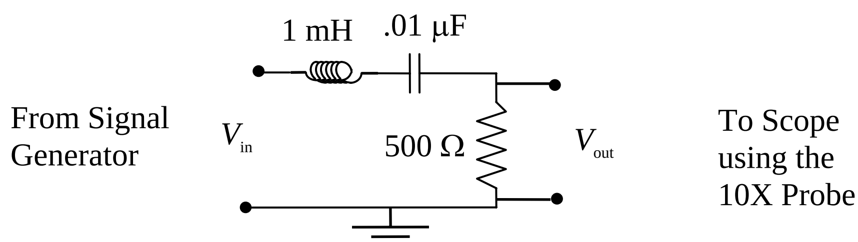
PRINCETON UNIVERSITY **PHYSICS 104 LAB**
Physics Department **Week #8**

EXPERIMENT VII
RLC RESONANT CIRCUITS

Introduction. Most of electronics is based on the responses of a few circuit elements to time-varying voltages and currents (alternating currents = AC). Resistors (R), capacitors (C), inductors (L), transistors, and diodes are by far the most common circuit components; understand them, and you have the basis for understanding an enormous variety of functions performed by electronic circuits. This week we concentrate on circuits containing R, L, and C. In Labs #10-11 you will learn about voltage rectification by a diode and amplification by a transistor circuit.

This Lab allows you to breadboard several circuits, drive them with a sinusoidal voltage from a signal generator, and study their behavior with an oscilloscope. What you learn is of general interest, because any complicated waveform can be decomposed (by Fourier analysis) into a superposition of sinusoids. Knowing the response of a circuit to sine waves allows you to find the response to any waveform.

1. *RLC* Series Resonant Circuit. In the following circuit a resistor R, an inductor L, and a capacitor C are in series, driven by an oscillatory voltage source, $V_{\text{in}} = V_0 \cos\omega t = \text{Re}\, e^{i\omega t}$, of angular frequency $\omega = 2\pi f$. The output voltage, V_{out}, is measured across resistor R.



Using the method of complex impedance, we note that $Z_R = R$, $Z_L = i\omega L$, and $Z_C = 1/i\omega C$, and that for a series arrangement the total impedance is

$$Z = Z_R + Z_L + Z_C = R + i\omega L + \frac{1}{i\omega C} = R + i\omega L - \frac{i}{\omega C} = R\left(1 + i\,\frac{\omega^2 LC - 1}{\omega RC}\right).$$

Hence, the output voltage is

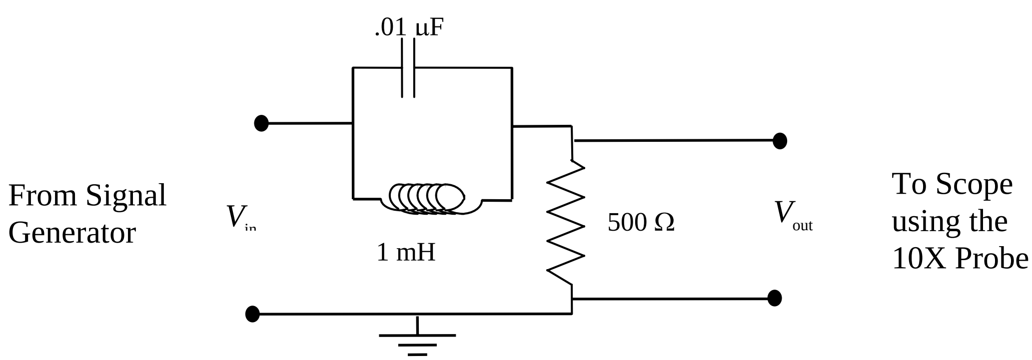
$$V_{\text{out}} = IZ_R = V_{\text{in}}\frac{Z_R}{Z} = V_{\text{in}}\frac{1}{1 + i\,\dfrac{\omega^2 LC - 1}{\omega RC}} = V_{\text{in}}\frac{1 - i\,\dfrac{\omega^2 LC - 1}{\omega RC}}{1 + \left(\dfrac{\omega^2 LC - 1}{\omega RC}\right)^2} = V_0\frac{e^{i\left(\omega t - \tan^{-1}\frac{\omega^2 LC - 1}{\omega RC}\right)}}{\sqrt{1 + \left(\dfrac{\omega^2 LC - 1}{\omega RC}\right)^2}}.$$

The output voltage is always less than the input voltage, except for the special case that $\omega = 2\pi f = 1/\sqrt{LC}$. This special frequency is called the resonant frequency, $f_{res} = 1/2\pi\sqrt{LC}$. Construct the series *RLC* circuit on a breadboard, and connect it to a Wavetek signal generator. Set the output of the signal generator to $1V_{p\text{-}p}$ and make a rough measurement of the resonant frequency f_{res} by tuning the input frequency through the peak. Compare the peak (resonant) frequency with your calculation. Do your results agree within the uncertainties of the component values?

The frequency response of the circuit is important. Plot $|V_{out}| / |V_{in}|$ as a function of frequency f, going to a frequency on either side of the resonant frequency at which the value $|V_{out}| / |V_{in}|$ is down by a factor of ten from that you measured at the resonant frequency (f_{res}). Identify the full width at half maximum, Δf, of the resonance curve.

If you are interested, you can measure the resonance curve for $R = 250\ \Omega$ in addition to the curve for $R = 500\ \Omega$. Figure 29-20 in Tipler illustrates what you might expect to happen.

2. *RLC* Series-Parallel Resonant Circuit. Construct the following circuit, where the capacitor and inductor are in parallel, and the combination is in series with a resistor, and measure the voltage across the resistor as a function of frequency over the same range as you used with the series resonant *RLC* circuit above, and make a plot of $|V_{out}| / |V_{in}|$ as a function of frequency. Does the shape of the curve look the same as it did in the series circuit? You will use an *LC* parallel circuit again later in the semester as part of a larger electronic circuit.

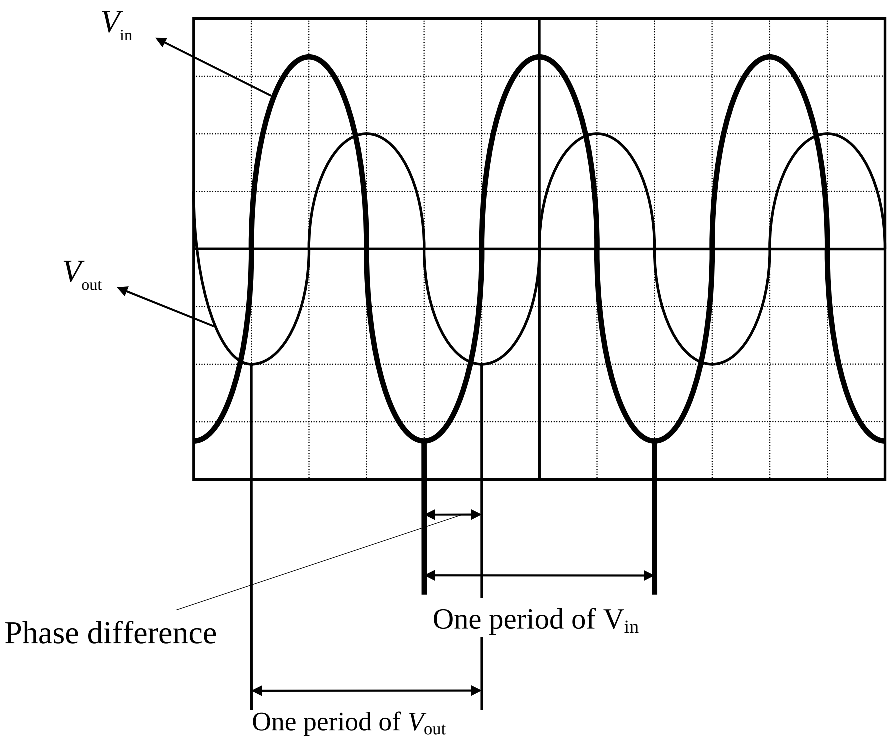


As usual, you should spend some time writing your interpretation of the results in your notebook. As you have probably noticed, thinking things through, and stating them in your own words, is the route to understanding at a deeper level. Writing makes you think.

3. Phases in AC Circuits. In what you have done thus far, an interesting part of the behavior of AC circuits has not been important, namely the phase difference between V_{in} and V_{out} across each of the components. Now go back and set up the series *RLC* circuit and measure the phase of the voltage across each component compared to the phase of the input signal from the generator. Record data for at least 5 frequencies, $f \ll f_{res}$, $f = f_{res} - \Delta f/2$, $f = f_{res}$, $f = f_{res} + \Delta f/2$ and $f \gg f_{res}$.

For this, you will have to build the circuit 3 times, once with each of *R*, *L* or *C* being connected to ground, such that V_{out} can be measured across this component. Does it matter in which order you place the other two components? [*The ground lead of the oscilloscope probe must be connected to the circuit for the oscilloscope to function, and this ground is the same at the ground of the signal generator. If the two grounds are connected at different places in the circuit, you could mistakenly "short out" one or more circuit components.*]

The figure below is an example of two waves (V_{in} and V_{out}) on CH1 and CH2 of an oscilloscope, in which we say that V_{in} *leads* V_{out} by 90^o.



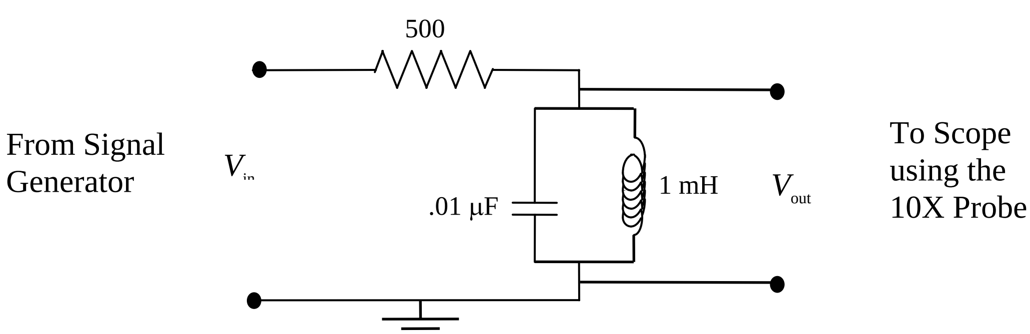
period : V_{in} 4 divisions
V_{out} 4 divisions

4 divisions = 360^o
1 division = 90^o

V_{in} *leads* V_{out} by 1 division (90^o).

You may also wish to view the phase difference between V_{in} and V_{out} via the technique of Lissajous: turn the horizontal sweep of the oscilloscope to x-y; vary the frequency of V_{in}, interpret the figure on the scope in terms of the phase difference.

In the parallel *RLC* circuit shown below you can also measure V_{out} as a function of frequency across the *LC* combination. Even though this circuit is almost identical to the other parallel *RLC* circuit you constructed earlier, there is one difference. The *LC* combination has been moved so that one end of it is at ground so that the scope probe can be connected properly across the LC loop.



Measure the voltage across the LC loop as a function of frequency, over a similar range as before, and make a plot of $|V_{out}| / |V_{in}|$ as a function of frequency. Does the shape of the curve look the same as it did when you measured V_{out} across R in the parallel circuit? And of course, describe the phase relation between V_{in} and V_{out}.

ELI the ICE man!

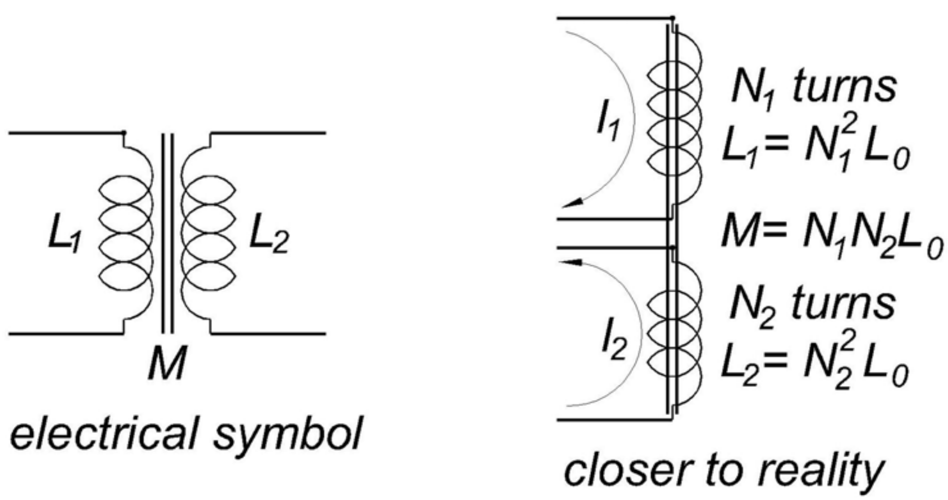
4. Optional: The Transformer.

A transformer is a combination of two inductors arranged such that (ideally) all of the magnetic flux through each of the inductors also passes through the other. The purpose of the transformer is to transform the oscillatory voltage V_1 that is applied to the first inductor into an oscillatory voltage V_2 that appears across the second inductor where it could be used to provide power to a load that is designed to operate at V_2 rather than V_1. The inductors do not consume power, so if the currents in the two inductors are I_1 and I_2, the output power $V_2 I_2$ must be provided by the input power source according to the relation

$$P_{in} = V_1 I_1 = P_{out} = V_2 I_2.$$

See part 2 of the Prelab exercise and sec. 29.7 of Tipler for additional discussion of this.

The electrical symbol for a transformer is shown on the left below, but the diagram on the right is a closer representation of the function of a transformer.



An oscillatory drive current I_1 in inductor L_1 (the primary) induces oscillatory current I_2 in inductor L_2 (the secondary) because there is a mutual inductance M between the two inductors. According to Lenz' law, the oscillatory magnetic field created by current I_2 opposes the changes in the magnetic field of current I_1; hence current I_2 is opposite to current I_1.

The (complex) voltage $V_1 = V_{1,0}e^{i\omega t}$ across inductor 1 is not simply $\dot{\Phi}_1 = L_1\dot{I}_1 = i\omega L_1 I_1$, where $\omega = 2\pi f$ is the angular frequency. The current I_2 acts back on inductor 1 and lowers the voltage there by amount $\dot{\Phi}_{12} = M\dot{I}_2 = i\omega M I_2$:

$$V_1 = i\omega L_1 I_1 - i\omega M I_2.$$

Similarly the voltage across inductor 2 is not simply $i\omega L_2 I_2$, but is reduced by the effect of mutual inductance with inductor 1:

$$V_2 = i\omega L_2 I_2 - i\omega M I_1.$$

In an ideal transformer the mutual inductance M is related to the self inductances L_1 and L_2 according to $M^2 = L_1 L_2$. To see this, suppose the first inductor of the transformer consists of N_1 turns, each with self inductance L_0, while the second inductor has N_2 turns, each of self inductance L_0 also. If a current I flows in a loop of self inductance L_0, then the magnetic flux through that loop is $\Phi_0 = L_0 I$. Since inductor 1 has N_1 turns, and the flux due to each loop passes through all the other, the total flux through one loop of that inductor is $\Phi = N_1 L_0 I$, and the total flux through all N_1 loops is therefore $\Phi_1 = N_1^2 L_0 I$. Then, since $\Phi_1 = L_1 I$, we learn that the self inductance of inductor 1 is

$$L_1 = N_1^2 L_0.$$

By a similar argument, we find that

$$L \ = N_2^2 L_0.$$

But when current I_1 flow in inductor 1, its flux, $\Phi = N_1 L_0 I_1$, is also linked by inductor 2. Hence, the flux Φ_2 in the N_2 turns of inductor 2 due a current I_1 in inductor 1 is $\Phi_2 = N_1 N_2 L_0 I_1$. The mutual inductance between the two inductors is defined by the relation $\Phi_2 = M I_1$, so we learn that

$$M = N_1 N_2 L_0.$$

You can readily verify that if you began with current I_2 in inductor 2, the resulting flux linked by inductor 1 would be $\Phi_1 = M I_2$. Thus, for an ideal transformer, the self and mutual inductances obey

$$M^2 = L_1 L_2.$$

In general, not all of the flux created by one inductor is linked by another, so in general we can only say that $M^2 \le L_1 L_2$.

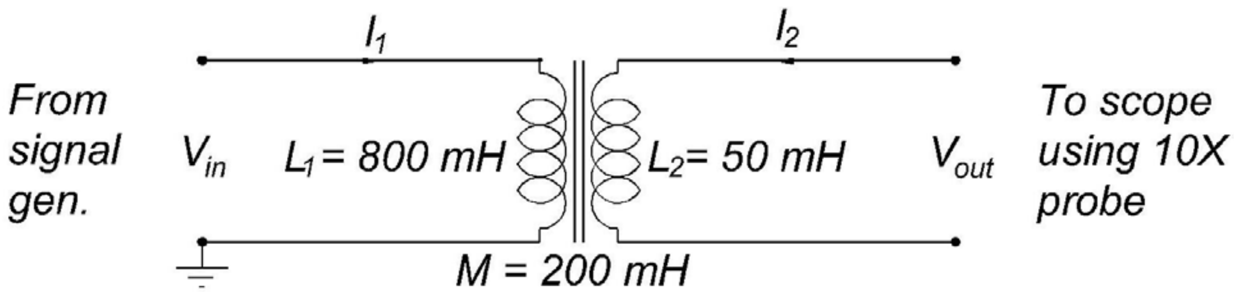
The transformer on your lab bench is close to ideal, in that $M^2 \approx L_1 L_2$. Typical values are L_1 = 800 mH, L_2 = 50 mH and M = 200 mH. If time permits you may want to measure these values yourselves, as described at the end of the Lab instructions.

Your transformer is somewhat less than ideal in that each of the inductors has a small but nonzero resistance, due to the many turns of wire required. Measure these resistances, R_1 and R_2, using a multimeter. Let L_1 and R_1 correspond to the primary side of the transformer (black leads), and L_2 and R_2 correspond to the secondary side (colored leads).

From part 2 of the Prelab exercise (see also sec. 29.7 of Tipler), we learn that for an ideal transformer driven by input voltage V_{in} (applied to the inductor 1), the output voltage V_{out} that appears across inductor 2 is given by

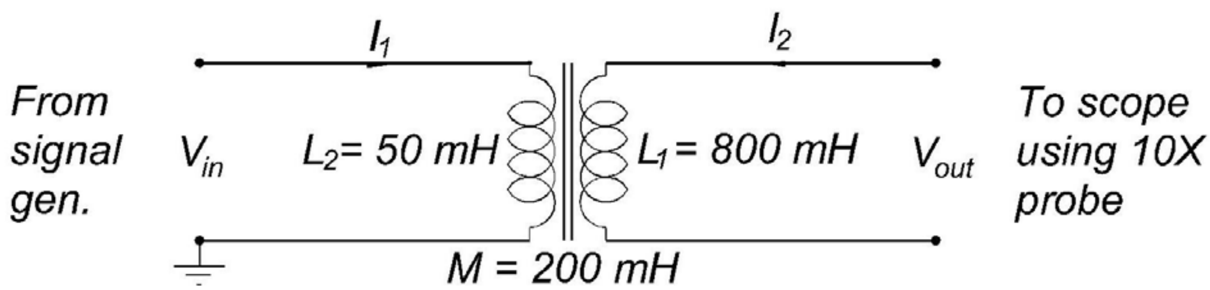
$$V_{out} = V_{in} \frac{N_2}{N_1},$$

provided that the load impedance is large compared to ωL_2, where $\omega = 2\pi f$ is the angular frequency of the input voltage source. Connect the Waketek signal generator to the primary side of the transformer, monitor this input voltage on Ch1 of your oscilloscope, and observe the output voltage on the secondary using the 10X probe.



Then, connect the signal generator to the secondary side and observe the voltage on the primary side. How do your results depend on the frequency of the signal generator?

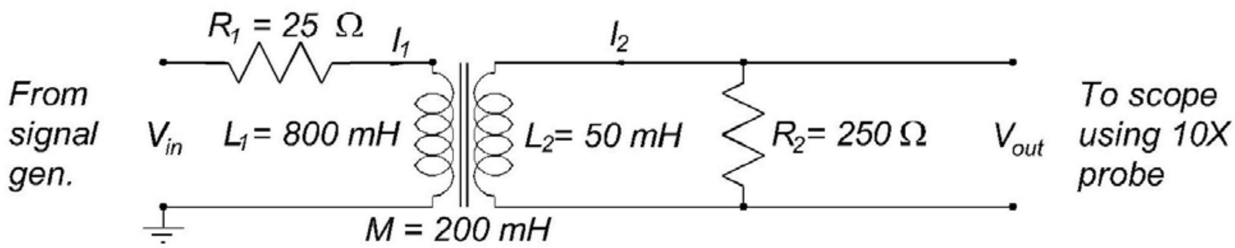
Your transformer was designed to be a "filament transformer", to provide 60 Hz power to the filament of vacuum tubes (back in the days when these devices were still common). It contains iron (the transformer is heavy for its size), whose magnetic properties are less than ideal at frequencies above 10^6 Hz.



The last experiment is to study the relation between the input and output currents and voltages. From part 2 of the Prelab exercise (and from eq. 29-63 of Tipler), the ideal behavior is that

$$V_{in} I_{in} = V_{out} I_{out}.$$

To be able to observe the output current, the impedance of the load that you apply to the transformer must be reasonable small. The impedance of the 10X probe is too high, so place a 250 Ω resistor across the secondary side of the transformer (using the breadboard), as shown in the diagram below. Turn up the Wavetek generator output to 20 $V_{p\text{-}p}$. Insert the multimeter, used in ammeter mode, into your circuit, first between the signal generator and one of the primary leads, and then between the 10X probe and one side of the 250 Ω resistor. Report the products $V_{in} I_{in}$ and $V_{out} I_{out}$ at several frequencies, say 10, 100, 1000, 10,000 and 100,000 Hz.



We can ignore the internal resistance of inductor 2 in the theory of this circuit, as this is small compared to 250 Ω. However, the analysis should include the internal resistance of inductor 1, which has been shown as 25 Ω on the figure above.

The analysis of the above circuit utilizes two loops, one containing V_{in}, R_1 and L_1, and the other containing L_2 and R_2. The impedance of the scope probe is so large that practically no current flows through the probe, so we leave the probe out of the analysis. The voltages in the two loops are coupled due to the mutual inductance M. The equation for the voltages in the first loop is

$$V_{\text{in}} = I_1 R_1 + V_1 = \left(R_1 + i\omega L_1\right) I_1 - i\omega M I_2,$$

where V_1 is the voltage across inductor 1 as given on p. 5. Similary, the equation for the voltages in the second loop is

$$0 = -i\omega M I_1 + \left(R_2 + i\omega L_2\right) I_2.$$

From the second of these equations we learn that

$$I_1 = I_2\left(\frac{L_2}{M} + \frac{R_2}{i\omega M}\right) = I_2\left(\frac{N_2}{N_1} + \frac{R_2}{i\omega M}\right).$$

The ideal behavior that $I_1 N_1 = I_2 N_2$ will hold only at frequencies large enough that $R_2 \ll \omega M$. To find the current I_2, we multiply the first voltage equation by $-i\omega M$, multiply the second voltage equation by $R_1 + i\omega L_1$, and subtract the first of the resulting equations from the second to find that

$$i\omega M V_{\text{in}} = I_2\left[R_1 R_2 + i\omega\left(L_1 R_2 + L_2 R_1\right) + \omega^2\left(M^2 - L_1 L_2\right)\right] = I_2\left[R_1 R_2 + i\omega\left(L_1 R_2 + L_2 R_1\right)\right],$$

assuming the transformer is ideal enough that $M^2 = L_1 L_2$. We are mainly interested in the output voltage across resistor 2,

$$V_{\text{out}} = I_2 R_2 = V_{\text{in}} \frac{i\omega M R_2}{R_1 R_2 + i\omega\left(L_1 R_2 + L_2 R_1\right)} = \frac{M}{L_1} V_{\text{in}} \frac{1}{1 + \dfrac{L_2 R_1}{L_1 R_2} + \dfrac{R_1}{i\omega L_1}} = \frac{N_2}{N_1} V_{\text{in}} \frac{1}{1 + \dfrac{L_2 R_1}{L_1 R_2} - \dfrac{i R_1}{\omega L_1}},$$

Your transformer was designed to operate at f = 60 Hz, *i.e.*, $\omega = 2\pi f = 377\,\text{Hz}$, in which case $R_1 \ll \omega L_1$. So for $f \geq 60\,\text{Hz}$ we can ignore the imaginary part in the above expression for V_{out}. Hence,

$$V_{\text{out}} \approx \frac{N_2}{N_1} V_{\text{in}} \frac{1}{1 + \dfrac{L_2 R_1}{L_1 R_2}}.$$

Multiplying this equation by the relation that $I_2 = I_1 N_1 / N_2$, we predict

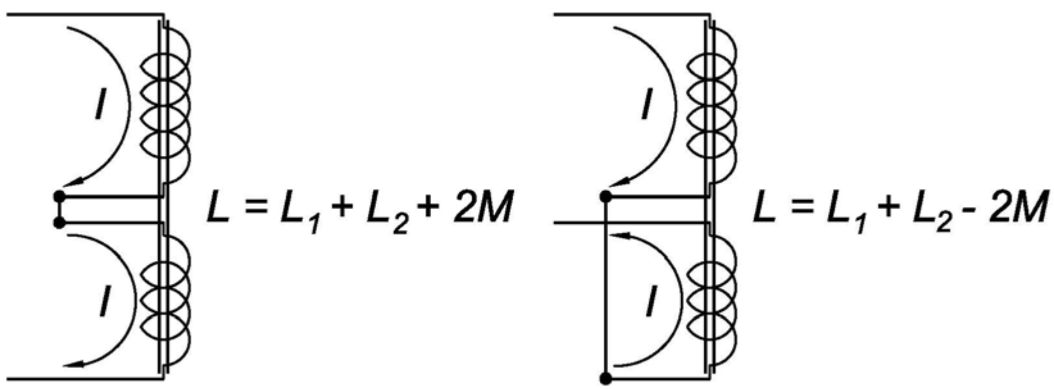
$$V_{\text{out}} I_2 \approx V_{\text{in}} I_1 \frac{1}{1+\frac{L_2 R_1}{L_1 R_2}},$$

at frequencies high enough that $\omega M \gg R_2$, but low enough that the transformer iron is not misbehaving.

Measurement of the inductances L_1, L_2 and M.

This part is optional because there is only one LCR multimeter per Lab. Consult with your AI for use of this meter.

With the LCR meter in L mode, you can directly measure the self inductances L_1 and L_2 by connecting the meter leads to the primary or secondary leads of the transformer. Measurement of the mutual inductance is more subtle. Connect the leads of the primary and secondary together in series, and use the LCR meter to measure the inductance of the combination.



If you get a value larger than either L_1 or L_2 then you probably have followed the hookup shown on the left side of the above figure. But if you found a value smaller than either L_1 or L_2 then you probably have followed the hookup shown on the right side of the above figure. In any case, you should measure the inductance using both hookups, which will provide you with two measurements of the mutual inductance *M*. The difference between these two measurements will give you an idea as to the accuracy of this technique.

The logic of this method is that when you hook up the transformer as shown on the left, you are creating an inductor of $N_1 + N_2$ turns, each of self inductance L_0. Then, as discussed earlier, the resulting inductance is $L = \left(N_1 + N_2\right)^2 L_0 = \left(N_1^2 + N_2^2 + 2N_2 N_2\right) L_0 = L_1 + L_2 + 2M$. But when you hook up the transformer as on the right, the turns of the two inductors oppose one another, and the effective total number of turns is only $\left|N_1 - N_2\right|$, in which case the observed inductance is $L = \left(N_1 - N_2\right)^2 L_0 = L_1 + L_2 - 2M$.

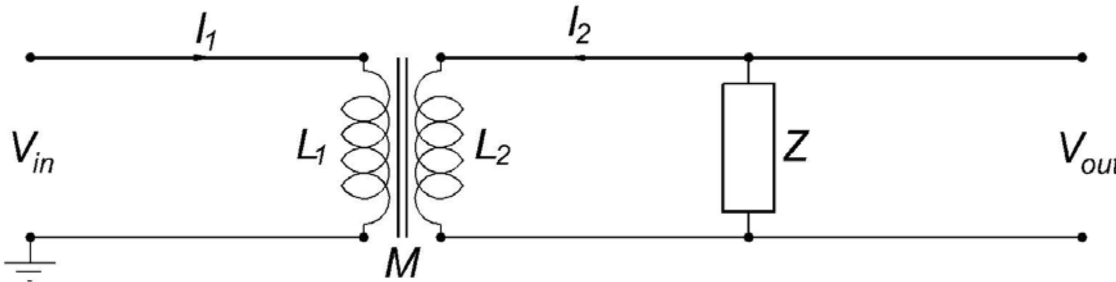
PRINCETON UNIVERSITY **PHYSICS 104 LAB**
Physics Department **Week #8**
Name:_____________________________ **Date/Time of Lab:_____________**

EXPERIMENT VII PRELAB PROBLEM SET

1. Analyze the series-parallel *RLC* circuit on page 2 of the AC circuits lab to find a (complex) expression for V_{out} / V_{in} of the form $Ae^{-i\delta}$, where $A = |V_{out}| / |V_{in}|$, and δ is the phase difference. Identify the resonant frequency of this circuit from this expression. If the tolerances in the components used are 10%, and 5% for *L* and *C* respectively, what is the uncertainty in the calculated value of the resonant frequency? (Try calculating the frequency using the largest possible values of *L*, *C*; and by using the smallest possible values of *L*, *C*).

(continued on reverse)

2. (Optional!) Analyze the idealized circuit shown below in which a transformer with self inductances $L_1 = N_1^2 L_0$ and $L_2 = N_2^2 L_0$ and mutual inductance $M = N_1 N_2 L_0$ has zero internal resistance. An oscillatory voltage source V_{in} of angular frequency ω is applied to the primary. The secondary has a load of impedance Z, which might be a resistor, a capacitor, an inductor, or some combination of these.



Write down the equations for the voltages in the two "loops" (first loop contains V_{in} and L_1 and carries current I_1, the second loop contains L_2 and Z and carries current I_2) which are coupled via the mutual inductance M. See p. 6 of the lab writeup if you need help.

Solve for the ratio I_1 / I_2 in terms of the numbers of turns N_1 and N_2 and other parameters. What relation must hold among those other parameters such that the ideal relation $I_1 N = I_2 N_2$ is true to a good approximation?

Solve for I_2 and then for $V_{\text{out}} = I_2 Z$ in terms of V_{in}, N_1 and N_2 and other parameters. You should now be able to confirm the ideal transformer equation that $V_{\text{in}} I_1 = V_{\text{out}} I_2$.