Princeton University

Ph205

Mechanics

Problem Set 10

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(1988)

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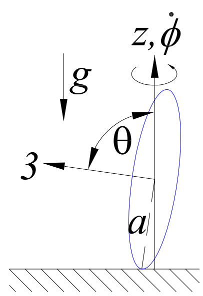
http://kirkmcd.princeton.edu/examples/

1. **Spinning Coin Revisited.**

It is possible to spin a coin on a horizontal table about a vertical diameter, with its center at rest. But, if the angular velocity becomes too low, the coin falls over and takes on the motion considered in Prob. 6, Set 9.

Consider a thin, uniform disk of mass m and radius a that spins without friction on a horizontal table (such that its center moves only vertically). Use Euler angles θ, ϕ and ψ, in the manner of Fig. 47, p. 110 of L.D. Landau and E.M. Lifshitz, *Mechanics*, $3^{\rm rd}$ ed. (Pergamon, 1976), http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf, and Lagrange's method to analyze the motion.



(a) Show that the steady precession rate about the vertical is,

$$\dot\phi_{\rm steady} = \frac{\omega_3 \pm \sqrt{\omega_3^2 + 4g\cos^2\theta/a\sin\theta}}{\cos\theta}, \tag{1}$$

where θ is the angle between the symmetry axis 3 of the disk to the vertical, and ω_3 is the angular velocity about that axis.

This relation becomes invalid at $\theta = 90^\circ$, when the coin is "on edge".

(b) Show that in this case the two possible classes of steady motion are,

- ω_3 arbitrary, $\dot\phi = 0, \Leftrightarrow$ rolling and slipping.
- $\omega_3 = 0$, $\dot\phi$ arbitrary, $\Leftrightarrow$ spinning on edge.

For the second case (spinning on edge), use a small-angle approximation, $\theta = \pi/2 + \epsilon$, to show that the motion is stable if $\dot\phi > 2\sqrt{g/a}$; otherwise the coin falls over.

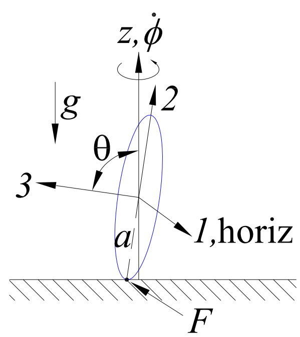
2. **Rolling Disk Revisited.**

Consider arbitrary motion of a disk that rolls without slipping on a horizontal plane. Experts will note that this example has 5 coordinates and 2 nonholonomic constraints. But, try it without resort to Lagrange multipliers.

The thin, uniform disk has mass m and radius a. Use a coordinate system that is similar to, but not quite the same as that of Euler:

- $\hat{\mathbf{z}}$ is vertical.
- Principal axis $\hat{\mathbf{1}}$ is always horizontal.
- Principal axis $\hat{\mathbf{2}}$ lies in a vertical plane that includes the center of the disk.
- Principal axis $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ is the symmetry axis of the disk. Axes $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$ lie in the same vertical plane.

The axes $(1, 2, 3)$ are principal axes, but they are not body axes (that are fixed with respect to the rotating disk).



Also define,

- θ = angle between $\hat{\mathbf{2}}$ and $\hat{\mathbf{z}}$.
- $\dot{\phi}$ = angular velocity of the disk (and of the $(1, 2, 3)$ axes) about the vertical.
- $\mathbf{F}$ = the (as yet unknown) force on the disk at the point of contact with the horizontal surface.

The "elementary" equations of motion are,

$$\mathbf{F}_{\text{total}} = m \frac{d\mathbf{v}_{\text{cm}}}{dt}, \qquad \boldsymbol{\tau}_{\text{cm}} = \frac{d\mathbf{L}_{\text{cm}}}{dt}. \tag{2}$$

The constraint of rolling without slipping can be written in terms of velocities (as a time-dependent version of Chasles' theorem),

$$\mathbf{v}_{\text{contact}} = 0 = \mathbf{v}_{\text{cm}} + \boldsymbol{\omega} \times \mathbf{a}, \tag{3}$$

where,

- $\mathbf{a} = -a\,\hat{\mathbf{2}}$ = vector from the center of mass to the point of contact.
- $\boldsymbol{\omega}_{(1,2,3)} = -\dot{\theta}\,\hat{\mathbf{1}} + \dot{\phi}\,\hat{\mathbf{z}}$ = angular velocity of the axes $(1, 2, 3)$.
- $\boldsymbol{\omega} = \boldsymbol{\omega}_{(1,2,3)} + \dot{\psi}\,\hat{\mathbf{3}}$ = total angular velocity of the disk.
- $\dot{\psi}$ = (spin) angular velocity of the disk relative to the $(1, 2, 3)$ axes.

Then, $\omega_3 = \omega_{(1,2,3)_3} + \dot{\psi}$.

Eliminate $\mathbf{F}$ and $\mathbf{v}_{\rm cm}$ to arrive at the equation of motion,

$$\frac{ma^2}{4}\frac{d}{dt}\left(-\dot{\theta}\,\hat{\mathbf{1}} + \dot{\phi}\sin\theta\,\hat{\mathbf{2}} + 2\omega_3\,\hat{\mathbf{3}}\right) = mga\cos\theta\,\hat{\mathbf{1}} - ma^2\,\hat{\mathbf{2}} \times \frac{d}{dt}\left(\omega_3\,\hat{\mathbf{1}} + \dot{\theta}\,\hat{\mathbf{3}}\right). \tag{4}$$

Note that for axis $\hat{\mathbf{i}}$,

$$\frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{(1,2,3)} \times \hat{\mathbf{i}}. \tag{5}$$

Show that for steady motion ($\dot{\phi},\ \omega_3$ constant, $\dot{\theta} = 0$),

$$\dot{\phi}^2\sin\theta\cos\theta - 6\omega_3\dot{\phi}\sin\theta - \frac{4g}{a}\cos\theta = 0. \tag{6}$$

At $\theta = \pi/2$ the disk is "on edge". Here, we are interested in the rolling motion $\dot{\phi} = 0$ but ω_3 arbitrary. Is this motion stable?

To answer this, consider $\theta = \pi/2 + \epsilon$ for small ϵ, small $\dot{\phi}$ and arbitrary ω_3. Ignore 2nd-order terms in the equation of motion, such as ϵ^2 and $\epsilon\,\dot{\theta}$, to show that the:

- $\hat{\mathbf{1}}$ terms $\Rightarrow \ddot{\epsilon} \approx -\frac{4}{5}\left(3\omega_3^2 - \frac{g}{a}\right)\epsilon$.
- $\hat{\mathbf{2}}$ terms $\Rightarrow \dot{\phi} \approx 2\,\epsilon\,\omega_3\epsilon$.
- $\hat{\mathbf{3}}$ terms $\Rightarrow \dot{\omega}_3 \approx 0$ (ω_3 constant).

Hence, the rolling "on edge" is stable if $\omega_3 > \sqrt{g/3a}$; otherwise the disk falls over into (generally unstable) motion of the form considered in Prob. 5, Set 9.

3. **Marble Rolling on a Turntable.** (Is this how you lost your marbles?)

A marble (uniform sphere of mass m and radius a) rolls without slipping on a horizontal turntable that rotates with constant angular velocity $\mathbf{\Omega}$ about the symmetry axis of the turntable.

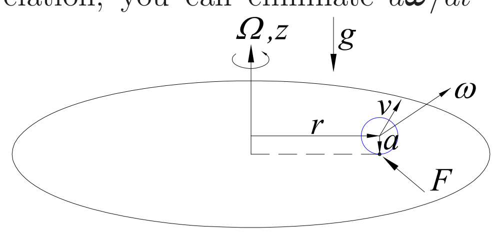
Use the vectorial approach of Prob. 2 to analyze the motion by "elementary" methods in the lab frame.

(a) What is the (nonholonomic) rolling constraint between $\mathbf{a}$, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{\Omega}$ and $\boldsymbol{\omega}$ where,

- $\mathbf{a}$ = vector from the center of the marble to the point of contact with the turntable.
- $\mathbf{r}$ = vector perpendicular to the symmetry axis of the turntable to the center of the marble.
- $\mathbf{v}$ = velocity of the center (of mass of) the marble.
- $\boldsymbol{\omega}$ = total angular velocity of the marble in the lab frame.

(b) Consider arbitrary motion of the marble (rolling without slipping on the turntable). Note that by differentiating the constraint relation, you can eliminate $d\boldsymbol{\omega}/dt$ from the equations of motion, leading to,

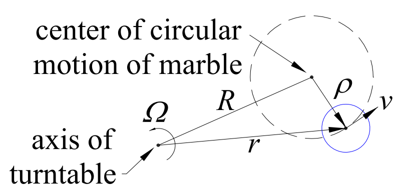
$$\frac{I + ma^2}{I}\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{\Omega} \times \mathbf{r}), \qquad (7)$$



with Ω constant. Show that this leads to,

$$\mathbf{v} = \frac{I}{I + ma^2}\mathbf{\Omega} \times \boldsymbol{\rho}, \quad \text{where} \quad \mathbf{r} = \mathbf{R} + \boldsymbol{\rho} \quad \text{and} \quad \mathbf{R} = \mathbf{r}_0 + \frac{I + ma^2}{I}\frac{\mathbf{\Omega} \times \mathbf{v}_0}{\Omega}, \qquad (8)$$

where $\boldsymbol{\rho}$ is the radius vector and $\mathbf{r}_0$ and $\mathbf{v}_0$ are the initial position of velocity of the center of the marble. This is motion in a circle of radius ρ about the axis parallel to $\mathbf{a}$ at distance $\mathbf{R}$ from the axis of the turntable.



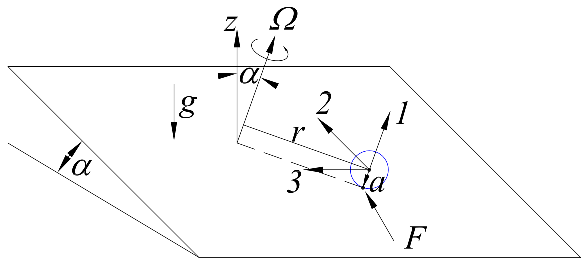
Ignore a possible "spin" angular velocity of the marble about the axis $\mathbf{a}$, which "spin" would be independent of Ω, and find the total angular velocity $\boldsymbol{\omega}$ of the marble.

You should find that the angular velocity component about the vertical is $\Omega I/(I + ma^2) = 2\Omega/7$ *for a uniform sphere..*

(c) Suppose that the plane of the turntable makes angle α to the horizontal.

Note that the normal force of the turntable on the marble is not necessarily $mg\cos\alpha$.

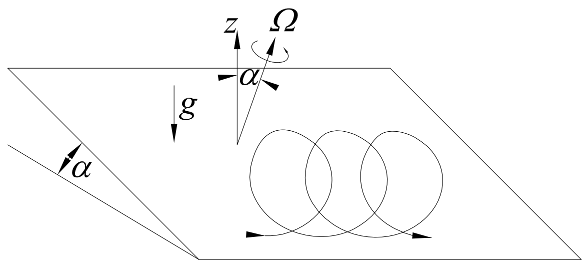
Consider axes $\hat{\mathbf{1}}$ perpendicular to the tilted turntable, $\hat{\mathbf{2}}$ pointing up the slope, and $\hat{\mathbf{3}}$ horizontal.



Deduce the equation of motion, slightly modified from that in part (b). Show that a solution is $\mathbf{v} = \mathbf{v}_{(b)} + \mathbf{v}_{\rm drift}$ where $\mathbf{v}_{(b)}$ is that found in part (b) (in the $(1, 2, 3)$ coordinate system) and,

$$\mathbf{v}_d = \frac{mga^2 \sin\alpha}{I\Omega}\,\hat{\mathbf{3}}. = \text{constant vector} \tag{9}$$

Hence, the motion involves a horizontal drift, as sketched in the figure below.

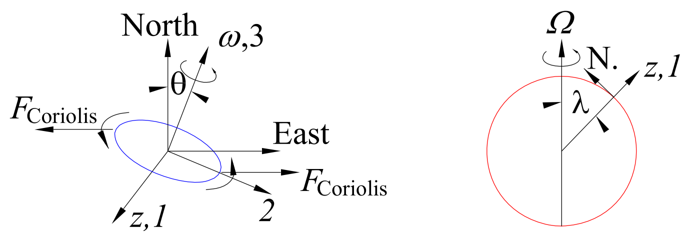


Why not try it?

4. **Gyrocompass.**

A gyrocompass is a spinning flywheel whose axis $\boldsymbol{\omega}$ of rotation is constrained to lie in a horizontal plane at the surface of the Earth.

If we analyze the motion in a frame fixed to the surface of the (spinning) Earth, the Coriolis force must be taken into account. When $\boldsymbol{\omega}$ makes angle θ to the North, as shown in the figure, the left side of the flywheel is moving up, and the Coriolis force on it is to the West. Similarly, the right side of the flywheel is moving down, and the Coriolis force on it is to the East. Hence, there is a net torque on the flywheel that tends to restore θ to zero, *i.e.*, to the North.



(a) Suppose the flywheel is a hoop of mass m and radius a. Calculate the total Coriolis torque about the center of the hoop to show that,

$$\tau_1 = ma^2\,\omega\,\Omega\sin\lambda\sin\theta, \qquad \tau_2 = -ma^2\,\omega\,\Omega\sin\lambda\cos\theta, \qquad \tau_3 = 0, \tag{10}$$

where $\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}$ are principal axes (but not body axes), with $\hat{\mathbf{1}}$ vertical upwards (*i.e.*, $\hat{\mathbf{1}} = \hat{\mathbf{z}}$), $\hat{\mathbf{2}}$ horizontal, and $\hat{\mathbf{3}}$ along the symmetry axis of the hoop; Ω is the angular velocity of the Earth, and λ is the colatitude of the gyrocompass.

Show that this torque causes the gyrocompass to make small oscillations in θ with angular frequency $\sqrt{2\,\omega\,\Omega\sin\lambda}$.

(b) Analyze the motion in an inertial frame, where the torque equation about the center of the hoop is $\boldsymbol{\tau} = d\mathbf{L}/dt$, where $\mathbf{L}$ is the angular momentum. In this frame, the torque is only due to the constraint forces on the axle of the gyrocompass, which keep the axle in the horizontal plane with respect to the Earth, but which do not make the gyro point North.

Note that $\mathbf{L} = \mathsf{I}\cdot\boldsymbol{\omega}_{\text{total}}$, where the total angular velocity has three pieces,

- Rotation about the gyro axle (axis $\hat{\mathbf{3}}$) with angular velocity $\boldsymbol{\omega}$.
- Rotation at angular velocity $\dot{\theta}$ about the local vertical axis ($\hat{\mathbf{z}} = \hat{\mathbf{1}}$) with respect to the Earth.
- Rotation at angular velocity Ω of the Earth about its axis.

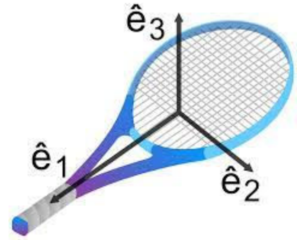
The principal axes $\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}$ introduced in part (a) rotate with angular velocity $\boldsymbol{\omega}_{123} = \dot{\boldsymbol{\theta}} + \boldsymbol{\omega}$.

Show that the components of the torque equation for with $\tau_i = 0$ (in the inertial frame) lead to $\dot{\omega} = 0$ and $\ddot{\theta} + (2\,\lambda\,\Omega\sin\lambda)\,\theta = 0$, which leads again to the result of part (a).

In practice, a motor is required to keep ω constant. And, the gyrocompass must have a mechanism to find the horizontal plane even when its supports are tipping, *as on an airplane or ship. This requires a "plumb bob", and a mechanism to defeat the effect of a possibly oscillation point of support...*

5. **The Tennis Racquet Theorem.**

Consider a rigid body whose principal moments of inertia are $I_1 < I_2 < I_3$. As discussed in sec. 37 of Landau's *Mechanics*, free rotation with angular velocity $\boldsymbol{\omega}$ pointing close to axis 2 is "unstable".



Examine the special case where the kinetic energy has the form $T = L^2/2I_2$, and $\mathbf{L}$ is the angular momentum about the center of mass. Use expressions for T and $\mathbf{L}$ to show that,

$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_1 I_2}, \qquad \omega_3^2 = \frac{I_2 - I_1}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_2 I_3}, \tag{11}$$

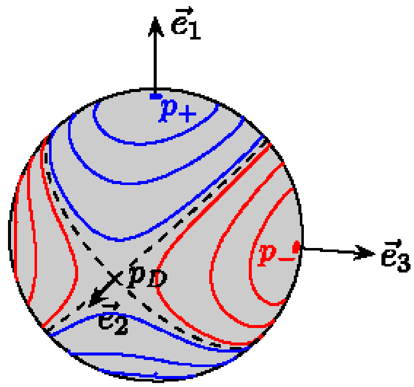
and that Euler's equations lead to,

$$\omega_1 = \omega_{1,\max}\,\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)], \tag{12}$$

$$\omega_2 = \omega_{2,\max}\tanh[k\,\omega_{2,\max}\,(t - t_0)], \qquad \omega_{2,\max} = \frac{L}{I_2}, \qquad k = \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}}, \tag{13}$$

$$\omega_3 = \omega_{3,\max}\,\mathrm{sech}[k\,\omega_{2,\max}\,(t - t_0)]. \tag{14}$$

As $t \to \infty$, $\omega_1,\ \omega_3 \to 0$ while $\omega_2 \to \omega_{2,\max}$, and the final rotation is about axis 2. Thus, for this special case of motion along "separating polhodes", a kind of stability occurs. That is, the motion consider in this problem is along the dashed lines in the figure below.



In practice, the special case is hard to achieve, since for any slight perturbation of the kinetic energy T away from $L^2/2I_2$, $\boldsymbol{\omega}$ will move towards axis $\mathbf{2}$ along a path close to one of the separating polhodes, then "bounce away" from the $\hat{\mathbf{2}}$ axis and move towards axis $-\hat{\mathbf{2}}$ along a path close to the other separating polhode, "bounce away" from this axis, and repeat the cycle.... To a viewer of the spinning tennis racquet, this cycle seems "unstable" because the axis of rotation migrates between $\hat{\mathbf{2}}$ and $-\hat{\mathbf{2}}$ every half cycle, although in the mathematical sense it is a "stable" orbit.

We infer from this problem that the cycle time for trajectories very close to the separating polhodes is very long, approaching infinity in the limit considered here. The long period of such cycles contributes to the impression by the "casual" observer that the motion is "unstable".

One of many YouTube videos on this theme: https://www.youtube.com/watch?v=1VPfZ_XzisU

6. **Ball and Paper**

A uniform ball initially rolls without slipping on a sheet of paper that is at rest on a horizontal surface (also at rest). Then, the paper is given an arbitrary, horizontal motion (which may include jerks), such that the ball eventually rolls off the paper onto the horizontal surface.

In the trivial case that the paper is not moved, the velocity of the ball once off the paper is the same as its initial velocity. Show that the final velocity of the ball (after it rolls off the paper) is the same as its initial velocity even when the paper is moved (and the ball rolls with slipping when it first comes off the paper).

7. **Gyroscope Revisited** (Optional Challenge Problem)

In Lecture 19 of the Notes, http://kirkmcd.princeton.edu/examples/Ph205/ph205l19.pdf the problem of a gyroscope with one point fixed was analyzed in the classic method of Lagrange, which avoids mention of the force on the pivot.

Give a discussion of that force when the gyro is launched from rest at angle θ_1, such that the pattern of the nutation is Fig. 11-17(c) below (from S.T. Thornton and J.B. Marion, *Classical Dynamics of Particles and Systems*, 5^{th} ed. (Brooks/Cole, 2004), kirkmcd.princeton.edu/examples/EM/marion.pdf

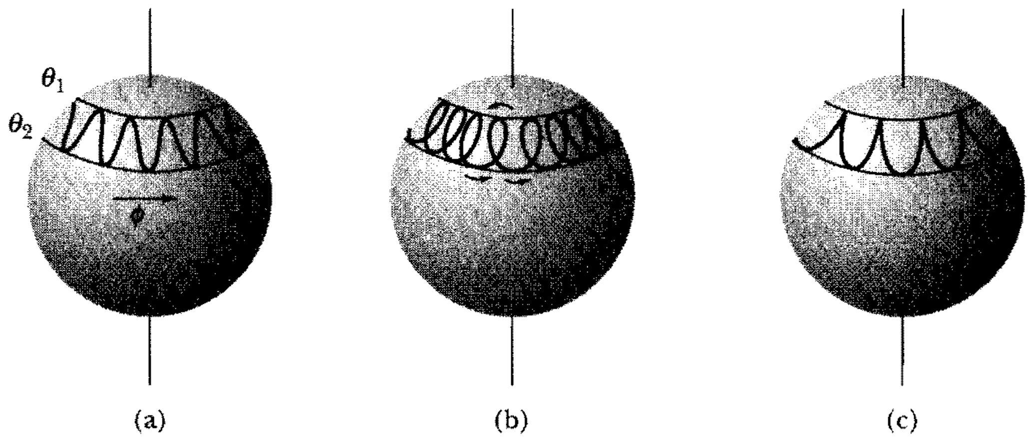


FIGURE 11-17 The rotating top also nutates between the limit angles θ_1 and θ_2. In (a) $\dot{\phi}$ does not change sign. In (b) $\dot{\phi}$ does change sign, and we see looping motion. In (c) the initial conditions include $\dot{\theta} = \dot{\phi} = 0$; this is the normal cusp-like motion when we spin a top and release it.

Use the center of mass as the reference point for torque, the inertia tensor and angular momentum, as well as the pivot point.

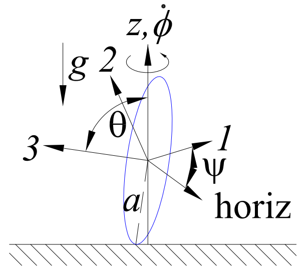
Solutions

1. **Spinning Coin Revisited.**

This appears as Prob. 11-58, p. 353 of W. Chester, *Mechanics* (Allen & Unwin, 1979), http://kirkmcd.princeton.edu/examples/mechanics/chester_mechanics_79.pdf

We consider a coin spinning without friction on a horizontal table. Unlike Prob. 6, Set 9, the instantaneous axis is not necessarily the diameter in the vertical plane.

We use Lagrange's method to find the motion in terms of three Euler angles, θ = angle of the symmetry axis of the coin to the vertical, ϕ = azimuthal angle of the horizontal diameter of the coin, and ψ = angle to the horizontal of body axis 1, where body axes 1 and 2 are in the plane of the disk and axis 3 is the symmetry axis.



(a) As deduced in eq. (35.2), p. 111 of L.D. Landau and E.M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, 1976), http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf, the kinetic energy of rotation about the center of the disk is,

$$T_{\rm rot} = \frac{I_1}{2}\left(\dot\theta^2 + \dot\phi^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot\phi\cos\theta + \dot\psi\right)^2, \tag{15}$$

where for a thin, uniform disk of mass m and radius a, $I_1 = I_2 = ma^2/4$ and $I_3 = ma^2/2$.

The center of the disk is at height $z = a\sin\theta$ above the table, so the kinetic energy of the motion of the center of mass of the disk is,

$$T_{\rm cm} = \frac{m\dot z^2}{2} = \frac{ma^2\dot\theta^2\cos^2\theta}{2}, \tag{16}$$

and the potential energy can be written as,

$$V = mga\sin\theta. \tag{17}$$

The energy $E = T + V = T_{\rm cm} + T_{\rm rot} + V$ is conserved, and as the Lagrangian $\mathcal{L} = T - V$ does not depend on either ϕ or ψ, the generalized momenta P_ϕ and P_ψ are also conserved,

$$P_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = \frac{\partial T_{\rm rot}}{\partial\dot\phi} = \left(I_1\sin^2\theta + I_3\cos^2\theta\right)\dot\phi + I_3\,\dot\psi\,\cos\theta, \tag{18}$$

$$P_\psi = \frac{\partial\mathcal{L}}{\partial\dot\psi} = \frac{\partial T_{\rm rot}}{\partial\dot\psi} = I_3\left(\dot\phi\,\cos\theta + \dot\psi\right). \tag{19}$$

Note that the total angular momentum vector is (from eq. (35.1), p. 111 of the above link),

$$\boldsymbol{\omega} = \left(\dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi, \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi, \dot\phi \cos\theta + \dot\psi\right), \tag{20}$$

so we recognize that the conserved, generalized momentum P_ψ is the angular momentum about axis 3,

$$P_\psi = I_3 \left(\dot\phi \cos\theta + \dot\psi\right) = I_3\,\omega_3 = L_3. \tag{21}$$

The forces on the spinning coin are only vertical, so the angular momentum L_z about the vertical axis is also conserved. Since angle ϕ is the azimuth about the vertical axis, we anticipate that the conserved, generalized momentum P_ϕ is L_z. To verify this, we consider a moment when angle $\psi = 0$, axis 1 is horizontal, and axis 2 is in a vertical plane containing the center of the coin. Then, the total angular momentum in the $(1, 2, 3)$ system is $\boldsymbol{\omega} = \left(\dot\theta, \dot\phi \sin\theta, \dot\phi \cos\theta + \dot\psi\right)$, the angular momentum is,

$$\mathbf{L} = \mathsf{I} \cdot \boldsymbol{\omega} = \left(I_1\dot\theta, I_2\,\dot\phi \sin\theta, I_3\left(\dot\phi \cos\theta + \dot\psi\right)\right), \tag{22}$$

and,

$$L_z = L_2 \sin\theta + L_3 \cos\theta = I_1\,\dot\phi \sin^2\theta + I_3\left(\dot\phi \cos^2\theta + \dot\psi \cos\theta\right) = P_\phi. \tag{23}$$

Of the equations of motion, only that for coordinate θ remains to be discussed,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = I_1\,\ddot\theta = \frac{\partial\mathcal{L}}{\partial\theta} = I_1\,\dot\phi^2 \sin\theta\cos\theta - I_3\left(\dot\phi \cos\theta + \dot\psi\right)\dot\phi \sin\theta - mga\cos\theta$$

$$= \sin\theta\left(I_1\,\dot\phi^2 \cos\theta - I_3\,\omega_3\,\dot\phi - \frac{mga\cos\theta}{\sin\theta}\right) = \frac{ma^2\sin\theta}{4}\left(\dot\phi^2\cos\theta - 2\omega_3\,\dot\phi - \frac{4g\cos\theta}{a\sin\theta}\right). \tag{24}$$

For steady motion, $\ddot\theta = 0$, and,

$$\dot\phi_{\text{steady}} = \frac{\omega_3 \pm \sqrt{\omega_3^2 + 4g\cos^2\theta/a\sin\theta}}{\cos\theta}. \tag{25}$$

This relation becomes invalid at $\theta = 90°$, when the coin is "on edge".

For $\theta = 0$, eq. (24) provides no constraint on the steady motion, so $\dot\phi = \omega_3$ (in this case) is arbitrary (in the idealization of zero friction).

(b) For steady motion at $\theta = 90°$, eq. (24) reduces to $I_3\,\omega_3\,\dot\phi = 0$, which implies that,

- Either ω_3 arbitrary, $\dot\phi = 0, \Leftrightarrow$ rolling and slipping,
- Or $\omega_3 = 0$, $\dot\phi$ arbitrary, $\Leftrightarrow$ spinning on edge.

To discuss the stability of the second case, spinning on edge, we consider a small departure, $\theta = \pi/2 + \epsilon$ from the steady motion. Then, $\ddot\theta = \ddot\epsilon$, $\cos\theta \approx -\epsilon$, $\sin\theta \approx 1$, and the equation of motion (24) becomes,

$$I_1\,\ddot\epsilon \approx -I_1\,\dot\phi^2\,\epsilon - I_3\,\omega_3\,\dot\phi + mga\epsilon = -\left(I_1\,\dot\phi^2 - mga\right)\epsilon - L_3\,\dot\phi, \tag{26}$$

which is stable (simple harmonic motion in ϵ with constant $\dot\phi$) for,

$$\dot\phi > \sqrt{\frac{mga}{I_1}} = 2\sqrt{\frac{g}{a}}. \tag{27}$$

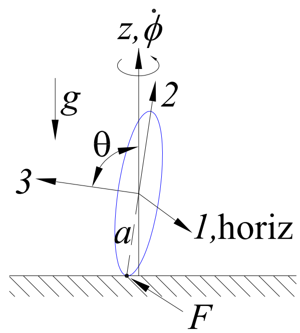
2. **Rolling Disk Revisited.**

We consider arbitrary motion of a thin, uniform disk of mass m and radius a that rolls without slipping on a horizontal plane.

We use a coordinate system that is similar to, but not quite the same as that of Euler:

- $\hat{\mathbf{z}}$ is vertical.
- Principal axis $\hat{\mathbf{1}}$ is always horizontal.
- Principal axis $\hat{\mathbf{2}}$ lies in a vertical plane that includes the center of the disk.
- Principal axis $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ is the symmetry axis of the disk.

The axes $(1, 2, 3)$ are principal axes, but they are not body axes (that are fixed with respect to the rotating disk).



We also define,

- θ = angle between $\hat{\mathbf{3}}$ and $\hat{\mathbf{z}}$.
- $\dot{\phi}$ = angular velocity of the disk, and of the $(1, 2, 3)$ axes, about the vertical.
- $\mathbf{F}$ = the force on the disk at the point of contact with the horizontal surface.

The "elementary" equations of motion are,

$$\mathbf{F}_{\text{total}} = \mathbf{F} - mg\,\hat{\mathbf{z}} = m\frac{d\mathbf{v}_{\text{cm}}}{dt}\,, \qquad \boldsymbol{\tau}_{\text{cm}} = \mathbf{a} \times \mathbf{F} = \frac{d\mathbf{L}_{\text{cm}}}{dt}\,. \tag{28}$$

The constraint of rolling without slipping can be written in terms of velocities (as a time-dependent version of Chasles' theorem),

$$\mathbf{v}_{\text{contact}} = 0 = \mathbf{v}_{\text{cm}} + \boldsymbol{\omega} \times \mathbf{a}, \tag{29}$$

where,

- $\mathbf{a} = -a\,\hat{\mathbf{2}}$ = vector from the center of mass to the point of contact.
- $\boldsymbol{\omega}_{(1,2,3)} = \dot{\theta}\,\hat{\mathbf{1}} + \dot{\phi}\,\hat{\mathbf{z}}$ = angular velocity of the axes $(1, 2, 3)$.
- $\dot{\psi}$ = (spin) angular velocity of the disk relative to the $(1, 2, 3)$ axes.
- $\boldsymbol{\omega} = \boldsymbol{\omega}_{(1,2,3)} + \dot{\psi}\,\hat{\mathbf{3}}$ = total angular velocity of the disk.

We also have that the principal moments of inertial about the center of the disk are $I_1 = I_2 = I_3/2 = ma^2/4$, and,

$$\hat{\mathbf{z}} = \sin\theta\,\hat{\mathbf{2}} + \cos\theta\,\hat{\mathbf{3}}, \tag{30}$$

$$\boldsymbol{\omega} = \dot\theta\,\hat{\mathbf{1}} + \dot\phi\,\hat{\mathbf{z}} + \dot\psi\,\hat{\mathbf{3}} = \dot\theta\,\hat{\mathbf{1}} + \dot\phi\sin\theta\,\hat{\mathbf{2}} + \left(\dot\psi + \dot\phi\cos\theta\right)\,\hat{\mathbf{3}}, \tag{31}$$

$$\mathbf{L}_{\rm cm} = \mathsf{I}_{\rm cm}\cdot\boldsymbol{\omega} = \left[I_1\dot\theta\,\hat{\mathbf{1}} + I_1\dot\phi\sin\theta\,\hat{\mathbf{2}} + I_3\left(\dot\psi + \dot\phi\cos\theta\right)\,\hat{\mathbf{3}}\right]. \tag{32}$$

We note that $\omega_3 = \dot\psi + \dot\phi\cos\theta$.

From the rolling constraint (29), we have that,

$$\mathbf{v}_{\rm cm} = -\boldsymbol{\omega}\times\mathbf{a} = a\boldsymbol{\omega}\times\hat{\mathbf{2}}, \qquad \frac{d\mathbf{v}_{\rm cm}}{dt} = a\frac{d}{dt}(\boldsymbol{\omega}\times\hat{\mathbf{2}}). \tag{33}$$

The equation of motion of the center of mass can now be rewritten as,

$$\mathbf{F} = mg\,\hat{\mathbf{z}} + m\frac{d\mathbf{v}_{\rm cm}}{dt} = mg(\sin\theta\,\hat{\mathbf{2}} + \cos\theta\,\hat{\mathbf{3}}) + ma\frac{d}{dt}(\boldsymbol{\omega}\times\hat{\mathbf{2}}), \tag{34}$$

and the torque equation about the center of mass can be rewritten as,

$$\begin{aligned}\boldsymbol{\tau}_{\rm cm} = \mathbf{a}\times\mathbf{F} &= -a\,\hat{\mathbf{2}}\times\left[mg(\sin\theta\,\hat{\mathbf{2}} + \cos\theta\,\hat{\mathbf{3}}) + ma\frac{d}{dt}(\boldsymbol{\omega}\times\hat{\mathbf{2}})\right]\\ &= -mga\cos\theta\,\hat{\mathbf{1}} + ma^2\,\hat{\mathbf{2}}\times\frac{d}{dt}\left(\omega_3\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{3}}\right)\\ &= \frac{d\mathbf{L}_{\rm cm}}{dt} = \frac{d}{dt}\left(I_1\dot\theta\,\hat{\mathbf{1}} + I_1\dot\phi\sin\theta\,\hat{\mathbf{2}} + I_3\omega_3\,\hat{\mathbf{3}}\right).\end{aligned} \tag{35}$$

Note that for axis $\hat{\mathbf{i}}$,

$$\frac{d\,\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{(1,2,3)}\times\hat{\mathbf{i}} = \left(\dot\theta\,\hat{\mathbf{1}} + \dot\phi\,\hat{\mathbf{z}}\right)\times\hat{\mathbf{i}} = \left(\dot\theta\,\hat{\mathbf{1}} + \dot\phi\sin\theta\,\hat{\mathbf{2}} + \dot\phi\cos\theta\,\hat{\mathbf{3}}\right)\times\hat{\mathbf{i}}, \tag{36}$$

$$\frac{d\,\hat{\mathbf{1}}}{dt} = \dot\phi\cos\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{2}}}{dt} = -\dot\phi\cos\theta\,\hat{\mathbf{1}} + \dot\theta\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{3}}}{dt} = \dot\phi\sin\theta\,\hat{\mathbf{1}} - \dot\theta\,\hat{\mathbf{2}}. \tag{37}$$

For steady motion, $\dot\phi$, ω_3 constant, $\dot\theta = 0$, we have,

$$\frac{d\,\hat{\mathbf{1}}}{dt} = \dot\phi\cos\theta\,\hat{\mathbf{2}} - \dot\phi\sin\theta\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{2}}}{dt} = -\dot\phi\cos\theta\,\hat{\mathbf{1}}, \qquad \frac{d\,\hat{\mathbf{3}}}{dt} = \dot\phi\sin\theta\,\hat{\mathbf{1}}, \tag{38}$$

and only the $\hat{\mathbf{1}}$-component of eq. (35) is nonzero,

$$-\,mga\cos\theta - ma^2\omega_3\,\dot\phi\sin\theta = \left(-I_1\dot\phi^2\sin\theta\cos\theta + I_3\,\omega_3\,\dot\phi\sin\theta\right) \tag{39}$$

$$\dot\phi^2\sin\theta\cos\theta - 6\omega_3\dot\phi\sin\theta - \frac{4g}{a}\cos\theta = 0. \tag{40}$$

At $\theta = \pi/2$ the disk is "on edge". Here, we are interested in the rolling motion $\dot\phi = 0$ but ω_3 arbitrary. Is this motion stable?

To answer this, we consider $\theta = \pi/2 + \epsilon$ for small ϵ, small $\dot\phi$ and arbitrary ω_3. Then, $\dot\theta = \dot\epsilon$, $\ddot\theta = \ddot\epsilon$, $\cos\theta \approx -\epsilon$, $\sin\theta \approx 1$, and on ignoring small terms like $\epsilon\,\dot\phi$, and we have,

$$\frac{d\,\hat{\mathbf{1}}}{dt} \approx -\epsilon\,\dot\phi\,\hat{\mathbf{2}} - \dot\phi\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{2}}}{dt} \approx \epsilon\,\dot\phi\,\hat{\mathbf{1}} + \dot\epsilon\,\hat{\mathbf{3}}, \qquad \frac{d\,\hat{\mathbf{3}}}{dt} \approx \dot\phi\,\hat{\mathbf{1}} - \dot\epsilon\,\hat{\mathbf{2}}, \tag{41}$$

and eq. (35) becomes, (again ignoring 2$^{\rm nd}$-order terms),

$$\begin{aligned}
\boldsymbol{\tau} &\approx \epsilon\,mga\,\hat{\mathbf{1}} + \hat{\mathbf{2}} \times ma^2 \frac{d}{dt}\left(\omega_3\,\hat{\mathbf{1}} - \dot\epsilon\,\hat{\mathbf{3}}\right) \\
&\approx \epsilon\,mga\,\hat{\mathbf{1}} + \hat{\mathbf{2}} \times ma^2 \left(\dot\omega_3\,\hat{\mathbf{1}} - \ddot\epsilon\,\hat{\mathbf{3}} - \epsilon\,\omega_3\,\dot\phi\,\hat{\mathbf{2}} - \omega_3\,\dot\phi\,\hat{\mathbf{3}} - \dot\epsilon\,\dot\phi\,\hat{\mathbf{1}}\right) \\
&= \epsilon\,mga\,\hat{\mathbf{1}} + ma^2 \left(-\dot\omega_3\,\hat{\mathbf{3}} - \ddot\epsilon\,\hat{\mathbf{1}} - \omega_3\,\dot\phi\,\hat{\mathbf{1}} + \dot\epsilon\,\dot\phi\,\hat{\mathbf{3}}\right) \\
&= \left(\epsilon\,mga - ma^2(\ddot\epsilon + \omega_3\,\dot\phi)\right)\,\hat{\mathbf{1}} - ma^2\left(\dot\omega_3 - \dot\epsilon\,\dot\phi\right)\,\hat{\mathbf{3}} \\
&= \frac{d\mathbf{L}}{dt} \approx \frac{d}{dt}\left(I_1\,\dot\epsilon\,\hat{\mathbf{1}} + I_1\,\dot\phi\,\hat{\mathbf{2}} + I_3\,\omega_3\,\hat{\mathbf{3}}\right) \\
&\approx I_1\,\ddot\epsilon\,\hat{\mathbf{1}} + I_1\,\ddot\phi\,\hat{\mathbf{2}} + I_3\,\dot\omega_3\,\hat{\mathbf{3}} - I_1\,\dot\epsilon\,\dot\phi\,\hat{\mathbf{3}} + I_1\,\dot\phi\,\dot\epsilon\,\hat{\mathbf{3}} + I_3\,\omega_3\,\dot\phi\,\hat{\mathbf{1}} - I_3\,\dot\epsilon\,\omega_3\,\hat{\mathbf{2}} \\
&= \left(I_1\,\ddot\epsilon + I_3\,\omega_3\,\dot\phi\right)\,\hat{\mathbf{1}} + \left(I_1\,\ddot\phi - I_3\,\dot\epsilon\,\omega_3\right)\,\hat{\mathbf{2}} + I_3\,\dot\omega_3\,\hat{\mathbf{3}}.
\end{aligned} \tag{42}$$

Then,

- $\hat{\mathbf{3}}$ terms $\Rightarrow (ma^2 + I_3)\,\dot\omega_3 \approx ma^2\,\dot\epsilon\,\dot\phi \approx 0, \quad \Rightarrow \quad \omega_3 \approx$ constant, since $\epsilon\,\dot\phi$ is of 2$^{\rm nd}$ order.
- $\hat{\mathbf{2}}$ terms $\Rightarrow I_1\,\ddot\phi \approx I_3\,\dot\epsilon\,\omega_3 \quad \Rightarrow \quad \dot\phi = I_3\,\epsilon\,\omega_3/I_1 = 2\,\epsilon\,\omega_3$.
- $\hat{\mathbf{1}}$ terms $\Rightarrow (ma^2+I_1)\,\ddot\epsilon \approx -(ma^2+I_3)\,\omega_3\dot\phi + \epsilon\,mga \approx -\epsilon(I_3(ma^2+I_3)\omega_3^2/I_1 - mga)$.

Hence, the rolling "on edge" is stable if $\omega_3 > \sqrt{I_1 mga/I_3(ma^2+I_3)} = \sqrt{g/3a}$; otherwise the disk falls over into (generally unstable) motion of the form considered in Prob. 5, Set 9.

This problem is Ex. 11.6.5, p. 339 of W. Chester, *Mechanics* (Allen & Unwin, 1979),
http://kirkmcd.princeton.edu/examples/mechanics/chester_mechanics_79.pdf
Additional aspects of this problem are discussed in
http://kirkmcd.princeton.edu/examples/rollingdisk.pdf

3. **Marble Rolling on a Turntable.**

This problem first appeared on pp. 280-283 of S. Earnshaw, *Dynamics*, 3rd ed. (Cambridge, 1844), http://kirkmcd.princeton.edu/examples/mechanics/earnshaw_44.pdf
See also, http://kirkmcd.princeton.edu/examples/mechanics/gray_18.pdf, pp. 514-516
http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf, pp. 303-307
http://kirkmcd.princeton.edu/examples/mechanics/weltner_ajp_47_984_79.pdf
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http://kirkmcd.princeton.edu/examples/mechanics/gersten_ajp_60_43_92.pdf
http://kirkmcd.princeton.edu/examples/mechanics/sokirko_ajp_62_151_94.pdf
http://kirkmcd.princeton.edu/examples/mechanics/ehrlich_ajp_63_351_95.pdf
http://kirkmcd.princeton.edu/examples/mechanics/weckesser_ajp_65_736_97.pdf
http://kirkmcd.princeton.edu/examples/mechanics/zengel_ajp_85_901_17.pdf
http://kirkmcd.princeton.edu/examples/mechanics/borisov_ejp_39_065001_18.pdf

A marble (uniform sphere of mass m and radius a) rolls without slipping on a horizontal turntable that rotates with constant angular velocity $\mathbf{\Omega}$ about the symmetry axis of the turntable.

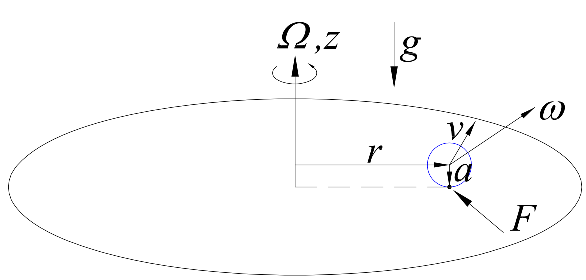
We make a vectorial analysis in the lab frame.

(a) The marble rolls without slipping, so the (nonholonomic) rolling constraint (and its time derivative) can be written as,

$$\mathbf{v}_{\rm contact} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{a} = \mathbf{\Omega} \times \mathbf{r}, \qquad \frac{d\mathbf{v}_{\rm contact}}{dt} = \frac{d\mathbf{v}}{dt} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{a} = \mathbf{\Omega} \times \frac{d\mathbf{r}}{dt} = \mathbf{\Omega} \times \mathbf{v}, \tag{43}$$

where $\mathbf{v}_{\rm contact}$ is the velocity, in the lab frame, of the point of contact on the turntable of the ball (and NOT the instantaneous velocity of the point of contact on the ball),

- $\mathbf{a} = -a\,\hat{\mathbf{z}}$ = vector from the center of the marble to the point of contact with the turntable.
- $\mathbf{r}$ = vector perpendicular to the symmetry axis of the turntable to the center of the marble.
- $\mathbf{v} = d\mathbf{r}/dt$ = velocity of the center (of mass of) the marble.
- $\boldsymbol{\omega}$ = total angular velocity of the marble in the lab frame.



The equations of motion of the marble are,

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} + \mathbf{F}_{\rm contact}, \qquad I\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\tau}_{\rm cm} = \mathbf{a}\times\mathbf{F}_{\rm contact} = \mathbf{a}\times\left(m\frac{d\mathbf{v}}{dt} - m\mathbf{g}\right), \quad (44)$$

where the moment of inertia about the center of the (uniform) marble is $I = 2ma^2/5$.

(b) In the first part of this problem, the turntable is horizontal, so $\mathbf{a}$ (and $-\hat{\mathbf{z}}$) is parallel to $\mathbf{g}$, and the torque equation simplifies to,

$$I\frac{d\boldsymbol{\omega}}{dt} = m\mathbf{a}\times\frac{d\mathbf{v}}{dt}\,. \quad (45)$$

To combine eqs. (43) and (45), we write,

$$\mathbf{a}\times I\frac{d\boldsymbol{\omega}}{dt} = \mathbf{a}\times\left(m\mathbf{a}\times\frac{d\mathbf{v}}{dt}\right) = m\left(\mathbf{a}\cdot\frac{d\mathbf{v}}{dt}\right)\mathbf{a} - ma^2\frac{d\mathbf{v}}{dt} = -ma^2\frac{d\mathbf{v}}{dt}$$

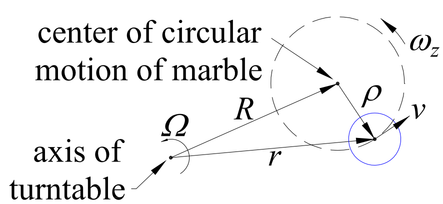
$$= I\left(\frac{d\mathbf{v}}{dt} - \boldsymbol{\Omega}\times\mathbf{v}\right), \quad (46)$$

$$\frac{d\mathbf{v}}{dt} = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\mathbf{v} = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\frac{d\mathbf{r}}{dt}. \quad (47)$$

The acceleration $d\mathbf{v}/dt$ is constant in magnitude and perpendicular to the velocity $\mathbf{v}$, which corresponds to circular motion of the center of the marble.
We integrate eq. (47) to find that the speed $v = |\mathbf{v}|$ of the marble is constant,

$$\mathbf{v} = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times(\mathbf{r}-\mathbf{R}) = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\boldsymbol{\rho}, \qquad v = \frac{I}{I+ma^2}\Omega\,\rho, \quad (48)$$

where $\mathbf{R}$ is the (constant) vector from the center of the turntable to the center of the circle (on the turntable) in which the marble moves in the lab frame, and $\boldsymbol{\rho} = \mathbf{r} - \mathbf{R}$ is the vector from the center of the circle to the point of contact of the marble.



From eq. (48) we have, since $\boldsymbol{\Omega}$ is perpendicular to $\boldsymbol{\rho}$,

$$\boldsymbol{\Omega}\times\mathbf{v} = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{\rho}) = -\frac{I}{I+ma^2}\Omega^2\,\boldsymbol{\rho}, \quad (49)$$

$$\boldsymbol{\rho} = \frac{I+ma^2}{I\Omega^2}\mathbf{v}\times\boldsymbol{\Omega}, \qquad \rho = \frac{I+ma^2}{I\Omega}v, \quad (50)$$

$$\mathbf{R} = \mathbf{r} - \boldsymbol{\rho} = \mathbf{r} - \frac{I+ma^2}{I\Omega^2}\mathbf{v}\times\boldsymbol{\Omega}. = \mathbf{r}_0 + \frac{I+ma^2}{I\Omega^2}\boldsymbol{\Omega}\times\mathbf{v}_0. \quad (51)$$

If we ignore a possible "spin" angular velocity $\boldsymbol{\omega}$ about the z-axis (*i.e.*, no z-component of the angular velocity relative to the c.m.), then the angular velocity component ω_z is just that due to the motion of the marble in the circle of radius ρ,

$$\omega_z = \frac{v}{\rho} = \frac{I\,\Omega}{I + ma^2} \qquad (\omega_{z,\rm spin} = 0), \tag{52}$$

and so for a solid, uniform marble, $\omega_z = 2\,\Omega/7$ and $\rho = 7v/2\Omega$, independent of the distance R of the center of the circular orbit of the marble from the center of the turntable.

For what it's worth, from the rolling constraint (43) we can now write,[1]

$$\mathbf{a} \times (\boldsymbol{\omega} \times \mathbf{a}) = a^2\boldsymbol{\omega} - (\mathbf{a}\cdot\boldsymbol{\omega})\mathbf{a} = a^2\boldsymbol{\omega} - a^2\omega_z\,\hat{\mathbf{z}}$$
$$= \mathbf{a} \times (\boldsymbol{\Omega} \times \mathbf{r} - \mathbf{v}) = a\Omega\,\mathbf{r} - \mathbf{a} \times \left(\frac{I}{I + ma^2}\boldsymbol{\Omega} \times \boldsymbol{\rho}\right) = a\Omega\,\mathbf{r} - \frac{a\Omega I}{I + ma^2}\,\boldsymbol{\rho}, \tag{53}$$
$$\boldsymbol{\omega} = \omega_z\,\hat{\mathbf{z}} + \frac{\Omega}{a}\left(\mathbf{r} - \frac{I}{I + ma^2}\,\boldsymbol{\rho}\right) = \omega_z\,\hat{\mathbf{z}} + \frac{\Omega}{a}\frac{ma^2\mathbf{r} + I\mathbf{R}}{I + ma^2}\,. \tag{54}$$

A simple, special case is that the center of the marble remains at rest in lab frame, *i.e.*, $\mathbf{v} = 0$, such that the radius ρ of the circle is zero, according to eq. (50). If the marble has no "spin" about the z-axis, its angular velocity $\boldsymbol{\omega}$ is in the radial direction, $\hat{\mathbf{r}}$, and the rolling constraint (43) reduces to $\omega\,a = \Omega\,r$ (as also follows from eq. (54) with $\omega_z = 0 = \rho$).

A video of the motion for a horizontal turntable is at
https://www.youtube.com/watch?v=3oM7hX3UUEU&authuser=0

So far, we have explained the motion of the marble, without identifying the forces and torques on the ball that "cause" the motion according to Newton.

It is convenient to consider components of the contact force $\mathbf{F}$ and the torque $\boldsymbol{\tau} = \mathbf{a} \times \mathbf{F}$ about the center of the ball in terms of units vectors $\hat{\boldsymbol{\rho}}$, $\hat{\mathbf{v}}$ and $\hat{\mathbf{z}}$.

Since the marble moves in a horizontal plane, $F_z = mg$ is the normal force on the ball from the turntable.

Since the marble moves in a circle of radius ρ at constant speed $v = 2\Omega\rho/7$ (for a uniform sphere), the tangential force is zero, $F_v = 0$, while the centripetal force is $F_\rho = -mv^2/\rho = -2m\Omega v/7 = -4\Omega^2\rho/49$.

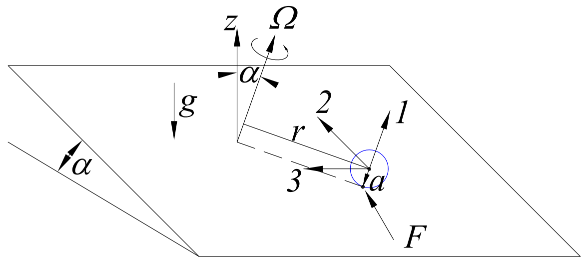
The torque on the marble about its center is (for a uniform sphere),

$$\boldsymbol{\tau} = \mathbf{a} \times \mathbf{F} = aF_\rho\,\hat{\mathbf{v}} = \frac{2m\Omega av}{7}\,\hat{\mathbf{v}} = I\dot{\boldsymbol{\omega}} = \frac{2ma^2}{5}\,\dot{\boldsymbol{\omega}}\,, \tag{55}$$

which is consistent with the time derivative of the last form of eq. (54).

[1]For a uniform sphere with $I = 2ma^2/5$, eq. (54) can be written as $\boldsymbol{\omega} = \omega_z\,\hat{\mathbf{z}} + (5\Omega/7a)(\mathbf{r} + 2\mathbf{R}/5)$, and $\dot{\boldsymbol{\omega}} = (5\Omega/7a)\,\mathbf{v}$.

(c) We now suppose that the plane of the turntable makes angle α to the horizontal. We consider principal axes $\hat{\mathbf{1}}$ perpendicular to the tilted turntable, $\hat{\mathbf{2}}$ pointing up the slope, and $\hat{\mathbf{3}}$ horizontal.



Then, the vertical axis is $\hat{\mathbf{z}} = \cos\alpha\,\hat{\mathbf{1}} + \sin\alpha\,\hat{\mathbf{2}}$, and the vector from the center of the marble to the point of contact with the tilted turntable is $\mathbf{a} = -a\,\hat{\mathbf{1}}$.

The rolling constraint can still be written as eq. (43), and the equations of motion are again given by eq. (44). However, since $\mathbf{a}$ is no longer (anti)parallel to $\mathbf{g}$, eq. (45) becomes,

$$I\frac{d\boldsymbol{\omega}}{dt} = m\mathbf{a}\times\frac{d\mathbf{v}}{dt} - m\mathbf{a}\times\mathbf{g} = m\mathbf{a}\times\frac{d\mathbf{v}}{dt} + mag\sin\alpha\,\hat{\mathbf{3}}. \tag{56}$$

We combine the second of eq. (43) with (56) to find,

$$\mathbf{a}\times I\frac{d\boldsymbol{\omega}}{dt} = \mathbf{a}\times\left(m\mathbf{a}\times\frac{d\mathbf{v}}{dt} + mag\sin\alpha\,\hat{\mathbf{3}}\right) = -ma^2\frac{d\mathbf{v}}{dt} + ma^2 g\sin\alpha\,\hat{\mathbf{2}}$$
$$= I\left(\frac{d\mathbf{v}}{dt} - \boldsymbol{\Omega}\times\mathbf{v}\right), \tag{57}$$
$$\frac{d\mathbf{v}}{dt} = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\mathbf{v} + \frac{ma^2 g\sin\alpha}{I+ma^2}\,\hat{\mathbf{2}}. \tag{58}$$

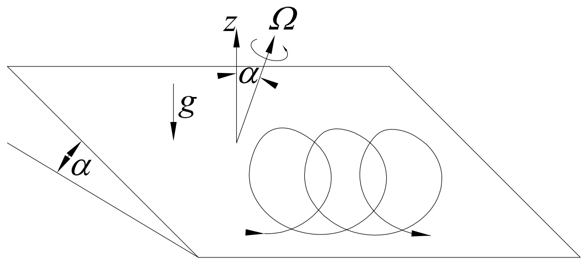
We try a solution of the form $\mathbf{v} = \mathbf{v}(t) + \mathbf{v}_{\rm drift}$ for constant drift velocity $\mathbf{v}_{\rm drift}$.

$$\frac{d\mathbf{v}(t)}{dt} = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\mathbf{v}(t) + \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\mathbf{v}_{\rm drift} + \frac{ma^2 g\sin\alpha}{I+ma^2}\,\hat{\mathbf{2}}, \tag{59}$$
$$\mathbf{v}(t) = \frac{I}{I+ma^2}\boldsymbol{\Omega}\times(\mathbf{r}-\mathbf{R}) = \mathbf{v}_{(b)}, \qquad \frac{I}{I+ma^2}\boldsymbol{\Omega}\times\mathbf{v}_{\rm drift} = -\frac{ma^2 g\sin\alpha}{I+ma^2}\,\hat{\mathbf{2}}, \tag{60}$$

where $\mathbf{v}_{(b)}$ is the form found in eq. (48) above. Finally, noting that $\boldsymbol{\Omega} = \Omega\,\hat{\mathbf{1}}$,

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{v}_{\rm drift}) = -\Omega^2\mathbf{v}_{\rm drift} = -\boldsymbol{\Omega}\times\frac{ma^2 g\sin\alpha}{I}\,\hat{\mathbf{2}} = -\frac{ma^2 g\Omega\sin\alpha}{I}\,\hat{\mathbf{3}}, \tag{61}$$
$$\mathbf{v}_{\rm drift} = \frac{ma^2 g\sin\alpha}{I\Omega}\,\hat{\mathbf{3}}, \qquad \mathbf{r} = \mathbf{R} + \mathbf{v}_{\rm drift}t + \boldsymbol{\rho}. \tag{62}$$

Hence, the motion involves a horizontal drift, as sketched in the figure below.

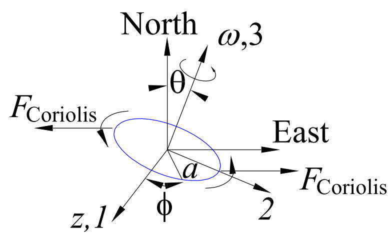


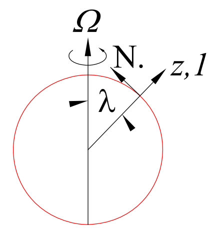
This like the motion of a charged particle in crossed, uniform **E** *and* **B** *fields (just as the circular motion in part (b) is like that of a charged particle in a uniform magnetic field).*

4. **Gyrocompass.**

A gyrocompass is a spinning flywheel whose axis $\boldsymbol{\omega}$ of rotation is constrained to lie in a horizontal plane at the surface of the Earth.

If we analyze the motion in a frame fixed to the surface of the (spinning) Earth, the Coriolis force must be taken into account. When $\boldsymbol{\omega}$ makes angle θ to the North, as shown in the figure, the left side of the flywheel is moving up, and the Coriolis force on it. is to the West. Similarly, the right side of the flywheel is moving down, and the Coriolis force on it is to the East. Hence, there is a net torque on the flywheel that tends to restore θ to zero, *i.e.*, to the North.





(a) We first analyze the motion in a frame fixed to the surface of the Earth, which latter rotates about its axis with angular velocity Ω with respect to the "fixed stars". We suppose the flywheel is a hoop of mass m and radius a. It is subject to torques about its center of mass due to the Coriolis (of order Ω, and the centrifugal force (of order Ω^2 which latter we neglect here.

In the frame fixed to the Earth, the flywheel has angular velocity ω about its symmetry axis, $\hat{\mathbf{3}}$, which axis is constrained to lie a the horizontal plane containing the center of the wheel. The symmetry axis makes (variable) angle θ to the local North direction (in the horizontal plane).

We also define principal axes $\hat{\mathbf{1}}$ to be vertical (also called $\hat{z}$, and $\hat{2}$ in the horizontal plane.

Then, an element $d\phi$ of the hoop at angle ϕ from the $\hat{\mathbf{1}}$ axis in the 1-2 plane has velocity relative to the 123 axes,

$$\mathbf{v}_{\rm rel} = a\omega(-\sin\phi\,\hat{\mathbf{1}} + \cos\phi\,\hat{\mathbf{2}}), \tag{63}$$

while these axes have angular velocity $-\dot{\theta}\,\hat{\mathbf{1}}$ relative to the coordinate system fixed to the Earth. In calculating the Coriolis force and torque, we neglect the small velocity of the hoop due to small $\dot{\theta}$.

The (constant) angular velocity $\boldsymbol{\Omega}$ of the Earth is,

$$\boldsymbol{\Omega} = \Omega(\cos\lambda\,\hat{\mathbf{z}} + \sin\lambda\,\hat{\mathbf{N}}) = \Omega(\cos\lambda\,\hat{\mathbf{1}} - \sin\lambda\sin\theta\,\hat{\mathbf{2}} + \sin\lambda\cos\theta\,\hat{\mathbf{3}}), \tag{64}$$

where $\hat{\mathbf{N}} = -\sin\theta\,\hat{\mathbf{2}} + \cos\theta\,\hat{\mathbf{3}}$ points to the local North. The Coriolis force on the mass element $m\,d\phi/2\pi$, at position $a(\cos\phi\,\hat{\mathbf{1}} + \sin\phi\,\hat{\mathbf{2}})$, is (neglecting the small velocity associated with $\dot{\theta}$),

$$d\mathbf{F} = -2\,dm\,\boldsymbol{\Omega}\times\mathbf{v}_{\rm rel} \tag{65}$$

$$= \frac{m\,d\phi}{\pi}a\,\omega\,\Omega[\sin\lambda\cos\theta\cos\phi\,\hat{\mathbf{1}} + \sin\lambda\cos\theta\sin\phi\,\hat{\mathbf{2}} + (\sin\lambda\sin\theta\sin\phi - \cos\lambda\cos\phi)\,\hat{\mathbf{3}}].$$

The torque about the center of the wheel on the mass element is,

$$
\begin{aligned}
d\boldsymbol{\tau} &= a(\cos\phi\,\hat{\mathbf{1}} + \sin\phi\,\hat{\mathbf{2}}) \times d\mathbf{F} \\
&= \frac{m\,d\phi}{\pi} a^2\,\omega\,\Omega[\sin\lambda\sin\theta\sin^2\phi - \cos\lambda\cos\phi\sin\phi)\,\hat{\mathbf{1}} \\
&\qquad + \sin\lambda\cos\theta\sin\phi - \sin\lambda\cos\theta\cos^2\phi\,\hat{\mathbf{2}} \\
&\qquad +(\sin\lambda\cos\theta\sin\phi\cos\phi - \sin\lambda\cos\theta\cos\phi\sin\phi)\,\hat{\mathbf{3}}].
\end{aligned} \tag{66}
$$

Integrating over $d\phi$, we find the total Coriolis torque to be,

$$
\boldsymbol{\tau} = ma^2\,\omega\,\Omega(\sin\lambda\sin\theta\,\hat{\mathbf{1}} - \sin\lambda\cos\theta\,\hat{\mathbf{2}}) \tag{67}
$$

Torque component τ_2 tends to rotate the plane of the wheel out of the vertical, which action is compensated by the constraint mechanism of the gyrocompass. Torque component 1 leads to the equation of motion,

$$
\tau_1 = ma^2\,\omega\,\Omega\sin\lambda\sin\theta = \frac{dL_1}{dt} = \frac{d}{dt}(-I_1\,\dot{\theta}) = -\frac{ma^2}{2}\ddot{\theta}, \tag{68}
$$

which implies simple harmonic motion for small θ with angular frequency $\sqrt{2\,\omega\,\Omega\sin\lambda}$.

(b) We now analyze the motion in an inertial frame, where the torque about the center of the gyrocompass is only due to the constraint forces on the axle of the gyrocompass, which keep the axle in the horizontal plane with respect to the Earth (but which do not make the gyro point North). That is, $\boldsymbol{\tau} = \tau_2\,\hat{\mathbf{2}}$ in this frame (ignoring the small torque component τ_3 needed to keep the spin angular velocity $\omega\,\hat{\mathbf{3}}$ constant).

This is discussed in Ex. 11.6.4, p. 337 of W. Chester, *Mechanics* (Allen & Unwin, 1979), http://kirkmcd.princeton.edu/examples/mechanics/chester_mechanics_79.pdf

Here, the angular momentum is $\mathbf{L} = \mathsf{I}\cdot\boldsymbol{\omega}_{\rm total}$, where the total angular velocity has three pieces,

- Rotation about the gyro axle (axis $\hat{\mathbf{3}}$) with angular velocity $\boldsymbol{\omega}$.
- Rotation at angular velocity $-\dot{\theta}$ about the local vertical axis ($\hat{\mathbf{z}} = \hat{\mathbf{1}}$) with respect to the Earth.
- Rotation at (constant) angular velocity $\boldsymbol{\Omega}$ of the Earth about its axis, given in eq. (64) above.

That is,

$$
\boldsymbol{\omega}_{\rm total} = \omega\,\hat{\mathbf{3}} - \dot{\theta}\,\hat{\mathbf{1}} + \boldsymbol{\Omega} = \omega\,\hat{\mathbf{3}} + \boldsymbol{\omega}_{123}, \qquad \frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{i}}, \tag{69}
$$

where the principal axes $\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}}$ introduced in part (a) rotate with angular velocity,

$$
\boldsymbol{\omega}_{123} = -\dot{\theta}\,\hat{\mathbf{1}} + \boldsymbol{\Omega} = (-\dot{\theta} + \Omega\cos\lambda)\,\hat{\mathbf{1}} - \Omega\sin\lambda\sin\theta\,\hat{\mathbf{2}} + \Omega\sin\lambda\cos\theta\,\hat{\mathbf{3}}. \tag{70}
$$

The angular momentum is,

$$\mathbf{L} = \mathsf{I} \cdot \boldsymbol{\omega}_{\rm total} = I_1(-\dot\theta + \Omega\cos\lambda)\,\hat{\mathbf{1}} - I_2\Omega\sin\lambda\sin\theta\,\hat{\mathbf{2}} + I_3(\omega + \Omega\sin\lambda\cos\theta)\,\hat{\mathbf{3}}. \quad (71)$$

The torque equation is now,

$$\begin{aligned} \boldsymbol{\tau} = \tau_2\,\hat{\mathbf{2}} = \frac{d\mathbf{L}}{dt} = \frac{\partial\mathbf{L}}{\partial t} + \boldsymbol{\omega}_{123} \times \mathbf{L} \\ = -I_1\,\ddot\theta\,\hat{\mathbf{1}} - I_2\,\Omega\,\dot\theta\sin\lambda\cos\theta\,\hat{\mathbf{2}} + I_3(\dot\omega - \Omega\,\dot\theta\sin\lambda\sin\theta)\,\hat{\mathbf{3}} \\ +I_1(-\dot\theta + \Omega\cos\lambda)(\Omega\sin\lambda\sin\theta\,\hat{\mathbf{3}} + \Omega\sin\lambda\cos\theta\,\hat{\mathbf{2}}) \\ +I_2(-\Omega\sin\lambda\sin\theta)((-\dot\theta + \Omega\cos\lambda)\,\hat{\mathbf{3}} - \Omega\sin\lambda\cos\theta\,\hat{\mathbf{1}}) \\ +I_3(\omega + \Omega\sin\lambda\cos\theta)((\dot\theta - \Omega\cos\lambda)\,\hat{\mathbf{2}} - \Omega\sin\lambda\sin\theta\,\hat{\mathbf{1}}). \end{aligned} \quad (72)$$

The 1-component of eq. (72) is, ignoring terms in the very small quantity Ω^2,

$$0 = -I_1\,\ddot\theta - I_3\,\omega\,\Omega\sin\lambda\sin\theta, \quad (73)$$

$$\ddot\theta = -\frac{I_3}{I_1}\omega\,\Omega\sin\lambda\sin\theta \approx -(2\,\omega\,\Omega\sin\lambda)\theta, \quad (74)$$

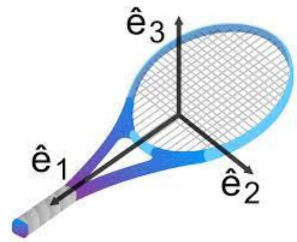
so small oscillations in θ have angular frequency $\sqrt{2\,\omega\,\Omega\sin\lambda}$ as found in part (a). The 3-component of eq. (72) is,

$$0 = I_3(\dot\omega - \Omega\,\dot\theta\sin\lambda\sin\theta) + I_2\Omega\,\dot\theta\sin\lambda\sin\theta \approx I_3\,\dot\omega, \qquad \omega \approx \text{constant}, \quad (75)$$

noting that $\dot\theta \ll \omega$, such that terms in $\omega\,\dot\theta$ are negligible.

5. **The Tennis Racquet Theorem.**

Consider a rigid body whose principal moments of inertia are $I_1 < I_2 < I_3$. We have claimed that free rotation with angular velocity $\boldsymbol{\omega}$ pointing close to axis 2 is "unstable".



For a kind of exception to this behavior, we consider the special case where the kinetic energy has the form $T = L^2/2I_2$, and $\mathbf{L}$ is the angular momentum about the center of mass. In general,

$$T = \frac{I_1\,\omega_1^2}{2} + \frac{I_2\,\omega_2^2}{2} + \frac{I_3\,\omega_3^2}{2}, \tag{76}$$

$$\mathbf{L} = I_1\omega_1\,\hat{\mathbf{1}} + I_2\omega_2\,\hat{\mathbf{2}} + I_3\omega_3\,\hat{\mathbf{3}}, \qquad L^2 = I_1^2\,\omega_1^2 + I_2^2\,\omega_2^2 + I_3^2\,\omega_3^2, \tag{77}$$

so for the special case, $2I_2T = I_1I_2\,\omega_1^2 + I_2^2\,\omega_2^2 + I_2I_3\,\omega_3^2 = L^2$. From eq. (77), we can also write $I_3^2\,\omega_3^2 = L^2 - I_1^2\,\omega_1^2 - I_2^2\,\omega_2^2$, which leads to,

$$2I_2I_3T = I_1I_2I_3\,\omega_1^2 + I_2^2I_2\,\omega_2^2 + I_2(L^2 - I_1^2\,\omega_1^2 - I_2^2\,\omega_2^2) = I_3L^2, \tag{78}$$

$$\omega_1^2I_1I_2(I_3 - I_2) = (I_3 - I_2)(L^2 - I_2^2\,\omega_2^2), \tag{79}$$

$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_1I_2}, \qquad \text{and with } 1 \leftrightarrow 3, \qquad \omega_3^2 = \frac{I_2 - I_1}{I_3 - I_1}\frac{L^2 - I_2^2\omega_2^2}{I_2I_3}. \tag{80}$$

Then, Euler's equation for $\dot\omega_2$ in torque-free motion leads to,

$$\dot\omega_2 = -\frac{I_1 - I_3}{I_2}\omega_1\omega_3 = -\frac{I_1 - I_3}{I_2}\sqrt{\frac{I_3 - I_2}{I_3 - I_1}\frac{I_2 - I_1}{I_3 - I_1}}\frac{L^2 - I_2^2\omega_2^2}{I_2\sqrt{I_1I_3}}$$

$$= \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}}\left(\frac{L^2}{I_2^2} - \omega_2^2\right), \tag{81}$$

$$\frac{d\,\omega_2}{\omega_{2,\max}^2 - \omega_2^2} = k\,dt, \qquad \text{with} \qquad k = \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1I_3}}, \qquad \omega_{2,\max} = \frac{L}{I_2}, \tag{82}$$

$$\frac{1}{\omega_{2,\max}}\tanh^{-1}\frac{\omega_2}{\omega_{2,\max}} = k(t - t_0), \qquad \omega_2 = \omega_{2,\max}\tanh[k\,\omega_{2,\max}\,(t - t_0)], \tag{83}$$

using Dwight 140.1, http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

Then, from eq. (80),

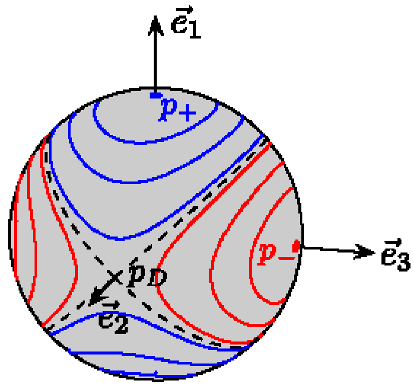
$$\omega_1^2 = \frac{I_3 - I_2}{I_3 - I_1}\frac{I_2}{I_1}(\omega_{2,\max}^2 - \omega_2^2) = \frac{I_3 - I_2}{I_3 - I_1}\frac{I_2}{I_1}\omega_{2,\max}^2\,\text{sech}^2[k\,\omega_{2,\max}\,(t - t_0)], \tag{84}$$

$$\omega_1 = \omega_{2,\max}\sqrt{\frac{I_2}{I_1}\frac{I_3 - I_2}{I_3 - I_1}}\text{sech}[k\,\omega_{2,\max}\,(t - t_0)] = \omega_{1,\max}\,\text{sech}[k\,\omega_{2,\max}\,(t - t_0)]. \tag{85}$$

And, exchanging indices 1 and 3,

$$\omega_3 = \omega_{2,\max} \sqrt{\frac{I_2}{I_3} \frac{I_2 - I_1}{I_3 - I_1}} \operatorname{sech}[k\, \omega_{2,\max}\ (t - t_0)] = \omega_{3,\max} \operatorname{sech}[k\, \omega_{2,\max}\ (t - t_0)]. \qquad (86)$$

As $t \to \infty$, ω_1, $\omega_3 \to 0$, while $\omega_2 \to \omega_{2,\max}$, and the final rotation is about axis 2. Thus, for this special case of motion along "separating polhodes", a kind of stability occurs. That is, the motion consider in this problem is along the dashed lines in the figure below (from http://kirkmcd.princeton.edu/examples/mechanics/vandamme_physica_d338_17_17.pdf).



In practice, the special case is hard to achieve, since for any slight perturbation of the kinetic energy T away from $L^2/2I_2$, $\boldsymbol{\omega}$ will move towards axis **2** along a path close to one of the separating polhodes, then "bounce away" from the $\hat{\mathbf{2}}$ axis and move towards axis $-\hat{\mathbf{2}}$ along a path close to the other separating polhode, "bounce away" from this axis, and repeat the cycle.... To a viewer of the spinning tennis racquet, this cycle seems "unstable" because the axis of rotation migrates between $\hat{\mathbf{2}}$ and $-\hat{\mathbf{2}}$ every half cycle, although in the mathematical sense it is a "stable" orbit.

We infer from this problem that the cycle time for trajectories very close to the separating polhodes is very long, approaching infinity in the limit considered here. The long period of such cycles contributes to the impression by the "casual" observer that the motion is "unstable".

The tennis-racquet theorem was first deduced by L. Poinsot, *Théorie Nouvelle de la Rotation des Corps* (Bachlier, 1851),

http://kirkmcd.princeton.edu/examples/mechanics/poinsot_motion_34.pdf

http://kirkmcd.princeton.edu/examples/mechanics/poinsot_motion_34_english.pdf

6. **Ball and Paper**

This is Prob. 9.29 of D. Morin, Introduction to Classical Mechanics
http://kirkmcd.princeton.edu/examples/mechanics/morin_07.pdf
See also Prob. 8.71.

The velocity of the ball changes only if it slips (either at first on the paper or later on the horizontal surface). Then, the force of horizontal, sliding friction changes the (horizontal) momentum of the ball by $\Delta\mathbf{P}$ during some time interval.

The torque about the center of the ball due to the sliding friction changes the angular momentum of the ball (about its center) by amount

$$\Delta\mathbf{L}_h = \mathbf{r} \times \Delta\mathbf{P}_h = -r\,\hat{\mathbf{z}} \times \Delta\mathbf{P}, \tag{87}$$

where r is the radius of the ball and $\hat{\mathbf{z}}$ is vertically upwards.[2,3]

Equation (87) holds while the ball rolls with slipping, and also holds once rolling without slipping eventually occurs.[4]

For rolling without slipping, $\omega_h = v/r$, where $\boldsymbol{\omega}_h$ is the horizontal component of the angular velocity of the ball, and $\mathbf{v} = \boldsymbol{\omega}_h \times \mathbf{r}$ (hence $\boldsymbol{\omega}_h = -\hat{\mathbf{z}} \times \mathbf{v}/r$) is the (horizontal) velocity of the center of the ball on the horizontal surface. The angular momentum of the ball (about its center) has horizontal component $\mathbf{L}_h = I\boldsymbol{\omega}_h$ with moment of inertia $I = kmr^2$ (about the center of the ball), where $k = 2/5$ for a uniform sphere, and $k < 1$ for any ball in which all mass is within its nominal radius r. Then,

$$\mathbf{L}_h = I\boldsymbol{\omega}_h = -kmr^2\,\hat{\mathbf{z}} \times \frac{\mathbf{v}}{r} = -kr\,\hat{\mathbf{z}} \times \mathbf{P}, \tag{88}$$

where $\mathbf{P} = m\mathbf{v}$ is the linear momentum of the ball.

The relation (88) holds for both the initial rolling without slipping of the ball on the paper, as well as for the final rolling without slipping on the horizontal surface, so we infer that,

$$\Delta\mathbf{L}_h = -kr\,\hat{\mathbf{z}} \times \Delta\mathbf{P}, \tag{89}$$

for the change, $\Delta\mathbf{L}_h$, in the horizontal angular momentum between the initial and final states of the ball when rolling without slipping. But, since k cannot be 1 for a uniform ball, both eqs. (87) and (89) can only be satisfied if $\Delta\mathbf{P} = 0$, and the momentum (and velocity) of the ball in its final motion are the same as its initial momentum (and velocity).

[2]Note that the change in angular momentum, $\Delta\mathbf{L} = \Delta\mathbf{L}_h$ is in a horizontal plane even if the initial angular momentum of the ball is not horizontal, which is possible for rolling without slipping as discussed in Prob. 9.27 of Morin.

[3]There is no torque component in the z-direction, so the vertical angular velocity ω_z is constant throughout this problem.

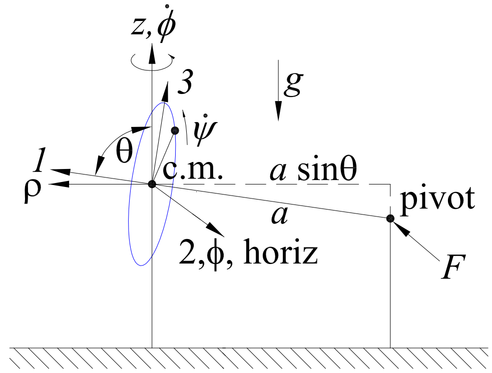
[4]This behavior was discussed in Set 4, Prob. 4(c) under the heading of "English".
http://kirkmcd.princeton.edu/examples/ph205set4.pdf
The path of the ball with initial velocity $\mathbf{v}$ not perpendicular to its initial, horizontal angular velocity $\boldsymbol{\omega}_h$ is a parabola until rolling without slipping commences, after which it is a straight line.

A special case is that the ball is initially at rest on the paper. Then, the ball comes to rest after it rolls of the paper, no matter how the paper is moved.

A famous "trick" is to yank the paper/tablecloth out from under the ball/tableware so quickly that the effect of frictional forces is negligible, and the ball/tableware remains at rest, and at its initial position.

7. **Gyroscope Revisited**

We consider a gyroscope of mass M with one point fixed (the pivot point) as sketched below.



The center of mass is at fixed distance a from the pivot point along axis $\hat{\mathbf{1}}$, which makes angle θ to the vertical. If the gyro spins on the ground, angle θ to the vertical can only be less than $\pi/2$.

We consider the components of force $\mathbf{F}$ of the pivot on the axle of the gyro in a cylindrical coordinate system (ρ, ϕ, z) where $\hat{\mathbf{z}}$ is vertical upwards, $\boldsymbol{\rho}$ is horizontal and pointing away from the pivot point, while $\boldsymbol{\phi}$ is horizontal and pointing in the counterclockwise sense of steady precession as seen from above. Note that (ρ, ϕ, z) is a rotating coordinate system, with the pivot on the z-axis.

The angular momentum of the gyro is most simply expressed by introducing a set of principal axes, $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$. For these, we take $\hat{\mathbf{1}}$ to be along the axle of the gyro, pointing away from the pivot, $\hat{\mathbf{2}}$ is horizontal and in the $\hat{\boldsymbol{\phi}}$ direction, while $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ is in the plane of the "flywheel" of the gyro and makes angle $\pi/2 - \theta$ to the $\hat{\mathbf{z}}$-axis.

The unit vectors $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}})$ and $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ are related by,

$$\hat{\mathbf{1}} = \sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{z}}, \qquad \hat{\mathbf{3}} = -\cos\theta\,\hat{\boldsymbol{\rho}} + \sin\theta\,\hat{\mathbf{z}}, \tag{90}$$

$$\hat{\boldsymbol{\rho}} = \sin\theta\,\hat{\mathbf{1}} - \cos\theta\,\hat{\mathbf{3}}, \qquad \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{1}} + \sin\theta\,\hat{\mathbf{3}}. \tag{91}$$

The equation of motion of the center of mass of the gyro is,

$$M\frac{d\mathbf{v}_{\rm cm}}{dt} = \mathbf{F} - Mg\,\hat{\mathbf{z}}, = F_\rho\,\hat{\boldsymbol{\rho}} + F_\phi\,\boldsymbol{\phi} + (F_z - Mg)\,\hat{\mathbf{z}}, \tag{92}$$

where g is the acceleration due to gravity. The equation of motion of the angular momentum of the gyro with respect to its center of mass is,

$$\begin{aligned}\frac{d\mathbf{L}_{\rm cm}}{dt} &= \boldsymbol{\tau}_{\rm cm} = -a\,\hat{\mathbf{1}} \times \mathbf{F} = -a(\sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{z}}) \times \mathbf{F} \\ &= a[F_\phi\cos\theta\,\hat{\boldsymbol{\rho}} + (F_z\sin\theta - F_\rho\cos\theta)\,\boldsymbol{\phi} - F_\phi\sin\theta\,\hat{\mathbf{z}}],\end{aligned} \tag{93}$$

where,

$$\begin{aligned}\mathbf{L}_{\rm cm} &= I_1\,\omega_1\,\hat{\mathbf{1}} + I_2\,\omega_2\,\hat{\mathbf{2}} + I_3\,\omega_3\,\hat{\mathbf{3}} \\ &= (I_1\,\omega_1\sin\theta - I_2\,\omega_3\cos\theta)\,\hat{\boldsymbol{\rho}} + I_2\,\omega_2\,\hat{\boldsymbol{\phi}} + (I_1\,\omega_1\cos\theta + I_2\,\omega_3\sin\theta)\,\hat{\mathbf{z}},\end{aligned} \tag{94}$$

I_i is the moment of inertia of the gyro about principal axis i, using its center of mass as the reference point, with $I_3 = I_2$ because of the axial symmetry of the gyro, and $\boldsymbol{\omega}$ is the total angular velocity of the gyro.

Note that according to eq. (93),

$$\frac{d\mathbf{L}_{\rm cm}}{dt} \cdot \hat{\mathbf{1}} = 0, \tag{95}$$

which will be shown below to imply that ω_1 is a constant of the motion.

Angular Velocities

The two equations of motion, (92)-(93), are related by the constraint that the pivot point is at rest,

$$0 = \mathbf{v}_{\rm pivot} = \mathbf{v}_{\rm cm} + \boldsymbol{\omega} \times (-a\,\hat{\mathbf{1}}), \qquad \mathbf{v}_{\rm cm} = \boldsymbol{\omega} \times a\,\hat{\mathbf{1}}. \tag{96}$$

The angular velocity $\boldsymbol{\omega}$ can also be thought of as composed of two parts,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{123} + \dot{\psi}\hat{\mathbf{1}}, \tag{97}$$

where $\boldsymbol{\omega}_{123}$ is the angular velocity of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$, and $\dot{\psi}\,\hat{\mathbf{1}}$ is the "spin" angular velocity of the disk relative to that triad; the relative/spin angular-velocity vector can only have a component along $\hat{\mathbf{1}}$ by definition. The angular velocity of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ has component $\dot{\theta}$ about the horizontal axis $\hat{\mathbf{2}} = \hat{\boldsymbol{\phi}}$ through the c.m., and is defined to have component $\dot{\phi}$ about the vertical axis $\hat{\mathbf{z}}$ through the c.m. Since axis $\hat{\mathbf{2}}$ is always horizontal, $\boldsymbol{\omega}_{123}$ has no component along the (horizontal) axis $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\boldsymbol{\rho}}$ through the c.m. Hence, the angular velocity of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ can be written as,

$$\boldsymbol{\omega}_{123} = \dot{\phi}\,\hat{\mathbf{z}} + \dot{\theta}\,\hat{\boldsymbol{\phi}} = \dot{\phi}\cos\theta\,\hat{\mathbf{1}} + \dot{\theta}\,\hat{\mathbf{2}} + \dot{\phi}\sin\theta\;\hat{\mathbf{3}}. \tag{98}$$

Combining eqs. (97) and (98), we can write the total angular velocity vector as,

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \omega_2\,\hat{\mathbf{2}} + \omega_3\,\hat{\mathbf{3}}, \tag{99}$$

where,

$$\omega_1 = \dot{\psi} + \dot{\phi}\cos\theta, \qquad \omega_2 = \dot{\theta}, \qquad \omega_3 = \dot{\phi}\sin\theta. \tag{100}$$

The time rate of change of $\hat{\mathbf{1}}$ at the center of mass is therefore,

$$\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{1}} = \dot{\phi}\sin\theta\,\hat{\mathbf{2}} - \dot{\theta}\,\hat{\mathbf{3}}, \tag{101}$$

The angular velocity of the triad $(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}})$ is simply $\dot{\phi}\,\hat{\mathbf{z}}$, so that,

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = \dot{\phi}\,\hat{\mathbf{z}} \times \boldsymbol{\phi} = -\dot{\phi}\,\boldsymbol{\rho}, \tag{102}$$

$$\frac{d\hat{\boldsymbol{\rho}}}{dt} = \dot{\phi}\,\hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} = \dot{\phi}\,\hat{\boldsymbol{\phi}}. \tag{103}$$

A Constant of the Motion

It is also useful to note that,

$$\hat{\mathbf{1}} \times \frac{d\hat{\mathbf{1}}}{dt} = \dot{\theta}\,\hat{\mathbf{2}} + \dot{\phi}\sin\theta\,\hat{\mathbf{3}} = \hat{\mathbf{1}} \times (\boldsymbol{\omega} \times \hat{\mathbf{1}}) = \boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\hat{\mathbf{1}})\,\hat{\mathbf{1}} = \boldsymbol{\omega} - \omega_1\,\hat{\mathbf{1}} = \boldsymbol{\omega}_\perp \tag{104}$$

where $\boldsymbol{\omega}_\perp$, the component of the total angular velocity $\boldsymbol{\omega}$ that is perpendicular to the symmetry axis $\hat{\mathbf{1}}$. Hence, we can write,

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \hat{\mathbf{1}} \times \frac{d\hat{\mathbf{1}}}{dt} = \omega_1\,\hat{\mathbf{1}} + \boldsymbol{\omega}_\perp, \tag{105}$$

in addition to eq. (99).

Then, the angular momentum $\mathbf{L}_{\rm cm}$, eq. (94), about the center of mass of the gyro is also related by,

$$\mathbf{L}_{\rm cm} = I_1\,\omega_1\,\hat{\mathbf{1}} + I_2\,\boldsymbol{\omega}_\perp = I_1\,\omega_1\,\hat{\mathbf{1}} + I_2\hat{\mathbf{1}} \times \frac{d\hat{\mathbf{1}}}{dt}\,, \tag{106}$$

$$\frac{d\mathbf{L}_{\rm cm}}{dt} = I_1\,\dot{\omega}_1\,\hat{\mathbf{1}} + I_1\,\omega_1\frac{d\hat{\mathbf{1}}}{dt} + I_2\,\hat{\mathbf{1}} \times \frac{d^2\hat{\mathbf{1}}}{dt^2}\,. \tag{107}$$

Recalling eqs. (95) and (101), we have that,

$$0 = \frac{d\mathbf{L}_{\rm cm}}{dt}\cdot\hat{\mathbf{1}} = I_1\,\dot{\omega}_1, \tag{108}$$

which integrates to,[5,6]

$$\omega_1 = \text{constant}. \tag{111}$$

In general, the z-component, $\mathbf{L}_{\rm cm}\cdot\hat{\mathbf{z}}$, of the angular momentum about the c.m. is not a constant of the motion, due to the nonzero vertical torque about the c.m. caused by the ϕ-component of the force $\mathbf{F}$ of the pivot.

---

[5] Some people call $\omega_1\,\hat{\mathbf{1}} = (\dot{\psi} + \dot{\phi}\cos\theta)\,\hat{\mathbf{1}}$ the spin angular velocity, but I prefer to call only $\dot{\psi}\,\hat{\mathbf{1}}$ the spin.

[6] We could now write the angular equation of motion, eq. (93), introducing the notation that $I_i = k_i m a^2$, as,

$$\frac{d\mathbf{L}_{\rm cm}}{dt} = I_1\,\omega_1\frac{d\hat{\mathbf{1}}}{dt} + I_2\,\frac{d}{dt}\left(\hat{\mathbf{1}} \times \frac{d\hat{\mathbf{1}}}{dt}\right) = \boldsymbol{\tau}, \tag{109}$$

$$k_2 m\,\frac{d}{dt}\left(a\,\hat{\mathbf{1}} \times \frac{d(a\,\hat{\mathbf{1}})}{dt}\right) = \boldsymbol{\tau} - k_1 m a^2\,\omega_1\frac{d\hat{\mathbf{1}}}{dt} = \boldsymbol{\tau} + a\,\hat{\mathbf{1}} \times \left(k_1 m a \omega_1 \hat{\mathbf{1}} \times \frac{d\hat{\mathbf{1}}}{dt}\right), \tag{110}$$

noting that $d\,\hat{\mathbf{1}}/dt = -\hat{\mathbf{1}} \times (\hat{\mathbf{1}} \times d\,\hat{\mathbf{1}}/dt)$.

Formally, eq. (110) can be thought of as the torque equation for a point mass $k_2 m$ at the end of a massless rod of length a whose other end is the pivot point of the gyro, subject to two torques about the pivot point. One of these torques has the same value as the torque $\boldsymbol{\tau}$ on the gyro about its center of mass, and the other torque is due to a force $k_1 m r \omega_1 \hat{\mathbf{1}} \times d\hat{\mathbf{1}}/dt$ on the mass $k_2 m$.

This analogy may or may not be helpful, but is pursued on p. 324 of E.A. Milne, *Vectorial Mechanics* (Metheun; Interscience Publishers, 1948),
http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf

Components of Angular Equation of Motion about the C.M.

We now have all the ingredients to calculate the (ρ, ϕ, z) components of $d\mathbf{L}_{\rm cm}/dt$. First, we use eq. (100) in eqs. (94),

$$\mathbf{L}_{\rm cm} = (I_1\,\omega_1 \sin\theta - I_2\,\dot\phi \sin\theta\cos\theta)\,\hat{\boldsymbol{\rho}} + I_2\,\dot\theta\,\hat{\boldsymbol{\phi}} + (I_1\,\omega_1\cos\theta + I_2\,\dot\phi\sin^2\theta)\,\hat{\mathbf{z}}, \qquad (112)$$

and then recall eqs. (102)-(103) to find,

$$\begin{aligned}\frac{d\mathbf{L}_{\rm cm}}{dt} &= \left(I_1\,\omega_1\dot\theta\cos\theta - I_2\,\ddot\phi\sin\theta\cos\theta - 2I_2\,\dot\phi\dot\theta\cos^2\theta\right)\,\hat{\boldsymbol{\rho}} \\ &\quad + \left(I_1\,\omega_1\dot\phi\sin\theta - I_2\,\dot\phi^2\sin\theta\cos\theta + I_2\,\ddot\theta\right)\,\hat{\boldsymbol{\phi}} \\ &\quad - \left(I_1\,\omega_1\dot\theta\sin\theta - I_2\,\ddot\phi\sin^2\theta - 2I_2\,\dot\phi\dot\theta\sin\theta\cos\theta\right)\,\hat{\mathbf{z}} \qquad (113)\end{aligned}$$

Finally, recalling the expression for the torque about the center of mass from eq. (93), the ρ-, ϕ- and z-components of the angular equation of motion are,

$$I_1\,\omega_1\dot\theta\cos\theta - I_2\,\ddot\phi\sin\theta\cos\theta - 2I_2\,\dot\phi\dot\theta\cos^2\theta = aF_\phi\cos\theta, \qquad (114)$$

$$I_1\,\omega_1\dot\phi\sin\theta - I_2\,\dot\phi^2\sin\theta\cos\theta + I_2\,\ddot\theta = aF_z\sin\theta - aF_\rho\cos\theta, \qquad (115)$$

$$I_1\,\omega_1\dot\theta\sin\theta - I_2\,\ddot\phi\sin^2\theta - 2I_2\,\dot\phi\dot\theta\sin\theta\cos\theta = aF_\phi\sin\theta. \qquad (116)$$

Components of Angular Equation of Motion about the Pivot

While eqs. (92) and (93) are sufficient to determine the motion, they do not determine all of the force components in general. To remedy this we also consider the angular equation of motion about the pivot point. We note that the principal moments of inertia of the gyro about the pivot point are $I_{1p} = I_1$, $I_{2p} = I_{3p} = I_2 + Ma^2$ according to the principal-axis theorem, and so the angular momentum about this point is,

$$\begin{aligned}\mathbf{L}_p &= I_1\,\omega_1\,\hat{\mathbf{1}} + I_{2p}\,\omega_2\,\hat{\mathbf{2}} + I_{2p}\,\omega_3\,\hat{\mathbf{3}} \\ &= (I_1\,\omega_1\sin\theta - I_{2p}\,\omega_3\cos\theta)\,\hat{\boldsymbol{\rho}} + I_{2p}\,\omega_2\,\hat{\boldsymbol{\phi}} + (I_1\,\omega_1\cos\theta + I_{2p}\,\omega_3\sin\theta)\,\hat{\mathbf{z}} \qquad (117)\end{aligned}$$

Then we have,

$$\begin{aligned}\frac{d\mathbf{L}_p}{dt} &= \boldsymbol{\tau}_p = a\,\hat{\mathbf{1}}\times(-Mg\,\hat{\mathbf{z}}) = -a(\sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{z}})\times Mg\,\hat{\mathbf{z}} = Mga\sin\theta\,\boldsymbol{\phi} \\ &= \left(I_1\,\omega_1\dot\theta\cos\theta - I_{2p}\,\ddot\phi\sin\theta\cos\theta - 2I_{2p}\,\dot\phi\dot\theta\cos^2\theta\right)\,\hat{\boldsymbol{\rho}} \\ &\quad + \left(I_1\,\omega_1\dot\phi\sin\theta - I_{2p}\,\dot\phi^2\sin\theta\cos\theta + I_{2p}\,\ddot\theta\right)\,\hat{\boldsymbol{\phi}} \\ &\quad - \left(I_1\,\omega_1\dot\theta\sin\theta - I_{2p}\,\ddot\phi\sin^2\theta - 2I_{2p}\,\dot\phi\dot\theta\sin\theta\cos\theta\right)\,\hat{\mathbf{z}} \qquad (118)\end{aligned}$$

The ρ-, ϕ- and z-components of the angular equation of motion (118) are,

$$I_1\,\omega_1\dot\theta\cos\theta - I_{2p}\,\ddot\phi\sin\theta\cos\theta - 2I_{2p}\,\dot\phi\dot\theta\cos^2\theta = 0, \qquad (119)$$

$$I_1\,\omega_1\dot\phi\sin\theta - I_{2p}\,\dot\phi^2\sin\theta\cos\theta + I_{2p}\,\ddot\theta = Mga\sin\theta, \qquad (120)$$

$$-\left(I_1\,\omega_1\dot\theta\sin\theta - I_{2p}\,\ddot\phi\sin^2\theta - 2I_{2p}\,\dot\phi\dot\theta\sin\theta\cos\theta\right) = 0. \qquad (121)$$

Steady (Slow) Precession

In the case of steady precession, $\dot\phi = \omega_p =$ constant, $\theta = \theta_0 =$ constant, and $0 = \ddot\phi = \dot\theta = \ddot\theta = (d\mathbf{v}_{\rm cm}/dt)_\phi = (d\mathbf{v}_{\rm cm}/dt)_z$.

The equation of motion (92) of the c.m. of the gyro tells us that,

$$F_\rho = -M\omega_p^2 a\sin\theta_0, \qquad F_\phi = 0, \qquad F_z = Mg, \tag{122}$$

noting that the c.m. of the gyro undergoes uniform circular motion in a horizontal circle of radius $a\sin\theta_0$, with centripetal acceleration $(d\mathbf{v}_{\rm cm}/dt)_\rho = -\omega_p^2 a\sin\theta_0$. Then, the angular equations of motion (114) and (116) are trivially satisfied, while eq. (115) leads to the quadratic equation for ω_p,

$$I_1\,\omega_1\omega_p\sin\theta_0 - I_2\,\omega_p^2\sin\theta_0\cos\theta_0 = aMg\sin\theta_0 + a^2M\omega_p^2\sin\theta_0\cos\theta_0, \tag{123}$$

$$I_{2p}\,\omega_p^2\cos\theta_0 - I_1\,\omega_1\omega_p + Mga = 0, \tag{124}$$

$$\omega_p = \frac{I_1\,\omega_1}{2I_{2p}\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MgaI_{2p}\cos\theta_0}{I_1^2\omega_1}}\right), \tag{125}$$

where $I_{2p} = I_2 + Ma^2$ is the moment of inertia of the gyro about the $\hat{\mathbf{2}} = \hat{\boldsymbol\phi}$ axis with respect to the pivot point rather than the c.m.

If $\theta_0 < \pi/2$ (as for a spinning top on the ground) the precession is not stable, and the gyro will fall to the ground, unless,

$$\omega_p > \frac{2}{I_1}\sqrt{mgrI_{2p}\cos\theta_0} \qquad (\theta_0 < \pi/2). \tag{126}$$

Equation (125) indicates that there are two possible values for the angular velocity ω_p of steady precession, while experience emphasizes only one of these – the smaller value. This story is clarified by considering the case that ω_1 is large, and eq. (125) can be approximated as,

$$\omega_p \approx \frac{I_1\,\omega_1}{2I_{2p}\cos\theta_0}\left[1 \pm \left(1 - \frac{2MgaI_{2p}\cos\theta_0}{I_1^2\omega_1}\right)\right], \tag{127}$$

$$\omega_p \approx \frac{Mga}{I_1\omega_1} \quad \text{(slow)} \qquad \text{or} \qquad \frac{I_1\,\omega_1}{I_{2p}\cos\theta_0} \qquad \text{(fast)}. \tag{128}$$

The constant angular velocity of slow (familiar) steady precession is, for large ω_1,

$$\omega_p(\text{slow}) \approx \frac{Mga}{I_1\omega_1}, \tag{129}$$

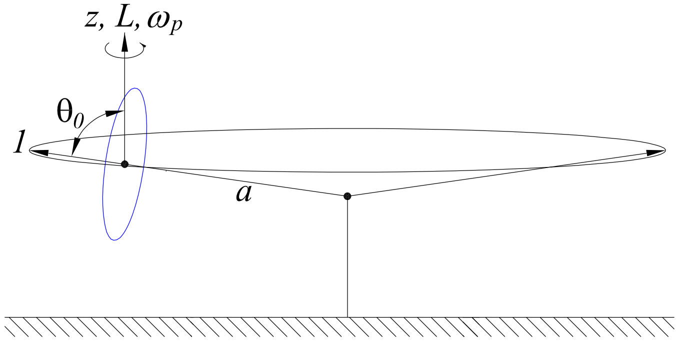
which is independent of the vertical angle θ_0 of the gyro.[7]

[7]For the special case of $\theta_0 = \pi/2$, there is only one steady motion, with $\omega_p = Mga/I_1\omega_1$, as for "slow" precession. This motion exist for any value of ω_1, although for small ω_1 the precession is "fast".

The less familiar fast precession (for $\theta_0 \neq \pi/2$),

$$\omega_p(\text{fast}) \approx \frac{I_1\,\omega_1}{I_{2p}\cos\theta_0}\,, \tag{130}$$

does not depend on gravity, and is essentially the free precession of the gyro in a zero-gravity environment. Indeed for very large ω_1, the torques due to the force $\mathbf{F}$ of the pivot are negligible, and the motion is torque-free precession with constant, total angular momentum $\mathbf{L}$ along in the $\hat{\mathbf{z}}$ direction.



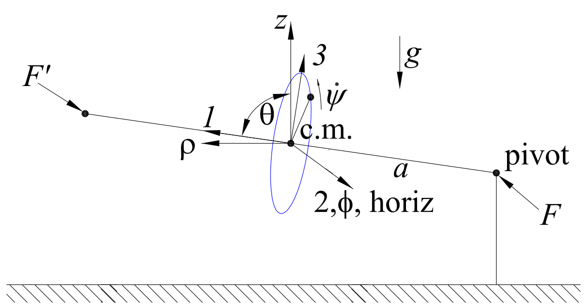
Recall from p. 191 of Lecture 18,[8] that for a symmetric top the torque-free precession angular velocity, according to an inertial observer, is $\omega_p = L/I_2$ with $L_1 = I_1\,\omega_1$ and $\hat{\mathbf{1}}$ at angle θ_0 to the total angular momentum $\mathbf{L}$. such that $L_1 = L\cos\theta_0$. That is, $\omega_p = I_1\omega_1/I_2\cos\theta_0$, which agrees with eq. (130) with the understanding that in the analysis of a gyro with a fixed point at the pivot rather than the c.m., $I_2 \to I_{2p}$.

The special case of steady motion with $\theta_0 = \pi/2$ is considered in
http://kirkmcd.princeton.edu/examples/disk_prelim.pdf

The Gyro Is Held Fixed by Additional force $\mathbf{F}'$

We now begin a survey of the forces associated with the gyro in various situations.

First, we consider that the gyro is prevented from moving by an additional force $\mathbf{F}'$ applied to an extension of the axle of the gyro by length a, as sketched below.



In this case, the c.m. of the gyro is at rest, and its angular momentum vector is constant.

The equation of motion of the center of mass of the gyro is now,

$$0 = M\frac{d\mathbf{v}_{\rm cm}}{dt} = \mathbf{F} + \mathbf{F}' - Mg\,\hat{\mathbf{z}}, = (F_\rho + F'_\rho)\,\hat{\boldsymbol{\rho}} + (F_\phi + F'_\phi)\,\boldsymbol{\phi} + (F_z + F'_z - Mg)\,\hat{\mathbf{z}}. \tag{131}$$

[8] http://kirkmcd.princeton.edu/examples/Ph205/ph205l18.pdf

The equation of motion of the gyro with respect to the (fixed) pivot point,

$$0 = \frac{d\mathbf{L}_p}{dt} = \boldsymbol{\tau}_p = 2a\,\hat{\mathbf{1}} \times \mathbf{F}' + a\,\hat{\mathbf{1}} \times (-Mg\,\hat{\mathbf{z}}) = a(\sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{z}}) \times (2\mathbf{F}' - Mg\,\hat{\mathbf{z}})$$
$$= -2aF'_\phi \cos\theta\,\hat{\boldsymbol{\rho}} + a[(Mg - 2F'_z)\sin\theta + 2F'_\rho\cos\theta]\,\boldsymbol{\phi} + 2aF'_\phi\sin\theta\,\hat{\mathbf{z}}, \tag{132}$$

When the c.m. is at rest, eq. (131) tells us that,

$$F'_\rho = -F_\rho, \qquad F'_\phi = -F_\phi, \qquad F'_z = Mg - F_z. \tag{133}$$

When the axle of the gyro is held fixed, the ρ- and z-components of eq. (132) then tells us that,

$$F'_\phi = 0 = F_\phi. \tag{134}$$

Using eq. (133) in the ϕ-component of eq. (132) we find,

$$(2F_z - Mg)\sin\theta - 2F_\rho\cos\theta = 0. \tag{135}$$

The ρ- and z-components of the forces are indeterminant in the sense that one could chose any value of, say, F'_ρ and a solution would exist. Of interest is the minimal solution in which,

$$F_\rho = 0 = F'_\rho, \quad F_z = \frac{Mg}{2} = F'_z. \tag{136}$$

That is, $\mathbf{F} = \mathbf{F}' = Mg\,\hat{\mathbf{z}}/2$ when the gyro is held fixed in place by the additional force $\mathbf{F}'$ with no ρ- or ϕ-components, just as could hold if the flywheel of the gyro were not spinning.

The Gyro Axis Is Constrained to Move in a Vertical Plane

In some demonstrations of gyroscopes, its axis is constrained to move in a vertical plane. If the gyro is released from rest it falls down without precessing.

We can use the analysis of the previous section, now supposing the addition force $\mathbf{F}'$ has only a ϕ-component, and that $\dot\phi = 0 = \ddot\phi$, but angle θ can vary.

The c.m. moves in a vertical Circe of radius a about the pivot point, so that,

$$\mathbf{v}_{\rm cm} = -a\dot\theta\,\hat{\mathbf{3}}. \tag{137}$$

$$\frac{d\mathbf{v}_{\rm cm}}{dt} = -a\ddot\theta\,\hat{\mathbf{3}} - a\dot\theta\frac{d\hat{\mathbf{3}}}{dt} - \dot\theta^2 a\,\hat{\mathbf{1}} = -a\ddot\theta\,\hat{\mathbf{3}} - 2a\dot\theta^2\,\hat{\mathbf{1}}$$
$$= -a\dot\theta(-\cos\theta\,\hat{\boldsymbol{\rho}} + \sin\theta\,\hat{\mathbf{z}}) - 2\dot\theta^2 a(\sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{z}})$$
$$= a\left(\ddot\theta\cos\theta - 2\dot\theta^2\sin\theta\right)\,\hat{\boldsymbol{\rho}} - a\left(\ddot\theta\sin\theta + 2\dot\theta^2\cos\theta\right)\,\hat{\mathbf{z}}, \tag{138}$$

noting that the angular velocity of the axes is now $\boldsymbol{\omega}_{123} = \dot\theta\,\hat{\mathbf{2}}$, and hence $d\hat{\mathbf{3}}/dt = \boldsymbol{\omega}_{123} \times \hat{\mathbf{3}} = \dot\theta\,\hat{\mathbf{1}}$, while the centripetal acceleration (along $-\hat{\mathbf{1}}$ has magnitude $\dot\theta^2 a$, and we recall eq. (90).

The equation of motion of the center of mass, eq. (131) with $\mathbf{F}' = F'_\phi \,\hat{\boldsymbol{\phi}}$, tells us that,

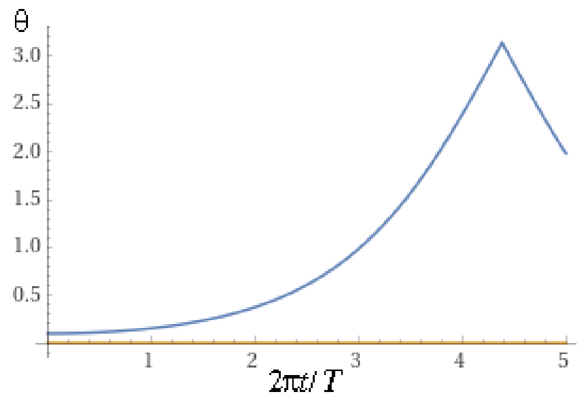
$$F_\rho = Ma\left(\ddot{\theta}\cos\theta - 2\dot{\theta}^2\sin\theta\right), \quad F'_\phi = -F_\phi, \quad F_z = Mg - Ma\left(\ddot{\theta}\sin\theta + 2\dot{\theta}^2\cos\theta\right). \tag{139}$$

To get additional information about the forces, we consider the angular equation of motion about the pivot point, $d\mathbf{L}_p/dt = \boldsymbol{\tau}_p$. We can take the dependence of $d\mathbf{L}_p/dt$ on the moments of inertia from eq. (118), and $\boldsymbol{\tau}_p$ from eq. (132) to find the ρ-, ϕ- and z-components of the angular equation of motion,

$$I_1\,\omega_1\dot{\theta} = -2aF'_\phi = 2aF_\phi, \tag{140}$$

$$I_{2p}\,\ddot{\theta} = Mga\sin\theta. \tag{141}$$

The differential equation (141), which holds even if the gyro is not spinning, can be integrated using Wolfram Alpha. We scale time t in the plot below by the period T of a simple pendulum of length I_{2p}/Ma, in which the gyro was released from rest at $t = 0$ and $\theta(0) = 0.1$.



The time for the gyro to fall to $\theta = \pi$ is about $0.7T$. As θ and $\dot{\theta}$ increase, the force F'_ϕ, needed to keep the motion in a vertical plane, increases in magnitude as $I_1\omega_1\dot{\theta}/2a$, where $\omega_1 = \dot{\psi}(0)$ is constant; this force is in the $-\hat{\boldsymbol{\phi}}$-direction, opposing the tendency of the gyro to precess.

The force exerted by the pivot can be computed from eq. (139) and the numerical solution for $\theta(t)$.

Gyro Is Launched from Rest at Angle θ_1

At length we take up discussion of forces and torques on a gyroscope with one point fixed, with emphasis on the case that it is released from rest.

To complete the equation of motion (92) of the center of mass, we compute $d\mathbf{v}_{\rm cm}/dt$ from the constraint that the pivot is at rest,

$$0 = \mathbf{v}_{\rm pivot} = \mathbf{v}_{\rm cm} + \boldsymbol{\omega}\times\mathbf{a}, \tag{142}$$

$$\begin{aligned}\mathbf{v}_{\rm cm} = -\boldsymbol{\omega}\times\mathbf{a} = \boldsymbol{\omega}_{123}\times a\,\hat{\mathbf{1}} &= \left(\dot{\phi}\,\hat{\mathbf{z}} + \dot{\theta}\,\hat{\boldsymbol{\phi}}\right)\times a(\sin\theta\,\hat{\boldsymbol{\rho}} + \cos\theta\,\hat{\mathbf{z}}) \\ &= a\dot{\theta}\cos\theta\,\hat{\boldsymbol{\rho}} + a\dot{\phi}\sin\theta\,\hat{\boldsymbol{\phi}} - a\dot{\theta}\sin\theta\,\hat{\mathbf{z}},\end{aligned} \tag{143}$$

$$\frac{d\mathbf{v}_{\rm cm}}{dt} = a\left(\ddot\theta\cos\theta - \dot\theta^2\sin\theta - \dot\phi^2\sin\theta\right)\hat{\boldsymbol\rho}$$
$$+a\left(\ddot\phi\sin\theta + 2\dot\phi\dot\theta\cos\theta\right)\hat{\boldsymbol\phi} - a\left(\ddot\theta\sin\theta + \dot\theta^2\cos\theta\right)\hat{\mathbf{z}}, \qquad (144)$$

noting that $\mathbf{a} = -a\,\hat{\mathbf{1}}$, and recalling eqs. (90) and (102)-(103). The ρ-, ϕ- and z-components of eq. (92) now tell us that,

$$F_\rho = Ma\left(\ddot\theta\cos\theta - \dot\theta^2\sin\theta - \dot\phi^2\sin\theta\right), \qquad (145)$$
$$F_\phi = Ma\left(\ddot\phi\sin\theta + 2\dot\phi\dot\theta\cos\theta\right), \qquad (146)$$
$$F_z = Mg - a\left(\ddot\theta\sin\theta + \dot\theta^2\cos\theta\right). \qquad (147)$$

The terms in $\ddot\theta$ are associated with the tangential acceleration $-a\ddot\theta\,\hat{\mathbf{3}}$ of the c.m. along a vertical arc of radius a centered on the pivot, while the terms in $\dot\theta^2$ are associated with the centripetal tangential acceleration $-a\dot\theta^2\,\hat{\mathbf{1}}$ of the c.m. in its motion along that arc. The motion of the c.m. in a horizontal circle of radius $a\sin\theta$ is association with tangential acceleration $a\sin\theta\ddot\phi\,\hat{\boldsymbol\phi}$ and centripetal acceleration $-a\sin\theta\dot\phi^2\,\hat{\boldsymbol\rho}$. The term in $\dot\phi\dot\theta$ does not seem to have a simple interpretation.

We recall that the angular equations of motion (114)-(115) with respect to the c.m. are,

$$I_1\,\omega_1\dot\theta - I_2\,\ddot\phi\sin\theta - 2I_2\,\dot\phi\dot\theta\cos\theta = aF_\phi, \qquad (148)$$
$$I_1\,\omega_1\dot\phi\sin\theta - I_2\,\dot\phi^2\sin\theta\cos\theta + I_2\,\ddot\theta = aF_z\sin\theta - aF_\rho\cos\theta, \qquad (149)$$

and those, eqs. (119)-(120) with respect to the pivot are,

$$I_1\,\omega_1\dot\theta - I_{2p}\,\ddot\phi\sin\theta - 2I_{2p}\,\dot\phi\dot\theta\cos\theta = 0, \qquad (150)$$
$$I_1\,\omega_1\dot\phi\sin\theta - I_{2p}\,\dot\phi^2\sin\theta\cos\theta + I_{2p}\,\ddot\theta = Mga\sin\theta. \qquad (151)$$

Note that use of eq. (145)-(147) in eqs. (114)-(115) converts them to eqs. (119)-(120).

The general motion of a gyroscope with one point fixed involves nutations about steady precession of the c.m. in a horizontal circle. Various forms of the nutations are sketched below, with the case of a gyro released from rest at vertical angle θ_1 corresponding to Fig. 11-17(c) below.

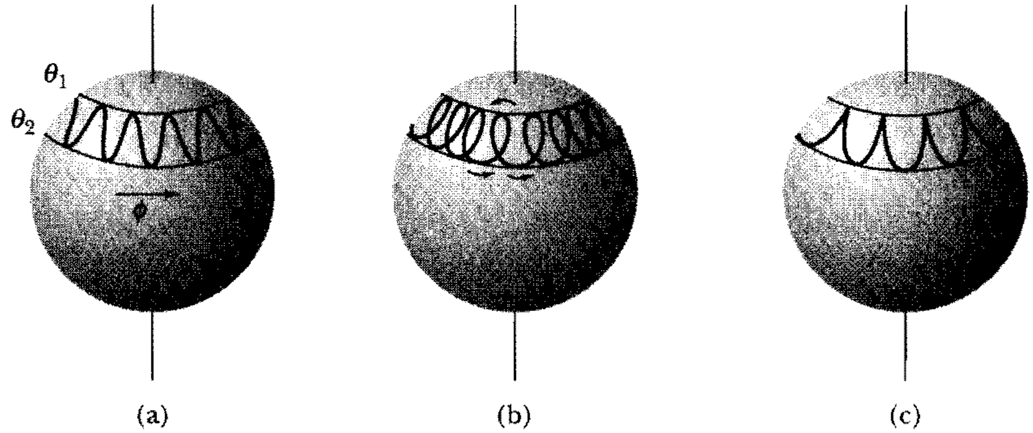


FIGURE 11-17 The rotating top also nutates between the limit angles θ_1 and θ_2. In (a) $\dot{\phi}$ does not change sign. In (b) $\dot{\phi}$ does change sign, and we see looping motion. In (c) the initial conditions include $\dot{\theta} = \dot{\phi} = 0$; this is the normal cusp-like motion when we spin a top and release it.

The initial angular momentum of the gyro is $\mathbf{L} = I_1\,\omega_1\,\hat{\mathbf{1}}$, along the axle, with ω_1 equal to the initial value of $\dot{\psi}$ (which is positive in the figure on p. 24).

Initially, the gyro falls vertically, and $0 < F_z < mg$, such that $\ddot{\theta} > 0$ the vertical angle θ increases with time.

The vertical force F_z also produces a torque about the c.m. of the gyro, $\tau_\phi = aF_z \sin\theta$. This torque pushes the angular momentum $\mathbf{L}$, and the gyro, in the $\boldsymbol{\phi}$ direction. Recall from eq. (93) that,

$$\frac{d\mathbf{L}_{\rm cm}}{dt} = \boldsymbol{\tau}_{\rm cm} = -a\,\hat{\mathbf{1}} \times \mathbf{F} = a[F_\phi \cos\theta\,\hat{\boldsymbol{\rho}} + (F_z \sin\theta - F_\rho \cos\theta)\,\boldsymbol{\phi} - F_\phi \sin\theta\,\hat{\mathbf{z}}], \quad (152)$$

The gyro takes on initial azimuthal angular acceleration $\ddot{\phi} > 0$ and azimuthal angular velocity $\dot{\phi} > 0$.

The c.m. of the gyro now has a component of acceleration in the $\boldsymbol{\phi}$ direction, so $F_\phi > 0$, and because there is now circular motion of the gyro around the pivot there is also a component F_ρ of the force in the $-\boldsymbol{\rho}$ direction. These forces generate torque components (about the c.m.), $\tau_z = -aF_\phi \sin\theta$ and $\tau'_\phi = -aF_\rho\ \cos\theta$.

The downward vertical torque τ_z pushes the angular momentum, and the gyro downwards, which gives a different perspective on the initial fall of the gyro. The additional azimuthal torque τ'_ϕ affects the azimuthal angular velocity $\dot{\phi}$ of the gyro, but never enough to change the sign of $\dot{\phi}$.

The angular equation (151) indicates that at the azimuthal angular velocity $\dot{\phi}$ increases, the vertical angular acceleration $\ddot{\phi}$ decreases, and can go negative. Then, the vertical force $F_z - Mg$ on the c.m. can switch from negative to positive, slowing the vertical fall of the gyro to a maximum vertical angle θ_2, and eventually pushing it back up (rising).

Meanwhile, as $\ddot{\phi}$ passes through zero and goes negative, the azimuthal angular velocity $\dot{\phi}$ reaches a maximum value, greater than that for steady precession, and eventually drops to zero as the gyro returns to vertical angle θ_1.

Then, the nutation cycle starts again.

The force components throughout this cycle are described in eqs. (145)-(147), but the interplay of forces and torques is so complex that it is hard to state a simple version of cause and effect at every moment in time.

Fictitious Forces, Torques and Changes in the Spin $\dot\psi$

We found in eq. (111) that $\omega_1 = \dot\psi + \dot\phi\cos\theta$ is a constant of the motion, with the implication that the angular velocity $\dot\psi\,\hat{\mathbf{1}}$ of the "flywheel" of the gyro changes as $\dot\phi$ or θ vary,

$$\ddot\psi\,\hat{\mathbf{1}} = \left(\dot\phi\dot\theta\sin\theta - \ddot\phi\cos\theta\right)\hat{\mathbf{1}}. \tag{153}$$

This is perhaps surprising in that we presume there is no torque about the center of the flywheel due to friction at the bearings of the flywheel. However, as we are analyzing the gyro with respect to its c.m. in the rotating coordinate system (ρ, ϕ, z), whose angular velocity with respect to the lab frame is $\mathbf{\Omega} = \boldsymbol{\omega}_{\rho\phi z} = \dot\phi\,\hat{\mathbf{z}} = \dot\phi(\cos\theta\,\hat{\mathbf{1}} + \sin\theta\,\hat{\mathbf{3}})$, we must remember the existence of "fictitious" forces,[9] such that the effective force on mass m, at distance $\mathbf{r} = r(\cos\psi\,\hat{\mathbf{2}} + \sin\psi\,\hat{\mathbf{3}})$ from the origin of the rotating frame (here taken to be the c.m. of the gyro), is given by,

$$F_{\text{effective}} = \mathbf{F} + \mathbf{F}_{\text{Centrifugal}} + \mathbf{F}_{\text{Coriolis}} + \mathbf{F}_{\text{Azimuthal}} + \mathbf{F}_{\text{Coord}}, \tag{154}$$

where $\mathbf{F}$ is the force on mass m in the inertial lab frame,

$$\mathbf{F}_{\text{Centrifugal}} = -m\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = m[\Omega^2\mathbf{r} - (\mathbf{\Omega}\cdot\mathbf{r})\,\mathbf{\Omega}], \tag{155}$$

$$\mathbf{F}_{\text{Coriolis}} = -2m\mathbf{\Omega}\times\mathbf{v} = -2m\mathbf{v}\times\mathbf{\Omega}, \tag{156}$$

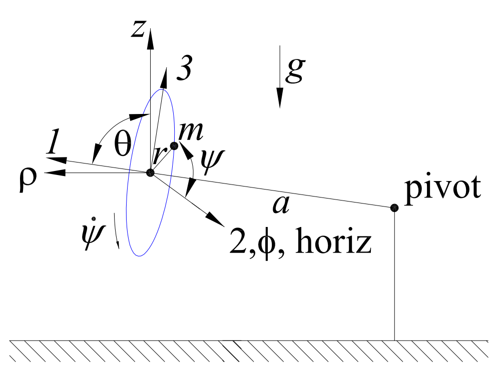
$$\mathbf{F}_{\text{Azimuthal}} = -m\dot{\mathbf{\Omega}}\times\mathbf{r} = -m\mathbf{r}\times\dot{\mathbf{\Omega}}, \tag{157}$$

$$\mathbf{F}_{\text{Coord}} = -m\frac{d^2\mathbf{R}}{dt^2}, \tag{158}$$

and,

$$\begin{aligned}\mathbf{v} = (\boldsymbol{\omega} - \mathbf{\Omega})\times\mathbf{r} &= \left(\dot\psi\,\hat{\mathbf{1}} + \dot\theta\,\hat{\mathbf{2}}\right)\times r(\cos\psi\,\hat{\mathbf{2}} + \sin\psi\,\hat{\mathbf{3}}) \\ &= r\left(\dot\theta\sin\psi\,\hat{\mathbf{1}} - \dot\psi\sin\psi\,\hat{\mathbf{2}} + \dot\psi\cos\psi\,\hat{\mathbf{3}}\right),\end{aligned} \tag{159}$$

is the velocity of mass m in the rotating frame, and $\mathbf{R}$ is the origin of the rotating coordinate system in the inertial lab frame.



[9] See, for example, p. 172 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l16.pdf

To understand eq. (153), we are interested in the 1-component of the torque $\mathbf{r}\times\mathbf{F}_{\text{effective}}$, integrated over mass points on a circle of radius r on the "flywheel" of the gyro.

The force $\mathbf{F}$ on mass m in the lab frame consists of $-mg\,\hat{\mathbf{z}}$ plus the internal force of the "flywheel" on the mass. The internal forces cannot lead to any motion of the c.m. or change in the angular momentum of the gyro, so we neglect them in the subsequent discussion.

The force of gravity on mass m, and the coordinate force, eq. (158), do not depend on the position $\mathbf{r}$ of the mass, so the torque $\mathbf{r}\times(\mathbf{F}+\mathbf{F}_{\text{coord}})$ has components that depend linearly on either $\cos\psi$ or $\sin\psi$ (via the factor of $\mathbf{r}$), which sum to zero on integration over ψ for a ring of radius r.

The centrifugal torque associated with mass m is,

$$\begin{aligned}\boldsymbol{\tau}_{\text{Cent}} &= \mathbf{r}\times\mathbf{F}_{\text{Cent}} = -m(\boldsymbol{\Omega}\cdot\mathbf{r})\,\mathbf{r}\times\boldsymbol{\Omega} \\ &= -mr^2\dot{\phi}^2\sin\theta\sin\psi(\sin\theta\cos\psi\,\hat{\mathbf{1}}+\cos\theta\sin\psi\,\hat{\mathbf{2}}-\cos\theta\cos\psi\,\hat{\mathbf{3}}),\end{aligned} \tag{160}$$

whose 1-component (and 3-component) varies as $\cos\psi\sin\psi$, which integrates over ψ to zero for a ring of radius r.

The Coriolis torque on mass m is,

$$\begin{aligned}\boldsymbol{\tau}_{\text{Cor}} &= \mathbf{r}\times\mathbf{F}_{\text{Cor}} = 2m[(\mathbf{r}\cdot\boldsymbol{\Omega})\,\mathbf{v}-(\mathbf{r}\cdot\mathbf{v}]\boldsymbol{\Omega}] \\ &= 2mr^2\dot{\phi}\sin\theta\sin\phi\left(\dot{\theta}\sin\psi\,\hat{\mathbf{1}}-\dot{\psi}\sin\psi\,\hat{\mathbf{2}}+\dot{\psi}\cos\psi\,\hat{\mathbf{3}}\right),\end{aligned} \tag{161}$$

whose 1-component integrates over ψ to a term proportional to $\dot{\psi}\dot{\theta}\sin\theta$ as in eq. (153).

Finally, the torque associated with the Azimuthal "fictitious" force on mass m is, with $\dot{\boldsymbol{\Omega}} = \ddot{\phi}\,\hat{\mathbf{z}} = \ddot{\phi}(\cos\theta\,\hat{\mathbf{1}}+\sin\theta\,\hat{\mathbf{3}})$,

$$\begin{aligned}\boldsymbol{\tau}_{\text{Az}} &= \mathbf{r}\times\mathbf{F}_{\text{Az}} = m\left[(\mathbf{r}\cdot\dot{\boldsymbol{\Omega}})\,\mathbf{r}-r^2\dot{\boldsymbol{\Omega}}\right] \\ &= -mr^2\ddot{\phi}\left[\cos\theta\,\hat{\mathbf{1}}+\sin\theta\,\hat{\mathbf{3}}-\sin\theta\sin\psi\left(\cos\psi\,\hat{\mathbf{2}}+\sin\psi\,\hat{\mathbf{3}}\right)\right],\end{aligned} \tag{162}$$

whose 1-component integrates over ψ to the term $-\ddot{\psi}\cos\theta$ in eq. (153).

Thus, the "fictitious" torques associated with the Coriolis force and the Azimuthal force in the rotating frame centered on the c.m. give a "Newtonian" explanation as to how the "spin" angular velocity $\dot{\psi}\,\hat{\mathbf{1}}$ of the gyro can vary during the nutations.

More extensive discussion along the lines of this solution is given in R. Usubamatov, *Theory of Gyroscopic Effects* (Springer, 2020),
http://kirkmcd.princeton.edu/examples/mechanics/usubamatov_20.pdf

An example of Coriolis forces in biology is discussed in K.T. McDonald, *Stabilization of Insect Flight via Sensors of Coriolis Force* (Feb. 17, 2007),
http://kirkmcd.princeton.edu/examples/stabilization.pdf