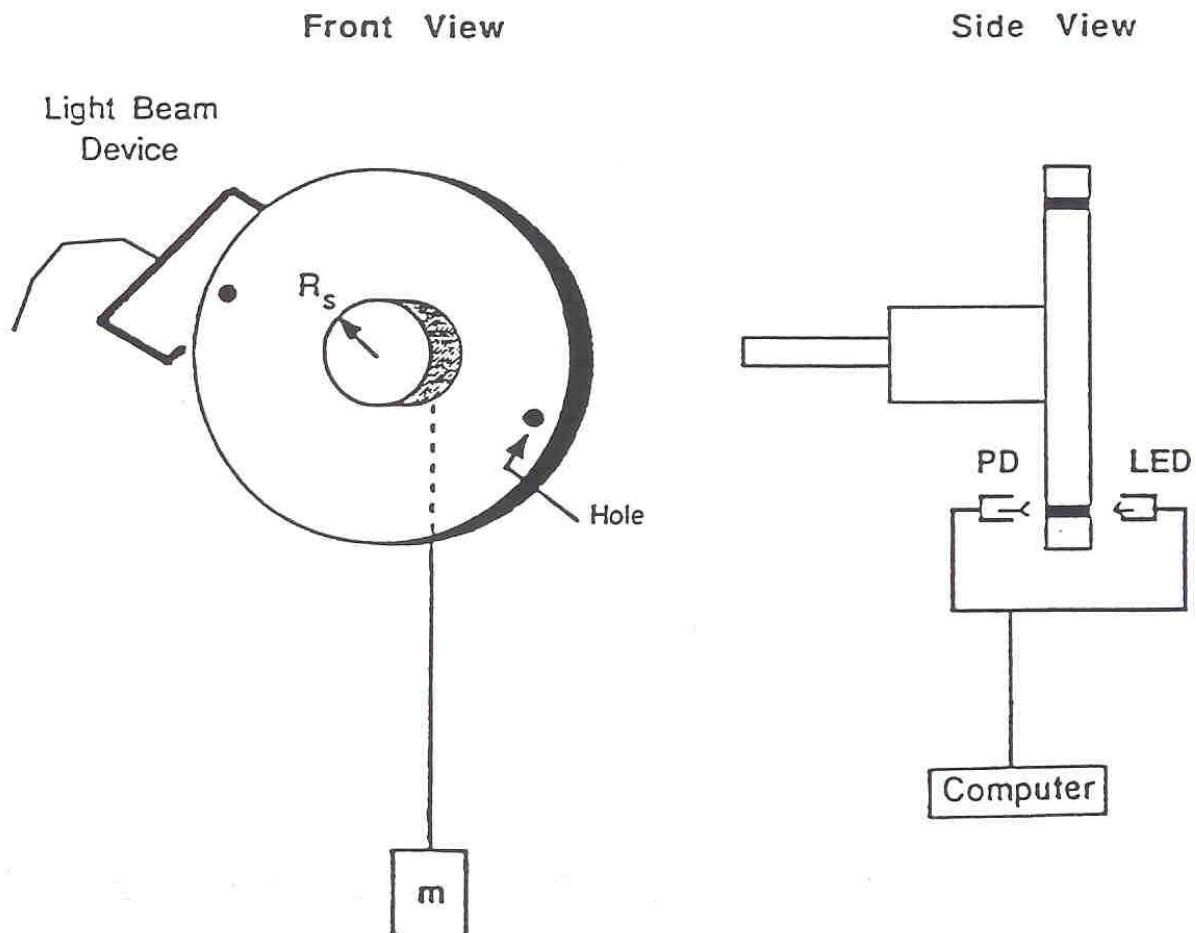
if you knew the tension T.

Figure 1: Sketch of the apparatus to measure the moment of inertia of a pulley.

The tension T cannot be measured directly, but can be inferred by its effect on the motion of the falling mass m:

$$mg - T = ma, \quad \text{or} \quad T = m(g-a), \tag{6}$$

where a is the vertical acceleration of mass m. To complete the analysis we note that since the string wraps around the pulley at radius r,

$$a = \alpha r, \tag{7}$$

and hence eq. (5) becomes

$$I_{meas} = \frac{mgr}{\alpha} - mr^2. \tag{8}$$

To measure the angular acceleration use the computer timer as sketched in Fig. 1. Position a LED-photodiode pair (photogate) so that the light beam intercepts the holes near the outer edge of the pulley. To check this, use the Photogate Status Check of program pt in directory c:\timer. When you spin the pulley the computer should indicate Photogate # 1 Unblocked briefly whenever one of the holes in the pulley passes by the photogate.

The string should be attached to a small screw in the hub of the pulley to prevent the hanging weight from hitting the floor.

For precision results, wrap the string around the smaller hub of the pulley in a single layer. The radius r used in your analysis must include the radius of the string! Do not wrap the string in the groove of the pulley.

To take data use option Motion Timer of the Main Menu. The computer starts taking data when the photogate first becomes unblocked, and stops only when you press Enter/Return. Therefore you must hit the Return key before the weight completely unwinds the string from the pulley.

Use option Display Table of Data to check the quality of your data. The times should be steadily decreasing. If not, take a new data set. When satisfied, Print Table of Data and also use File Options and Save Data to a File.

To analyze your data, Return to Data Analysis Options Menu, select Graph Data, and Velocity vs. Time. We wish to analyze the angular velocity of the pulley. The computer has recorded the time at the end of each half turn of the pulley, corresponding to π radians. Use Enter length (in meters) 3.1416. Then the graph will be labelled Velocity (m/s) but it will have the meaning of angular velocity in rad/s.

Among the Select Graph Style options, activate Point Protectors, Regression Line and Statistics. Accept the automatic scaling options and display your graph. If satisfactory, Print Graph.

The fitted slope, M, of the graph is just the angular acceleration α in rad/s^2 (provided your data is valid: the falling hanger didn't hit anything; the velocity graph fits well to a straight line...).

Measure mass m and radius r (including the thickness of the string) and then calculate the moment of inertia I_{meas} using eq. (8).

If your values of I_{calc} and I_{meas} differ by more than 1% you should try to resolve the discrepancy. If the difference is large, look for a gross error in value of R or H used in I_{calc}, etc. If the difference is only a few percent, you should also consider the sources of error in I_{meas}, for which an error analysis of eq. (8) tells us

$$\frac{\sigma_{I,meas}}{I} \approx \max\left\{\frac{\sigma_m}{m}, \ \frac{\sigma_r}{r}, \ \frac{\sigma_\alpha}{\alpha}\right\}. \tag{9}$$

5.2   Statics and Dynamics of a Block and Tackle

[Analyze the data from part 1 before proceeding.]

In this part you will examine the work-energy relation for a system that includes rotating objects.

Construct a block-and-tackle system as sketched in Fig. 2. The pulley you used in part 1 will be called the fixed pulley in that its axle is fixed. A string is attached to the axle of the fixed pulley, passes around the moving pulley, back up over the fixed pulley and down to the hanging mass. The string should be long enough that when the hanging mass (m_1) is near the fixed pulley the the moving pulley is close the the floor. The string should loop (twice if necessary to avoid slipping) around the groove in the hub of the fixed pulley (and, of course, in the groove of the moving pulley).
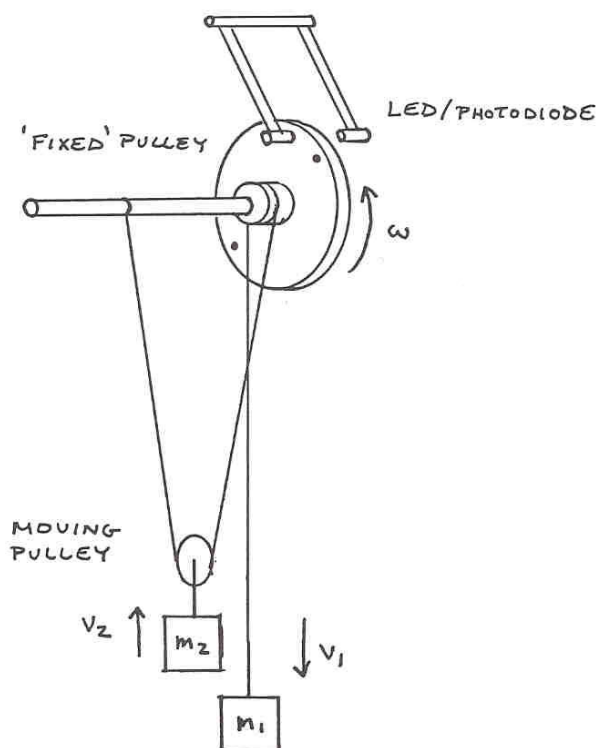

Figure 2: The block-and-tackle arrangement for part 2.

You may wish to wind the string used in part 1 around the hub of the fixed pulley and secure it out of the way of the groove with a piece of tape. Do not remove this string!

Weigh the moving pulley first. It should include a 100-g weight as a load.

Adjust the hanging weight until the system is in static equilibrium. Keep adding weights until the hanging mass would fall if only 1-2 g more were added. What is the relation between the hanging weight, the weight of the moving pulley and the tension in the string?

Next, add about 200 g to the hanging mass so that it falls down, raising the moving pulley in the process.

Use the computer to record the times at the end of successive half turns of the motion. Stop the data collection when the hanging mass nears the moving pulley – if the hanging mass hits the moving pulley during its fall the data thereafter will be bad.

Plot (angular) velocity vs. time to see if the data fall along a straight line. Repeat the data collection if necessary.

In the data analysis you will compare the final kinetic energy of the system to the change in potential energy. For this you need to know the final angular velocity, ω_f, and the distance h_1 through which the hanging weight fell.

The angular acceleration α is again the slope M of your graph of angular velocity vs. time which has been fit to the form $\omega = \mathsf{B} + \mathsf{M}t$. We infer that the weight started falling at time $t_0 = -\mathsf{B}/\mathsf{M}$. Let t_f be the time of the last velocity point on your graph. Then

$$\omega_f = \alpha \Delta t = \mathsf{M}\Delta t, \qquad \text{and} \qquad \theta_f = \frac{1}{2}\alpha \Delta t^2 = \frac{1}{2}\mathsf{M}\Delta t^2, \tag{10}$$

where

$$\Delta t = t_f - t_0 = t_f + \frac{\mathsf{B}}{\mathsf{M}}, \tag{11}$$

and the hanging weight fell by

$$h_1 = r\theta_f, \tag{12}$$

where r is the radius on the fixed pulley around which the string was wrapped. *Again, remember to include the radius of the string in the value of r. For better accuracy, use eq. (10) to calculate ω_f rather than reading it off the graph.*

Do you agree that while the hanging weight fell by h_1 the moving pulley rose by distance $h_2 = h_1/2$? Hence the change in potential energy was

$$\Delta \text{PE} = \left(m_1 - \frac{m_2}{2}\right)gh_1, \tag{13}$$

where m_1 is the hanging mass and m_2 is the mass of the moving pulley.

The final kinetic energy includes that due to the center-of-mass motion of the hanging mass and of the moving pulley, as well as that due to the rotation of the two pulleys (see Appendix 5.5.5). Do you agree that $v_1 = 2v_2 = \omega_f r$, and $\omega_2 = v_2/r_2$, where ω_2 is the angular velocity of the moving pulley and r_2 is the radius on the moving pulley at which the string is wrapped? The final kinetic energy is then

$$\text{KE}_f = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}I\omega_f^2 + \frac{1}{2}I_2\omega_2^2, \tag{14}$$

where I is the moment of inertia of the fixed pulley and I_2 is the moment of inertia of the wheel of the moving pulley.

However, the moment of inertia I_2 of the moving pulley is so small it can be ignored. Calculate

$$\text{KE}_f \approx \frac{1}{2}\left(\left(m_1 + \frac{m_2}{4}\right)r^2 + I\right)\omega_f^2, \tag{15}$$

using the value for I that you found in part 1. Compare your results for ΔPE and KE$_f$. You may omit an error analysis in part 2, but if energy is not conserved to within 5-10% look for a gross error somewhere.

5.3   Precession of a Gyroscope

[Analyze parts 1 and 2 before proceeding to parts 3 and 4.]

A gyroscope is a rapidly spinning object whose axle has freedom to change direction.

You will use a disk very similar to the pulley of part 1 as a gyroscope, spinning it up to high angular velocity with one of the electric spinners located at special tables in the lab.

Most people find it disconcerting to hold a gyroscope, but this is exactly what we encourage you to do. We hope it is left to the imagination as to what happens if you drop a gyroscope. If you feel bold, try (gently) twisting the axle of a spinning gyroscope before proceeding.

Gyroscopes are governed by the general law of rotation

$$\vec{\tau} = \frac{d\vec{L}}{dt}, \tag{16}$$

which is derived in Appendix 5.5.2. We must take careful note of directions as well as magnitudes in understanding this equation. In words, an applied torque vector causes a change in the angular momentum vector of the system in the direction of the applied torque. In addition, the magnitude of the rate of change of angular momentum is just the magnitude of the torque.

First, where is the angular momentum vector of a gyroscope? If the angular velocity ω of the spin of gyroscope about its axle is large we can ignore other components of the angular velocity that arise when the direction of the axle is changing. Then the angular momentum vector points along the axle of the gyroscope and we can write

$$\vec{L} = I\vec{\omega}, \tag{17}$$

following eq. (39). Use the right-hand rule (Appendix 5.5.1) to determine which way $\vec{L}$ and $\vec{\omega}$ point along the axle: if the fingers of your right hand curl in the sense of rotation of the gyroscope, then the thumb of your right hand points along $\vec{L}$ and $\vec{\omega}$.

Now suppose the forces on the gyroscope are sufficient to hold the support point on the axle fixed, but a torque is applied. Equation (16) tells us that that if the directions of the torque vector $\vec{\tau}$ and of the angular momentum vector $\vec{L}$ are not the same the direction of $\vec{L}$ will change. Then, since $\vec{L}$ has the direction of the axle, the latter direction must change also. As the direction of the torque is at right angles to the direction of the force that causes it, the direction toward which the axle moves is surprising (at first).

We now give a more quantitative argument. Suppose the axle (and hence the angular momentum vector $\vec{L}$) of the gyroscope is horizontal, and a vertical force F is applied to the axle at distance d from the point of support of the gyroscope. Then the resulting torque has magnitude $\tau = Fd$ and direction perpendicular to the axle but in the horizontal plane. This torque will 'precess' the angular momentum vector in the horizontal plane, always pulling it a right angles to its current direction. The angular momentum vector (and hence the axle) will slowly move in horizontal circle at a uniform rate.

We will describe the rate of change of direction of the axle as angular velocity Ω. Just as the magnitude of the velocity of a particle at radius r in uniform circular motion with angular velocity Ω is $dr/dt = \Omega r$, the rate of change of the rotating angular momentum is

$$\frac{dL}{dt} = \Omega L. \tag{18}$$

Combining the equations,

$$\tau = Fd = \frac{dL}{dt} = \Omega L = \Omega I \omega, \tag{19}$$

and hence we predict

$$\Omega = \frac{Fd}{I\omega} \tag{20}$$

for the precessional angular velocity of the gyroscope.

Confirm the concept of gyroscopic precession in the following experiment. *First, suspend a non-spinning gyroscope by its string and handle from one of the small cranes attached to the lab tables. Add a hanger and weights to the loop of string at free end of the axle until the axle is balanced horizontally. Determine this balance to within about 10 g, and screw down the retaining nut on the hanger to hold the weights fixed. Measure the distance d along the axle between the two support strings.*

Remove the hanger from the gyroscope, and the gyroscope from the crane. Put a single piece of masking tape on the brass disk of the gyroscope (if none is already present) to assist in later measurement of the spin rate. Spin up the gyroscope and restore it to the crane and add the hanging weight. At this time the gyroscope should have no net forces and no net torques on it, and so should just spin with its axle in a fixed direction. Use the right-hand rule to predict the direction of the angular momentum vector. For this, you must have paid attention to which way the gyroscope is spinning.

Now add another 500 g to the hanger, which will cause a torque about the point where the string from the crane attaches to the gyroscope. You may need to secure the extra weight with a piece of tape. Can you correctly predict the direction of the torque, and hence of the precession of the axle?

Measure the period T of precession of the gyroscope with a stopwatch (or use the Keyboard Timing Modes *of program* pt*). Measure the rotational frequency f of the spinning gyroscope with a stroboscope. The electrical spinners operate at about 3600 RPM, which is the maximum frequency of your gyroscope. If you work quickly you can complete the measurement of the period of precession before the gyroscope spin has dropped below 1800 RPM.*

It may be easier to measure the spin frequency with the stroboscope just before and after measuring the precession period T, and average the two values of f. Practice taking measurements with the stroboscope first. When the stroboscope flashes at frequency f the piece of tape on the spinning disk appears 'frozen'. But as a check that you have found the right value of f, adjust the frequency to 2f. You should now see two images of the tape. If you see only one, or cannot freeze the image at all near 2f, the value of f you started with was too low.

The period is related to the precession rate by $\Omega = 2\pi/T$, the spin angular velocity is related to the spin frequency f in revolutions per second by $\omega = 2\pi f$, and the vertical force $F = \Delta mg$ where Δm is the extra weight you added to the balanced gyroscope. Hence eq. (20) becomes

$$T = \frac{4\pi^2 I f}{\Delta mgd}. \tag{21}$$

The moment of inertia of the brass gyroscope is about 0.0128 kg-m^2. Compare the prediction (21) for the period T with your measured value.

Thus far, the axle of the gyroscope should have remained horizontal (unless it has lost a lot of its spin). If you want to change the angle of the axle to the vertical, you must give $d\vec{L}/dt$ a vertical component. This requires a vertical torque, which in turn requires a horizontal force. Push (but don't grab) the axle horizontally to see if the gyroscope responds as predicted.

Feel free to try these experiments while holding the axle of the gyroscope in your hand. Part 5.4.2 offers another chance to explore closely related phenomena.

You can also operate the gyroscope without the hanger. If you release the axle when it is horizontal the gyroscope again precesses with its axle always in a horizontal plane. More generally, the vertical angle of the axle will remain constant while the precessing axle sweeps out a cone. This violates the 'common sense' expectation that the vertical angle of the axle should change due to the pull of gravity. We hope you now appreciate how the gyroscope indeed responds to the torque induced by gravity, but according to the vector laws of rotating objects.

5.4   Angular-Momentum Toys

[You don't need to report on this part in your lab writeup.]

5.4.1   Rotating Stool and Hand Weights

A special stool ('Silver') is provided that can rotate about a vertical axis. Sit on the stool and hold the hand weights at arms' length. Have your partner set you spinning.

Pull the hand weights in, and your angular velocity should increase.

That is, the angular momentum, $I\omega$, about the vertical axis is conserved, and since $I \propto MR^2$ if you decrease R you decrease I and so ω must increase.

5.4.2   Rotating Stool and Spinning Bicycle Wheel

Use the small hand spinner to spin up the bicycle wheel. The wheel has lead weights around the rim to increase its moment of inertia.

Sit on the rotating stool holding the axis of the bicycle wheel horizontal initially. Then twist the axle of the wheel towards the vertical. Can you understand the subsequent motion on the basis of conservation of angular momentum?

5.4.3   Water Flow in a Tube

Pour water into the funnel attached to a copper tube that bends in a loop with its outlet directly below the funnel. What is the resulting motion of the system? Is angular momentum conserved in this device?

5.5   Appendix: Vector Formulation of the Laws of Rotating Objects

5.5.1   The Right-Hand Rule

The laws of rotational motion can be expressed compactly if we use the vector cross product. Thus the vector expression for torque $\vec{\tau}$ is

$$\vec{\tau} = \vec{r} \times \vec{F}, \tag{22}$$

where vector $\vec{r}$ points from some pivot point (chosen by you!) to the point of application of the force $\vec{F}$. Recall that the force of gravity of an extended object can be regarded as being applied at the center of mass.

The magnitude, τ, of the torque obeys

$$\tau = rF \sin\theta, \tag{23}$$

where r and F are the magnitudes of the radius and forces vectors, and angle θ ($\leq 180°$ by definition) is the angle between vectors $\vec{r}$ and $\vec{F}$.

Especially in parts 3 and 4 of this Lab we must also keep track of the directions of vectors defined by cross products. The torque vector $\vec{\tau}$ of eq. (22) is defined as being perpendicular to both $\vec{r}$ and $\vec{F}$ (or equivalently, perpendicular to the plane containing vectors $\vec{r}$ and $\vec{F}$).

This definition does not yet completely specify the direction of $\vec{\tau}$. It could be pointing either way along the axis perpendicular to the plane of $\vec{r}$ and $\vec{F}$. The **right-hand rule** is needed to complete the specification: position your **right** hand so that your fingers initially point along the direction of $+\vec{r}$, and so that they can be curled into the direction of $\vec{F}$ by a rotation of $\leq 180°$; your thumb then points in the direction of vector $\vec{\tau}$.

Literal application of the right-hand rule sometimes require dramatic contortions of your right arm. You must resist the temptation to use your left arm even if it feels better, because this would reverse the sign of the direction of $\vec{\tau}$. (Try it once to verify this.)

5.5.2   The Cross Product of $\vec{F} = m\vec{a}$

The law of rotational motion can be elegantly derived by taking the cross product of $\vec{F} = m\vec{a}$:

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times m\vec{a} = \vec{r} \times m\frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) \equiv \frac{d\vec{L}}{dt}, \tag{24}$$

where we have used the identity

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = \left(\frac{d\vec{r}}{dt} \times \vec{v}\right) + \left(r \times \frac{d\vec{v}}{dt}\right) = (\vec{v} \times \vec{v}) + \left(r \times \frac{d\vec{v}}{dt}\right) = r \times \frac{d\vec{v}}{dt}, \tag{25}$$

since the cross product of any vector with itself is zero.

The law of rotational motion, eq. (24), is thus

$$\vec{\tau} = \frac{d\vec{L}}{dt}, \qquad \text{where} \qquad \vec{L} \equiv \vec{r} \times m\vec{v} \equiv \text{ angular momentum.} \tag{26}$$

An extended body can be thought of as a collection of point masses m_i with positions and velocities $\vec{r}_i$ and $\vec{v}_i$ and angular momentum

$$\vec{L} = \sum_i \vec{r}_i \times m_i \vec{v}_i. \tag{27}$$

5.5.3   Rotation About a Symmetry Axis

There is an important simplification to eq. (27) for objects that are rotating about a symmetry axis (= axle) whose direction is constant.

Define ω to be the magnitude of the angular velocity of rotation about the axle. We can also form a vector called $\vec{\omega}$ with magnitude ω by assigning it the direction of the axle according to the right-hand rule. Namely, if the fingers of your right hand curl in the sense of rotation, then the thumb of your right hand points along the direction of $\vec{\omega}$.

The utility of the vector definition of $\vec{\omega}$ is that the velocity of any point $\vec{r}_i$ in the rotating object can be written

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i. \tag{28}$$

In this equation we have assumed that the origin of the coordinate system for the position vectors $\vec{r}_i$ lies on the axle.

We can verify that eq. (28) matches the more familiar relation for the magnitude of the velocity:

$$v_i = \omega R_i, \tag{29}$$

where R_i is the perpendicular distance between the point $\vec{r}_i$ and the axle. Namely, the definition of the vector cross product tells us that for eq. (28) we have

$$v_i = \omega r_i \sin \theta_i, \tag{30}$$

where θ_i is the angle between the axle and the vector $\vec{r}_i$. But the perpendicular distance R_i is just $r_i \sin \theta_i$, and eq. (29) follows.

Using eq. (28) we can rewrite eq. (27) as

$$\vec{L} = \sum_i \vec{r}_i \times m_i(\vec{\omega} \times \vec{r}_i) = \sum_i m_i[r_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\vec{r}_i], \tag{31}$$

using the vector identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}. \tag{32}$$

To go further, we note that position vector $\vec{r}_i$ can be broken up into components parallel and perpendicular to the axle:

$$\vec{r}_i = \vec{R}_i + r_i \cos \theta_i \hat{\omega}, \tag{33}$$

where $\vec{R}_i$ has magnitude $R_i = r_i \sin \theta_i$ and points perpendicular to the axle towards point i, and $\hat{\omega}$ is a unit vector along the axle. Since vectors $\vec{R}_i$ and $\hat{\omega}$ are perpendicular we have that $\vec{R}_i \cdot \hat{\omega} = 0$, so

$$(\vec{r}_i \cdot \vec{\omega})\vec{r}_i = r_i \omega \cos \theta_i (\vec{R}_i + r_i \cos \theta_i \hat{\omega}) = r_i \omega \cos \theta_i \vec{R}_i + r_i^2 \cos^2 \theta_i \vec{\omega}. \tag{34}$$

Inserting this into eq. (30) we have

$$\vec{L} = \sum_i m_i r_i^2 (1 - \cos^2\theta_i)\vec{\omega} + \omega\sum_i m_i r_i \cos\theta_i \vec{R}_i = I\vec{\omega} + \omega\sum_i m_i r_i \cos\theta_i \vec{R}_i, \qquad (35)$$

noting that

$$\sum_i m_i r_i^2 (1 - \cos^2\theta_i) = \sum_i m_i r_i^2 \sin^2\theta_i = \sum_i m_i R_i^2 \vec{\omega} = I. \qquad (36)$$

The moment of inertia I can also be written

$$I = \sum_i m_i R_i^2 = \int \rho R^2 d\text{Vol}, \qquad (37)$$

with ρ = mass density.

If the mass distribution is symmetric about the axle a marvelous simplification occurs in eq. (35). The symmetry implies that each mass m_i at $\vec{r}_i = \vec{R}_i + r_i \cos\theta_i \hat{\omega}$ has a partner of equal mass at position $-\vec{R}_i + r_i \cos\theta_i \hat{\omega}$, and hence

$$\sum_i m_i r_i \cos\theta_i \vec{R}_i = 0. \qquad (38)$$

Thus for objects rotating about a symmetry axis we have

$$\vec{L} = I\vec{\omega}, \qquad (39)$$

Finally, using eq. (39) for the angular momentum in the basic torque equation (26) we arrive at the familiar form

$$\tau = I\frac{d\omega}{dt} = I\alpha, \qquad (40)$$

where α is the angular acceleration.

5.5.4   Moments of Inertia of Some Simple Objects

An immediate application of eq. (37) is that a compact mass M at radius R from an axle has moment of inertia

$$I = MR^2 \qquad \text{(point mass on a sling)}. \qquad (41)$$

Similarly, a hoop of mass M all at radius R from an axle has

$$I = MR^2 \qquad \text{(hoop)}. \qquad (42)$$

For a uniform disk of mass M, outer radius R and thickness H the volume is $\pi R^2 H$, so the density is $\rho = M/(\pi R^2 H)$. The volume element is $d\text{Vol} = dR \cdot Rd\phi \cdot dh$, and

$$I = \frac{M}{\pi R^2 H}\int_0^R RdR \int_0^{2\pi} d\phi \int_0^H dH \; R^2 = \frac{1}{2}MR^2 \qquad \text{(uniform disk)}. \qquad (43)$$

For a uniform sphere of mass M, and radius R the volume is $(4/3)\pi R^3$, so the density is $\rho = 3M/(4\pi R^3)$. The volume element is $d\text{Vol} = dR \cdot R\sin\theta \, d\theta \cdot Rd\phi$, and the distance from a point (R, θ, ϕ) to the axis is $R\sin\theta$, so

$$I = \frac{3M}{4\pi R^3}\int_0^R R^2 dR \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi R^2 \sin^2\theta = \frac{2}{5}MR^2 \qquad \text{(uniform sphere)}. \qquad (44)$$

5.5.5   Parallel-Axis Theorem

If the momentum of inertia of an object of mass M about a symmetry axis is I_0, then the moment of inertia about a parallel axis at distance h from the symmetry axis is

$$I = I_0 + Mh^2. \tag{45}$$

To see this we use relation (37), rewritten as $I = \sum_i M_i \vec{R}_i'^2$, where $\vec{R}_i'$ is a vector perpendicular to the new axis from that axis to point i. If we define $\vec{h}$ as the vector perpendicular to the symmetry axis from the symmetry axis to the new axis and $\vec{R}_i$ as the vector perpendicular to the symmetry axis from the symmtery axis to point i then $\vec{R}_i' = \vec{R}_i - \vec{h}$. Furthermore, the center of mass of the object lies on the symmetry axis, so $\sum_i M_i \vec{R}_i = 0$. Then

$$I = \sum_i M_i(\vec{R}_i - \vec{h})^2 = \sum_i M_i R_i^2 + h^2 \sum_i M_i - 2\vec{h} \cdot \sum_i M_i \vec{R}_i = I_0 + Mh^2, \tag{46}$$

as claimed.

5.5.6   Kinetic Energy of Rotation

A moving object has kinetic energy

$$\text{KE} = \frac{1}{2} \sum_i m_i v_i^2 = \frac{1}{2} \sum_i m_i \vec{v}_i \cdot \vec{v}_i. \tag{47}$$

In general the motion of a rigid body can be described by the velocity $\vec{v}_{cm}$ of the center of mass and a rotation about some axis through the center of mass with angular velocity $\vec{\omega}$. If $\vec{r}_i$ is the position vector from the center of mass to point i then that point has velocity

$$\vec{v}_i = \vec{v}_{cm} + \vec{\omega} \times \vec{r}_i, \tag{48}$$

recalling eq. (28). On squaring this we find

$$\vec{v}_i \cdot \vec{v}_i = v_{cm}^2 + 2\vec{v}_{cm} \cdot \vec{\omega} \times \vec{r}_i + \omega^2 r_i^2 - (\vec{\omega} \cdot \vec{r}_i)^2, \tag{49}$$

using the vector identity

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c}). \tag{50}$$

Also note that $\vec{\omega} \cdot \vec{r}_i = \omega r_i \cos\theta_i$, and that $\omega^2 r_i^2 (1 - \cos^2\theta_i) = \omega^2 R_i^2$ where as before R_i is the perpendicular distance from the axis of rotation to point i. The kinetic energy can now be written

$$\text{KE} = \frac{1}{2} \sum_i m_i v_{cm}^2 + \vec{v}_{cm} \cdot \vec{\omega} \times \sum_i m_i \vec{r}_i + \frac{1}{2} \sum_i m_i R_i^2 \omega^2 = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I \omega^2, \tag{51}$$

using the fact that $\sum_i m_i \vec{r}_i = M \vec{r}_{cm} = 0$ since we have taken the center of mass to at the origin. Equation (51) is true whether or not the axis of rotation is a symmetry axis, but the interesting topic of rotations about non-symmetry axes will have to be pursued elsewhere.

Equation (51) contains the memorable result that the total kinetic energy of a moving object is the sum of the kinetic energy of the center of mass motion plus the kinetic energy of rotation about the center of mass.