Princeton University

Ph205

Mechanics

Problem Set 7

Kirk T. McDonald

(1988)

kirkmcd@princeton.edu

http://kirkmcd.princeton.edu/examples/

1. In a scattering experiment, the differential cross section is observed to be,

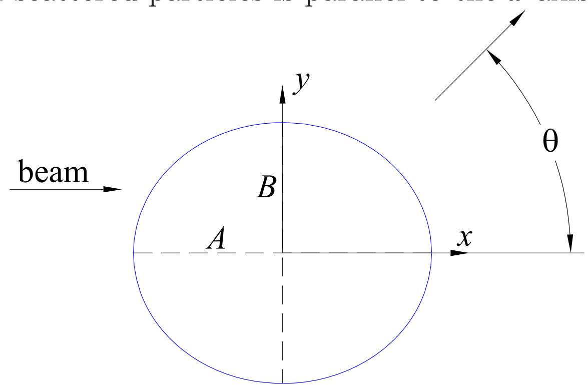
$$\frac{d\sigma}{d\cos\theta} = \frac{\pi B^2}{2}(1 + \epsilon\cos\theta), \qquad \sigma_{\rm total} = \pi B^2, \tag{1}$$

with $\epsilon \ll 1$. Supposing the scattering is elastic off a hard object, what is its shape?

If $\epsilon = 0$, the object would be a sphere, as in Prob. 6, Ph205 Problem Set 6. For nonzero, but small ϵ, the object is almost a sphere, say a (prolate) spheroid,

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{B^2} = 1, \tag{2}$$

where the beam of scattered particles is parallel to the x-axis.

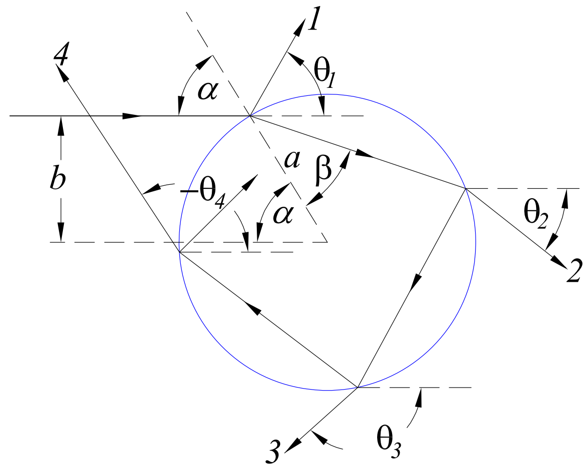


Find the differential cross section for scattering off a "hard" spheroid with arbitrary semiaxes A and B. What is A corresponding to eq. (1), for small ϵ?

2. **Rainbows, Haloes and Glories,** by R. Greenler (U. Cambridge, 1980), http://kirkmcd.princeton.edu/examples/optics/greenler_80.pdf

   (a) **Rainbows.**

   Consider the scattering of light by a (spherical) water drop. When light hits a boundary between air and water, both reflection and transmission are possible, so many scattered rays occur. The first 4 outgoing rays are shown in the sketch below.



   Ray 1 is mirror reflection off the surface of the drop, as in hard scattering off a sphere, Prob. 6, Ph205 Set 6.

   Ray 2 corresponds to Prob. 8, Ph205 Set 6.

   This problem concerns rays 3 and 4, which can lead to primary and secondary rainbows.

   Noting that impact parameter $b = a\sin\alpha$, as shown in the figured, the scattering cross section is related by,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi\sin\theta}\frac{db^2}{d\theta} = \frac{a^2\sin\alpha\cos\alpha}{\pi\sin\theta}\frac{d\alpha}{d\theta}\,. \tag{3}$$

   If $d\theta/d\alpha = 0$ for some b, then $d\sigma/d\Omega \to \infty$. That is, if many different α's, and hence different b's, lead to the same scattering angle θ, the scattered light gets very bright $\Rightarrow$ a rainbow.

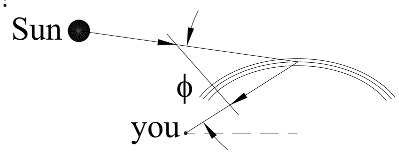
   Let m be the number of internal reflections before the ray emerges. Calculate $\theta = f(\alpha, \beta, m)$ from the geometry. Use Snell's law to relate angles α and β to the index of refraction of water (taking the index of air to be 1), and show that $d\theta/d\alpha = 0$ when,

$$\sin^2\alpha = \frac{(m+1)^1 - n^2}{(m+1)^2 - 1}\,. \tag{4}$$

   For water, $n \approx 4/3$. Evaluate α, β and θ for the first two rainbows, $m = 1$ and 2. (Ans: 138°, −129°.)

   The index n of refraction varies with wavelength; long $\lambda \Rightarrow$ small n. What is the order of colors in the first and second rainbows?

If you are watching the rainbow, what is the angle ϕ between the light you see from it and the Sun?

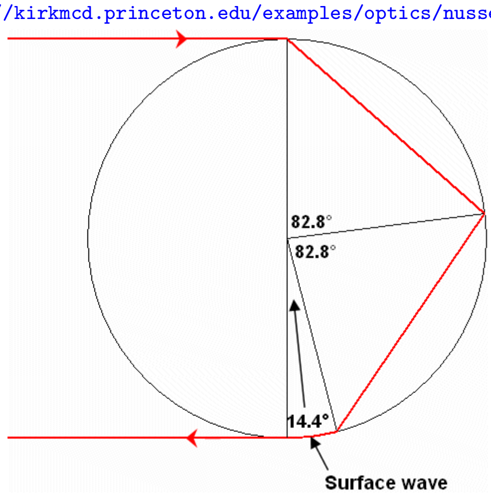


The explanation of the rainbow is attributed to Descartes.

(b) **Glories** (strictly cultural, no problem assigned.)

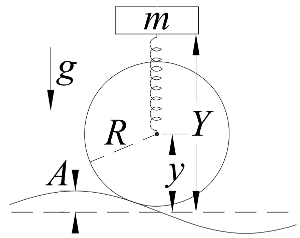
If you look at the shadow of an airplane on a cloud while flying, you see a "halo"/"glory" immediately outside the shadow. That is, there is an enhancement of the scattering by $\theta \approx 180°$ off water drops.

The first recorded observation of this phenomenon by a Westerner was in 1735, by a Spanish mountain climber in the Andes. But a good explanation was given only in 1977, H.M. Nussenzveig, *The Theory of the Rainbow*, Sci. Am. **236**(4), 116 (1977), http://kirkmcd.princeton.edu/examples/optics/nussenzveig_sa_236-4_116_77.pdf



Apparently, surface waves just inside the water drop can transport light for several degrees around the drop before it emerges. For impact parameter $b \approx a$, this is sufficient to makes the scattering angle of ray 3 of the previous figure emerge at $\theta \approx 180°$. See also the extensive web site, http://www.philiplaven.com/index1.html, from which the above photo and figure are taken.

3. A car is driven at constant horizontal velocity along a horizontal, "washboard" road such that the height of an axle above the average elevation of the road is $y \approx R + A\cos\omega t$, where R is the radius of a wheel and $A \ll R$.



Mass m of the car is supported above the center of the wheel by a vertical shock absorbed of rest length l and spring constant k. The damping of the shock absorber is proportional to the rate of change of its length,

$$F = -b(\dot{Y} - \dot{y}), \tag{5}$$

where Y is the height of the top of the shock absorber above the average elevation of the road.

Formulate, and solve, the differential equation for the vertical motion of mass m to show that the average height $\langle Y \rangle$ is,

$$\langle Y \rangle = R + l - \frac{g}{\omega_0^2}\,, \tag{6}$$

and that the amplitude of the oscillation of Y is,

$$\sqrt{\frac{\omega_0^4 + 4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}\,, \tag{7}$$

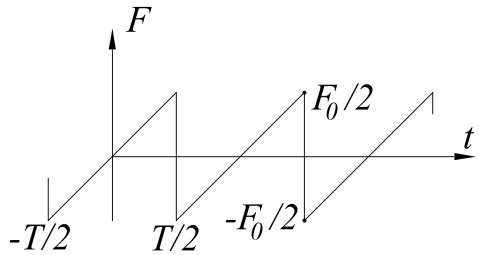
where $\omega_0^2 = k/m$ and $\beta = b/2m$.

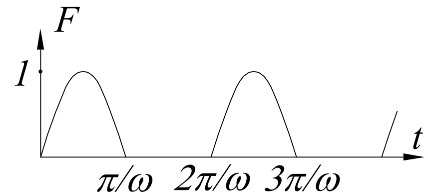
Suppose the shock absorber is critically damped. At what angular frequency ω is the amplitude of the oscillation in Y a maximum, and what is the maximum amplitude? *Ans:* $\text{Ampli}_{\text{max}} = 2\sqrt{3}\,A/3$.

4. (a) Find the Fourier expansion of the sawtooth waveform,

$$F(t) = \frac{F_0 t}{T} \qquad \left(-\frac{T}{2} < t < \frac{T}{2}\right). \tag{8}$$

$$Ans: \qquad F(t) = \frac{F_0}{\pi}\left(\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \cdots\right). \qquad (\omega = 2\pi/T). \tag{9}$$





(b) Find the Fourier expansion of the half-wave waveform,

$$F(t) = \begin{cases} \sin\omega t & \left(0 < t < \frac{\pi}{\omega}\right), \\ 0 & \left(\frac{\pi}{\omega} < t < \frac{2\pi}{\omega}\right). \end{cases} \tag{10}$$

$$Ans: \qquad F(t) = \frac{1}{\pi} + \frac{1}{2}\sin\omega t - \frac{2}{3\pi}\cos 2\omega t - \frac{2}{15\pi}\cos 4\omega t - \cdots \tag{11}$$

Which Fourier series of (a) or (b) converges faster?

5. (a) A mass m oscillates with spring constant k and damping constant b after being driven by the step function,

$$F(t) = \begin{cases} 0 & (t < 0), \\ F_0 & (t > 0). \end{cases} \tag{12}$$

Use Green's method to calculate the motion $x(t)$.

$$Ans: \qquad x = \frac{F_0}{m\omega_0^2}\left(1 - e^{-\beta t}\cos\omega_1 t - \frac{\beta}{\omega_1}e^{-\beta t}\sin\omega_1 t\right). \tag{13}$$

Sketch this for $\beta = 0$ and $\omega_0/4$.

Note that the damped oscillation makes a large overshoot of the equilibrium position $x = F_0/m\omega_0^2$. What is the maximum x of this overshoot, and what is the time t then?

(b) The same oscillator is subject to the impulse,

$$F(t) = \begin{cases} 0 & (t < 0), \\ F_0 & (0 < t < T), \\ 0 & (t > T). \end{cases} \tag{14}$$

Now what is $x(t)$?

Sketch the motion supposing the damping is strong enough that the initial oscillations have largely died out before the impulse ends at time T.

6. (a) A damped oscillator is subject to the driving force $F(t) = F_0\, e^{-\alpha t}$ for a positive constant α. Solve for the "steady" motion by making a suitable guess as to the form of $x(t)$.

   (b) Now suppose that the driving force is,

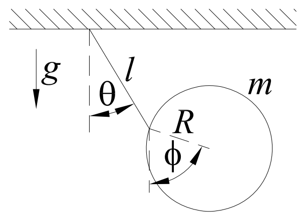
$$F(t) = \begin{cases} 0 & (t < 0), \\ F_0\, e^{-\alpha t} & (t > 0). \end{cases} \tag{15}$$

   Use Green's method to solve for the transient response (which should also include the "steady" motion of part (a) for $t > 0$).

$$\textit{Ans}: \qquad x(t > 0) = \frac{F_0/m}{\omega_0^2 + \alpha^2 - 2\alpha\beta}\left[e^{-\alpha t} + e^{-\beta t}\left(\frac{\alpha - \beta}{\omega_1}\sin\omega_1 t - \cos\omega_1 t\right)\right]. \tag{16}$$

   Sketch this for the case $\alpha = \beta$.

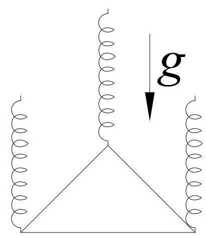
7. A hoop of mass m and radius R is attached to a massless, rigid rod of length l to for a compound pendulum. The hoop pivots freely about its connection to the end of the rod. Find the angular frequencies of the normal modes of oscillation for motion entirely in a vertical plane.



Hint: Make the small-angle approximation before deriving the equations of motion – but remember that you must keep terms of 2nd order to describe oscillatory motion.

For $R = l/2$, show that $\omega = \sqrt{(2 \pm 2\sqrt{2})g/l}$.

8. A thin plate in the form of an equilateral triangle is suspended by three springs (of the same spring constant and same rest length) from its corners, such that the equilibrium position of the plate is horizontal with the springs vertical.

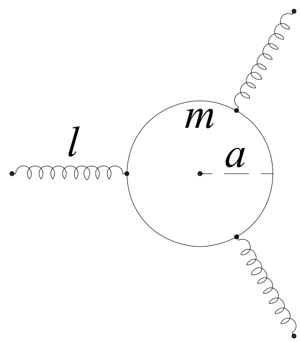


What are the angular frequencies of the normal modes of (small) oscillation in which the center of mass of the plate moves only vertically?

It suffices to guess the forms of the normal modes and then derive an equation of motion for each mode separately.

sl Ans: Two of the modes have the same frequency, which is twice that of a third mode. A fourth mode has frequency independent of the mass and of the spring constant.

9. A uniform disk of mass m and radius a rests on a frictionless, horizontal table. The disk is connected via three springs of constant k and rest length l_0 to three fixed points 120° apart. At equilibrium the springs have length $l > l_0$.

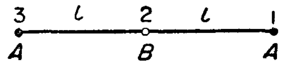


What are the angular frequencies of the three normal modes (including rotation) of small oscillations about equilibrium?

You might guess the modes and solve for them one by one, or use Lagrange's method. The problems on p. 60 of http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf may help with the geometry.

$$Ans: \qquad \omega_1 = \omega_2 = \sqrt{\frac{3k}{2m}\frac{2l - l_0}{l}}, \qquad \omega_3 = \sqrt{\frac{6k}{m}\frac{(l - l_0)(a + l)}{al}}. \tag{17}$$

10. (a) Consider the linear triatomic molecule ABA of Prob. 1, p. 72 of
http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf,
where atoms A are tied to atom B by springs of constant k. They solve the problem by guessing the modes, and using conservation of energy/momentum of the center of mass to reduce the problem to two degrees of freedom (ignoring the bending mode).



Work this problem via Lagrange's method by deducing the three coupled equations of motion for along the (x) as xis of the molecule, using coordinates x_1, x_2 and x_3. Assume oscillatory motion to derive the characteristic equation for ω^2, where ω is the angular frequency of small oscillations about equilibrium.

$$Ans: \qquad \omega^2 = 0, \qquad \frac{k}{m_A}, \qquad k\frac{2m_A + m_B}{m_A m_B}. \tag{18}$$

The case of $\omega = 0$ means that there is a nonoscillatory motion possible for this system, which is just translation of the entire system, without internal motion, along the x-axis.

(b) Suppose the middle atom B is tied to the origin by a spring, also of constant k. Now what are the frequencies of the normal modes of small oscillations?

Solutions

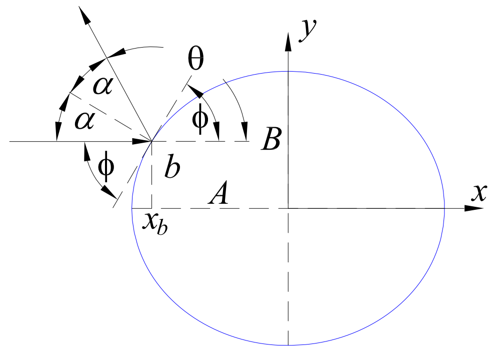
1. In elastic scattering off a prolate spheroid,

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{B^2} = 1, \tag{19}$$

the scattering angle θ is related by,

$$\theta = \pi - 2\alpha, \qquad \sin\alpha = \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \cos\frac{\theta}{2}. \tag{20}$$

where α is the angle between the incoming particle and the normal to the spheroid at the point of contact.



Also, at the point of contact where $y = b = B\sqrt{1 - x_b^2/A^2}$ and $x_b = -A\sqrt{1 - b^2/B^2}$,

$$\tan\phi = \frac{dy}{dx} = -B\frac{d}{dx}\sqrt{1 - \frac{x^2}{A^2}} = \frac{B\,|x_b|}{A\sqrt{A^2 - x_b^2}} = \frac{A\sqrt{B^2 - b^2}}{A^2 b/B} = \frac{B\sqrt{B^2 - b^2}}{Ab}$$

$$= \cot\alpha = \frac{\sqrt{1 - \sin^2\alpha}}{\sin\alpha}, \tag{21}$$

$$\cos\frac{\theta}{2} = \sin\alpha = \frac{1}{\sqrt{1 + (dy/dx)^2}} = \frac{Ab}{\sqrt{B^4 + (A^2 - B^2)b^2}}, \tag{22}$$

$$\cos^2\frac{\theta}{2}(B^4 + (A^2 - B^2)b^2 = A^2 b^2, \tag{23}$$

$$b^2 = \frac{B^4\cos^2\theta/2}{A^2 - (A^2 - B^2)\cos^2\theta/2} = \frac{B^4(1 + \cos\theta)}{2A^2 - (A^2 - B^2)(1 + \cos\theta)}, \tag{24}$$

The scattering differential cross section is,

$$\frac{d\sigma}{d\cos\theta} = \pi\frac{db^2}{d\cos\theta} = \pi\left(\frac{B^4}{2A^2 - (A^2 - B^2)(1 + \cos\theta)} + \frac{B^4(A^2 - B^2)(1 + \cos\theta)}{[2A^2 - (A^2 - B^2)(1 + \cos\theta)]^2}\right)$$

$$= \pi\frac{B^4[2A^2 - (A^2 - B^2)(1 + \cos\theta)] + B^4(A^2 - B^2)(1 + \cos\theta)}{[2A^2 + (A^2 - B^2)(1 + \cos\theta)]^2}$$

$$= \pi B^2\frac{2A^2 B^2}{[2A^2 - (A^2 - B^2)(1 + \cos\theta)]^2}$$

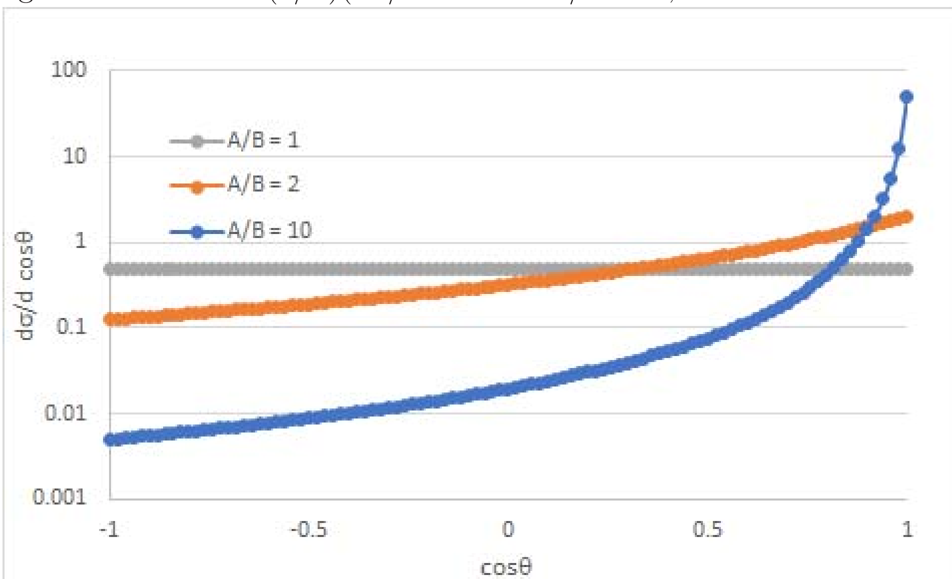
$$\approx \pi B^2\frac{2(1 + 2\delta)}{[2 + 2\delta - 2\delta\cos\theta)]^2} \approx \frac{\pi B^2}{2}(1 + 2\delta)(1 - 2\delta + 2\delta\cos\theta) \approx \frac{\pi B^2}{2}(1 + 2\delta\cos\theta), \tag{25}$$

where the approximation holds for small δ in $A = B(1+\delta)$.

If the differential cross section is observed to vary as $1 + \epsilon\cos\theta$, then $\delta = \epsilon/2$ and $A = B(1+\epsilon/2)$.

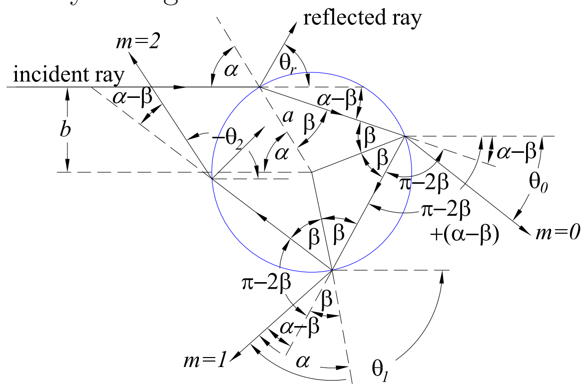
The total cross section is, of course, $\sigma = \pi B^2$.

The figure below shows $(1/\sigma)(d\sigma/d\cos\theta$ for $A/B = 1$, 2 and 10.



2. **Rainbows.**

In the scattering of light by a (spherical) water drop, let m be the number of internal reflections before the ray emerges.



Then, the emerging ray with $m = 1$ has scattering angle,

$$\theta_1 = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = 2(\alpha - \beta) + (\pi - 2\beta), \tag{26}$$

and the emerging ray with $m = 2$ has scattering angle,

$$\theta_2 = \theta_1 - (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = 2(\alpha - \beta) + 2(\pi - 2\beta), \tag{27}$$

and in general,

$$\theta_m = 2(\alpha - \beta) + m(\pi - 2\beta). \tag{28}$$

According to Snell's law,

$$\sin\alpha = n\sin\beta, \qquad \cos\alpha = n\cos\beta\frac{d\beta}{d\alpha} = \sqrt{n^2 - \sin^2\alpha}\,\frac{d\beta}{d\alpha}\,, \tag{29}$$

where $n \approx 4/3$ is the index of refraction of water. Hence,

$$\frac{d\theta_m}{d\alpha} = 2 - 2(m+1)\frac{d\beta}{d\alpha} = 2\left(1 - \frac{(m+1)\cos\alpha}{\sqrt{n^2 - \sin^2\alpha}}\right)\,, \tag{30}$$

for rainbow scattering with angle α_m related by,

$$n^2 - \sin^2\alpha_m = (m+1)^2(1 - \sin^2\alpha_m), \qquad \sin^2\alpha_m = \frac{(m+1)^1 - n^2}{(m+1)^2 - 1}\,. \tag{31}$$

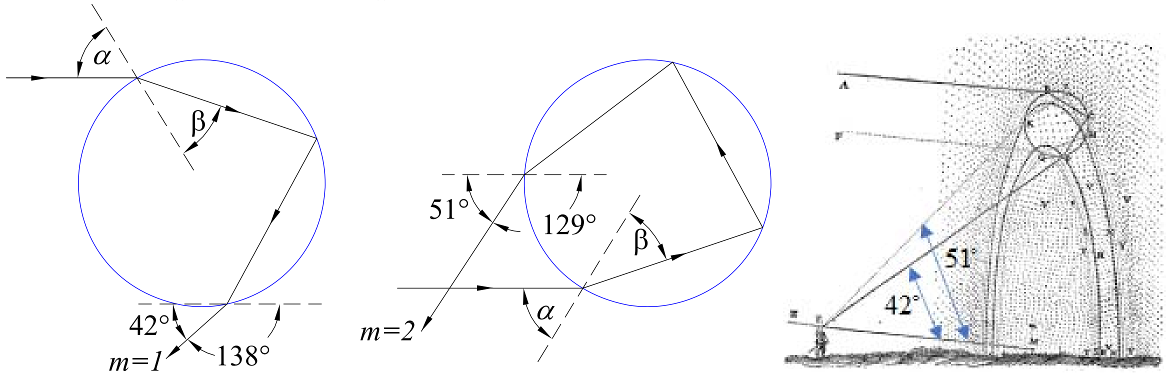
$$\sin^2\alpha_1 = \frac{4 - \frac{16}{9}}{3} = 0.95^2\,, \quad \alpha_1 = 59.4^\circ, \quad \sin\beta_1 = \frac{3}{4}\sin\alpha_1 = 0.65, \quad \beta_1 = 40.2^\circ, \tag{32}$$

$$\theta_1 = 180^\circ + 2\alpha_1 - 4\beta_1 = 138^\circ, \tag{33}$$

$$\sin^2\alpha_2 = \frac{9 - \frac{16}{9}}{8} = 0.86^2\,, \quad \alpha_2 = 71.8^\circ, \quad \sin\beta_2 = \frac{3}{4}\sin\alpha_1 = 0.71, \quad \beta_2 = 45.4^\circ, \tag{34}$$

$$\theta_2 = (180^\circ +)\ 2\alpha_2 - 6\beta_2 = -129^\circ. \tag{35}$$

The primary rainbow (m = 1) is due to light entering the upper part of water drops, relative to the observer, while the secondary rainbow (m = 2) is due to light entering the lower part of the drop.

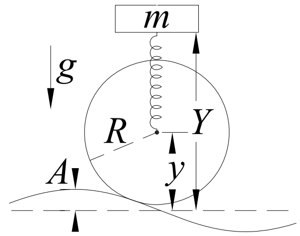


The angles of the rainbows relative to the incident light rays are 42° and 51°, as noted by Descartes in the right figure above, from p. 253 of *Discours de la Méthode* (Leyden, 1637), http://kirkmcd.princeton.edu/examples/mechanics/descartes_37.pdf.

Since $\sin\alpha = n\sin\beta$, a larger index $n(\lambda)$ implies a small angle β, for a given angle α. For the primary rainbow, rays associated with a larger index, *i.e.*, for smaller wavelength, emerge with a smaller angle. Hence the primary rainbow is blue at smaller angles and red at larger.

For the secondary rainbow, with its inverted internal geometry, the story is reversed, so red appear at larger angles and blue at smaller.

3. This problem is adapted from Prob. 51, p. 69 of K.R. Symon, *Mechanics* (Addison-Wesley, 1971), http://kirkmcd.princeton.edu/examples/mechanics/symon_71.pdf.



The differential equation for the vertical motion of mass m is,

$$m\ddot{Y} = -b(\dot{Y} - \dot{y}) - k(Y - y - l) - mg, \tag{36}$$

$$m\ddot{Y} + b\dot{Y} + kY = -b\dot{y} + k(y + l) - mg, \tag{37}$$

$$\ddot{Y} + \beta\dot{Y} + \omega_0^2 Y = -\beta\dot{y} + \omega_0^2(y + l) - g, \tag{38}$$

where $\beta = b/2m$ and $\omega_0^2 = k/m$.

The "washboard" road forces the axle to oscillate vertically above the average elevation of the road according to,

$$y = R + A\cos\omega t, \tag{39}$$

where R is the radius of a wheel and $A \ll R$. Hence,

$$\ddot{Y} + 2\beta\dot{Y} + \omega_0^2 Y = 2A\beta\omega\sin\omega t + \omega_0^2(R + A\cos\omega t) + \omega_0^2 l - g. \tag{40}$$

The time-average, steady-state value of Y is,

$$\langle Y \rangle = R + l - \frac{g}{\omega^2}\,. \tag{41}$$

We seek an oscillatory solution for forced motion of Y at angular frequency ω of the form $Y = \langle Y \rangle + Re(Y_0\, e^{i\omega t})$, for which (40) implies,

$$Re(-\omega^2 Y_0\, e^{i\omega t} - 2i\beta\omega Y_0\, e^{i\omega t} + \omega_0^2 Y_0\, e^{i\omega t}) = Re(-2iA\beta\omega\, e^{i\omega t} + A\omega_0^2\, e^{i\omega t}), \tag{42}$$

$$Y_0 = \frac{-2iA\beta\omega + A\omega_0^2}{\omega_0^2 - \omega^2 - 2i\beta\omega}\,, \qquad |Y_0| = A\sqrt{\frac{\omega_0^4 + 4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}\,. \tag{43}$$

In the particular case of critical damping, $\beta = \omega_0$, and the amplitude of the oscillation in Y is,

$$|Y_0| = A\sqrt{\frac{\omega_0^4 + 4\omega_0^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\omega_0^2\omega^2}} = \frac{A\omega_0}{\omega_0^2 + \omega^2}\sqrt{\omega_0^2 + 4\omega^2}. \tag{44}$$

This is maximal for,

$$\frac{d\,|Y_0|}{d\omega} = 0 = -\frac{2A\omega_0\omega}{(\omega_0^2 + \omega^2)^2}\sqrt{\omega_0^2 + 4\omega^2} + \frac{A\omega_0}{\omega_0^2 + \omega^2}\frac{4\omega}{\sqrt{\omega_0^2 + 4\omega^2}}\,, \tag{45}$$

$$0 = -2(\omega_0^2 + 4\omega^2) + 4(\omega_0^2 + \omega^2), \qquad \omega^2 = \frac{\omega_0^2}{2}\,, \qquad |Y_0|_{\max} = \frac{2\sqrt{3}\,A}{2}\,. \tag{46}$$

A version of this problem with an accelerating car is at
http://kirkmcd.princeton.edu/examples/washboard.pdf.

4. (a) The sawtooth is antisymmetric,

$$F(t) = \frac{F_0 t}{T} = -F(-t) \qquad \left(-\frac{T}{2} < t < \frac{T}{2}\right). \tag{47}$$

so its Fourier expansion will only contain sine terms, $F(t) = \sum_{n=1}^{\infty} B_n \sin n\omega t$ with $\omega = 2\pi/T$,

$$\begin{aligned} B_n &= \frac{2}{T}\int_{-T/2}^{T/2} F(t)\sin n\omega t\,dt = \frac{2F_0}{T^2}\int_{-T/2}^{T/2} t\sin\frac{2n\pi t}{T}\,dt = \frac{2F_0}{T^2}\frac{T^2}{4\pi^2 n^2}\int_{-n\pi}^{n\pi} x\sin x\,dx \\ &= \frac{F_0}{2n^2\pi^2}[\sin x - x\cos x]_{-n\pi}^{n\pi} = \frac{F_0}{2n^2\pi^2}(-1)^{n+1}2n\pi = \frac{(-1)^{n+1}F_0}{n\pi}, \end{aligned} \tag{48}$$

$$F(t) = \frac{F_0}{\pi}\left(\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \cdots\right). \qquad (\omega = 2\pi/T)\,. \tag{49}$$

(b) The half-wave function is neither symmetric nor antisymmetric, so its Fourier series will contain both cosine and sine terms,

$$F(t) = \left\{\begin{array}{ll} \sin\omega t & \left(0 < t < \frac{\pi}{\omega}\right) \\ 0 & \left(\frac{\pi}{\omega} < t < \frac{2\pi}{\omega}\right) \end{array}\right\} = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\cos n\omega t + B_n \sin n\omega t, \tag{50}$$

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} F(t)\cos n\omega t\,dt = \frac{2}{2\pi/\omega}\int_0^{\pi/\omega}\sin\omega t\cos n\omega t\,dt, \tag{51}$$

$$A_0 = \frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\,dt = \frac{\omega}{\pi}\left[-\frac{1}{\omega}\cos\omega t\right]_0^{\pi/\omega} = \frac{2}{\pi}\,, \tag{52}$$

$$A_1 = \frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\cos\omega t\,dt = \frac{\omega}{2\pi}\int_0^{\pi/\omega}\sin 2\omega t\,dt = \frac{\omega}{2\pi}\left[\frac{-1}{2\omega}\cos 2\omega t\right]_0^{\pi/\omega} = 0, \tag{53}$$

$$\begin{aligned} A_{2m} &= \frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\cos 2m\omega t\,dt = \frac{\omega}{\pi}\int_0^{\pi/\omega}\frac{1}{2}[\sin(2m+1)\omega t - \sin(2m-1)\omega t]\,dt \\ &= \frac{\omega}{2\pi}\left[\frac{-1}{(2m+1)\omega}\cos(2m+1)\omega t + \frac{1}{(2m-1)\omega}\cos(2m-1)\omega t\right]_0^{\pi/\omega} \\ &= \frac{1}{2\pi}\left(\frac{2}{2m+1} - \frac{2}{2m-1}\right) = -\frac{2}{(2m+1)(2m-1)\pi}\,, \end{aligned} \tag{54}$$

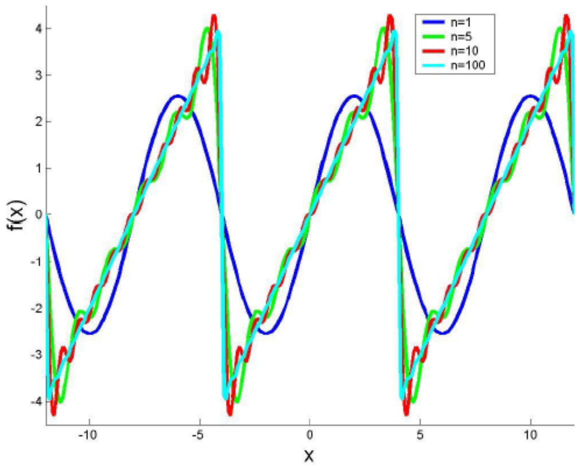
$$\begin{aligned} A_{2m+1} &= \frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\cos(2m+1)\omega t\,dt = \frac{\omega}{\pi}\int_0^{\pi/\omega}\frac{1}{2}[\sin(2m+2)\omega t - \sin 2m\omega t]\,dt \\ &= \frac{\omega}{2\pi}\left[\frac{-1}{(2m+2)\omega}\cos(2m+2)\omega t + \frac{1}{2m\omega}\cos 2m\omega t\right]_0^{\pi/\omega} = 0, \end{aligned} \tag{55}$$

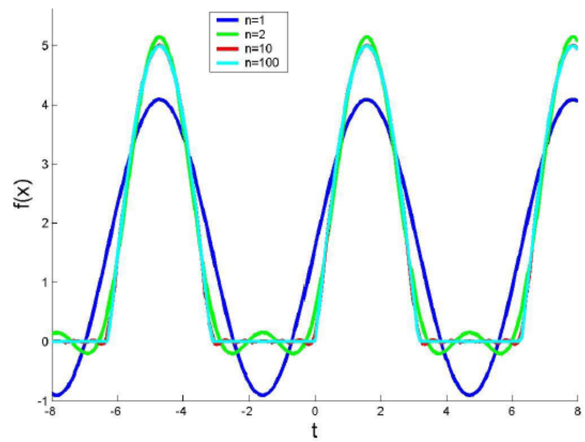
$$B_1 = \frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\sin\omega t\,dt = \frac{\omega}{\pi}\frac{1}{2}\frac{\pi}{\omega} = \frac{1}{2}\,, \tag{56}$$

$$\begin{aligned} B_{n>1} &= \frac{\omega}{\pi}\int_0^{\pi/\omega}\sin\omega t\sin n\omega t\,dt = \frac{\omega}{\pi}\int_0^{\pi/\omega}\frac{1}{2}[\cos(1-n)\omega t - \cos(n+1)\omega t]\,dt \\ &= \frac{\omega}{2\pi}\left[\frac{\sin(1-n)\omega t}{1-n} - \frac{\sin(n+1)\omega t}{n+1}\right]_0^{\pi/\omega} = 0, \end{aligned} \tag{57}$$

$$F(t) = \frac{1}{\pi} + \frac{1}{2}\sin\omega t - \frac{2}{3\pi}\cos 2\omega t - \frac{2}{15\pi}\cos 4\omega t - \cdots \tag{58}$$

The Fourier series of the sawtooth function converges somewhat more quickly than that of the half-wave, as shown in the figures below, from
http://kirkmcd.princeton.edu/examples/mechanics/TT_FourierSeries.pdf.





5. (a) The equation of motion of the oscillator is,

$$m\ddot{x} = -kx - b\dot{x} + F(t), \qquad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F}{m}\,, \qquad \beta = \frac{b}{2m}\,, \qquad \omega_0^2 = \frac{k}{m}\,, \tag{59}$$

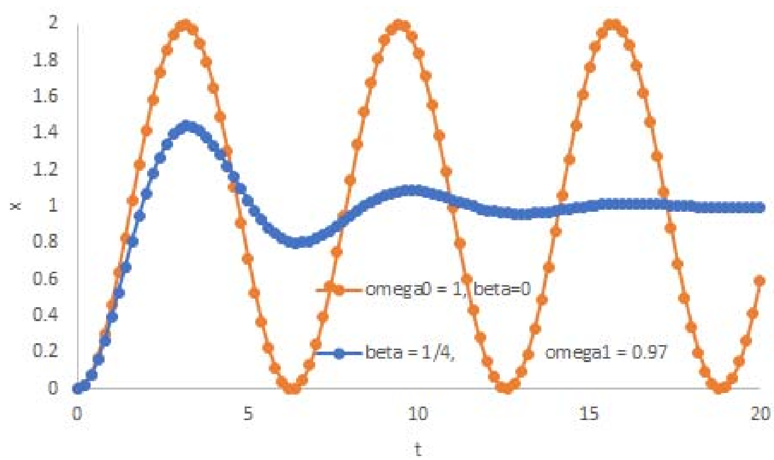
and the driving force is the step function,

$$F(t) = \begin{cases} 0 & (t < 0), \\ F_0 & (t > 0). \end{cases} \tag{60}$$

According to Green's method (p. 145 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l13.pdf), the motion is (for $t > 0$),

$$\begin{aligned} x(t > 0) &= \int_{-\infty}^{t} \frac{F(t')}{m\omega_1} e^{-\beta(t-t')} \sin\omega_1(t-t')\, dt' \qquad \left(\omega_1 = \sqrt{\omega_0^2 - \beta^2}\right) \\ &= \frac{F_0}{m\omega_1} \int_0^t e^{-\beta(t-t')} \sin\omega_1(t-t')\, dt' = \frac{F_0}{m\omega_1} \int_0^t e^{-\beta y} \sin\omega_1 y\, dy \\ &= \frac{F_0}{m\omega_1} \left[e^{-\beta y} \frac{-\beta\sin\omega_1 y - \omega_1\cos\omega_1 y}{\beta^2 + \omega_1^2} \right]_0^t \\ &= \frac{F_0}{m\omega_0^2} \left(1 - e^{-\beta t}\cos\omega_1 t - \frac{\beta}{\omega_1} e^{-\beta t} \sin\omega_1 t \right), \end{aligned} \tag{61}$$

using 577.1 of http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf.



The first maximum occurs for $\dot{x} = 0$,

$$0 = \beta\, e^{-\beta t}\cos\omega_1 t + (\beta^2/\omega_1)\, e^{-\beta t}\sin\omega_1 t + \omega_1\, e^{-\beta t}\sin\omega_1 t - \beta\, e^{-\beta t}\cos\omega_1 t, \tag{62}$$

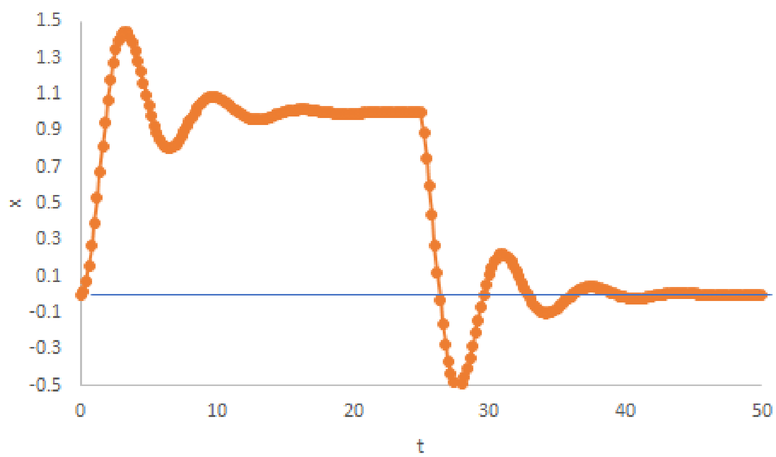
i.e., for $\omega_1 t = \pi$, with $x_{\max} = (F_0/m\omega_0^2)(1 + e^{-\beta\pi/\omega_1})$.

(b) The same oscillator is subject to the impulse,

$$F(t) = \begin{cases} 0 & (t < 0), \\ F_0 & (0 < t < T), \\ 0 & (t > T). \end{cases} \tag{63}$$

For $t < T$, the motion is again given by eq. (61), called $x_{(a)}(t)$ below,while for $t > T$ we have,

$$x(t > T) = \frac{F_0}{m\omega_1} \int_0^T e^{-\beta(t-t')} \sin\omega_1(t-t')\,dt' = \frac{F_0}{m\omega_1} \int_0^{t-T} e^{-\beta y} \sin\omega_1 y\,dy$$
$$= \frac{F_0}{m\omega_1} \left[e^{-\beta y} \frac{-\beta \sin\omega_1 y - \omega_1 \cos\omega_1 y}{\beta^2 + \omega_1^2} \right]_{t-T}^{t}$$
$$= \frac{F_0}{m\omega_0^2} \left(e^{-\beta(t-T)} \cos\omega_1(t-T) - \frac{\beta}{\omega_1} e^{-\beta(t-T)} \sin\omega_1(t-T) - e^{-\beta t}\cos\omega_1 t - \frac{\beta}{\omega_1} e^{-\beta t} \sin\omega_1 t \right)$$
$$= \frac{F_0}{m\omega_0^2} \left[1 - e^{-\beta t}\cos\omega_1 t - \frac{\beta}{\omega_1} e^{-\beta t}\sin\omega_1 t \right.$$
$$\left. - \left(1 - e^{-\beta(t-T)}\cos\omega_1(t-T) - \frac{\beta}{\omega_1} e^{-\beta(t-T)} \sin\omega_1(t-T) \right) \right] = x_{(a)}(t) - x_{(a)}(t-T). \qquad (64)$$



6. (a) The "steady" response of an oscillator to a driving force $F(t)$ may have the same form as the driving force.

Then, for the driving force $F(t) = F_0\, e^{-\alpha t}$ we "guess" that the "steady" motion has the form $x(t) = x_0\, e^{-\alpha t}$.

$$m\ddot{x} = -kx - b\dot{x} + F(t), \qquad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = (\alpha^2 - 2\alpha\beta + \omega_0^2)x_0\, e^{-\alpha t} = \frac{F_0}{m}\, e^{-\alpha t}, \tag{65}$$

$$x_0 = \frac{F_0/m}{\omega_0^2 + \alpha^2 - 2\alpha\beta}\, e^{-\alpha t}, \qquad \beta = \frac{b}{2m}, \qquad \omega_0^2 = \frac{k}{m}. \tag{66}$$

(b) If the driving force is,

$$F(t) = \begin{cases} 0 & (t < 0), \\ F_0\, e^{-\alpha t} & (t > 0), \end{cases} \tag{67}$$

we can use Green's method to solve for the transient response,

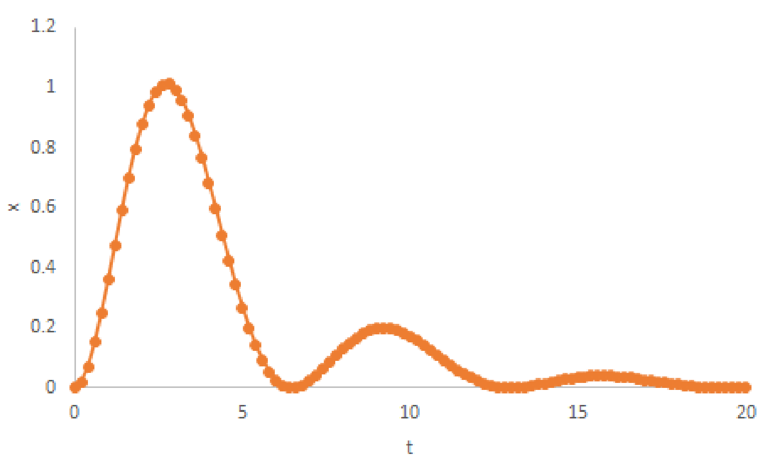
$$x(t > 0) = \int_{-\infty}^{t} \frac{F(t')}{m\omega_1}\, e^{-\beta(t-t')} \sin\omega_1(t - t')\, dt' \qquad \left(\omega_1 = \sqrt{\omega_0^2 - \beta^2}\right)$$

$$= \frac{F_0}{m\omega_1}\int_0^t e^{-\alpha t'}\, e^{-\beta(t-t')} \sin\omega_1(t-t')\, dt' = \frac{F_0}{m\omega_1}\int_0^t e^{-\alpha(t-y)}\, e^{-\beta y} \sin\omega_1 y\, dy$$

$$= \frac{F_0\, e^{-\alpha t}}{m\omega_1}\left[e^{(\alpha-\beta)y}\frac{(\alpha-\beta)\sin\omega_1 y - \omega_1\cos\omega_1 y}{(\alpha-\beta)^2 + \omega_1^2}\right]_0^t$$

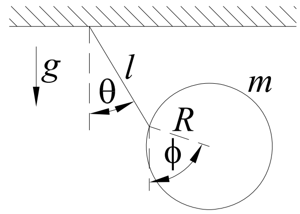
$$= \frac{F_0/m}{\omega_0^2 + \alpha^2 - 2\alpha\beta}\left[e^{-\alpha t} + e^{-\beta t}\left(\frac{\alpha-\beta}{\omega_1}\sin\omega_1 t - \cos\omega_1 t\right)\right]. \tag{68}$$

The motion for $\alpha = \beta = \omega_0/4$ is sketched below.



7. This problem is Ex. 1, p. 380 of Routh, *Elementary Rigid Dynamics*.

We seek the angular frequencies of small oscillations of the compound pendulum sketched below.



The equilibrium angles are $\theta_0 = 0 = \phi_0$, and it suffices to consider the kinetic and potential energy of the system to second order in small θ and ϕ. Taking the origin at the upper end of the rod of length l, the x-axis to be horizontal and the y-axis to be vertical, the coordinates of the center of mass of the hoop are,

$$x = l\sin\theta + R\sin\phi \approx l\theta + R\phi, \qquad \dot{x} \approx l\dot{\theta} + R\dot{\phi}, \tag{69}$$

$$y = l\cos\theta + R\cos\phi \approx l + R - \frac{l\theta^2}{2} - \frac{R\phi^2}{2}, \qquad \dot{y} \approx -l\theta\dot{\theta} - R\phi\dot{\phi}, \tag{70}$$

$$v^2 = \dot{x}^2 + \dot{y}^2 \approx \dot{x}^2 \approx l^2\dot{\theta}^2 + R^2\dot{\phi}^2 + 2lR\dot{\theta}\dot{\phi}. \tag{71}$$

The approximate equations of motion follow from the Lagrangian,

$$\mathcal{L} = T - V = \frac{mv^2}{2} + \frac{mR^2\dot{\phi}^2}{2} + mgy$$
$$\approx \frac{m}{2}(l^2\dot{\theta}^2 + 2R^2\dot{\phi}^2 + 2lR\dot{\theta}\dot{\phi}) + mg\left(l + R - \frac{l\theta^2}{2} - \frac{R\phi^2}{2}\right), \tag{72}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \approx ml^2\ddot{\theta} + mlR\ddot{\phi} = \frac{\partial\mathcal{L}}{\partial\theta} \approx -mgl\theta, \tag{73}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \approx 2mR^2\ddot{\phi} + mlR\ddot{\theta} = \frac{\partial\mathcal{L}}{\partial\phi} \approx -mgR\phi. \tag{74}$$

We seek oscillatory solutions of the form $\theta = \alpha\, e^{i\omega t}$, $\phi = \beta\, e^{i\omega t}$ for complex constants α and β, such that the equations of motion (73)-(74) reduce to,

$$-l^2\omega^2\alpha - lR\omega^2\beta = -gl\alpha, \qquad (l\omega^2 - g)\alpha + R\omega^2\beta = 0, \tag{75}$$

$$-2R^2\omega^2\beta - lR\omega^2\alpha = -gR\beta, \qquad l\omega^2\alpha + (2R\omega^2 - g)\beta = 0. \tag{76}$$

$$\begin{pmatrix} l\omega^2 - g & R\omega^2 \\ l\omega^2 & 2R\omega^2 - g \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{77}$$

The matrix equations (77) have a solution only if the determinant of the coefficient matrix vanishes,

$$0 = (l\omega^2 - g)(2R\omega^2 - g) - (l\omega^2)(R\omega^2) = lR\omega^4 - g(l + 2R)\omega^2 + g^2 \tag{78}$$

$$\omega^2 = \frac{g(l+2R) \pm \sqrt{g^2(l+2R)^2 - 4g^2lR}}{2lR} = \frac{g}{2lR}\left(l + 2R \pm \sqrt{l^2 + 4R^2}\right). \tag{79}$$

For $R = l/2$, we have $\omega^2 = (g/l)(2 \pm \sqrt{2})$.

We now consider the form of the normal modes of small oscillation for $R = l/2$, in which case eq. (77) becomes,

$$\begin{pmatrix} \omega^2 - g/l & \omega^2/2 \\ \omega^2 & \omega^2 - g/l \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{80}$$

The higher frequency mode has $\omega^2 = (g/l)(2 + \sqrt{2})$, and eq. (80) becomes,

$$\begin{pmatrix} 1+\sqrt{2} & 1+\sqrt{2}/2 \\ 2+\sqrt{2} & 1+\sqrt{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{81}$$

$$(1+\sqrt{2})\alpha + (1+\sqrt{2}/2)\beta = 0, \tag{82}$$

$$\frac{\beta}{\alpha} = -2\frac{1+\sqrt{2}}{2+\sqrt{2}} = -2\frac{1+\sqrt{2}}{2+\sqrt{2}}\frac{2-\sqrt{2}}{2-\sqrt{2}} = -\sqrt{2}. \tag{83}$$

In the higher frequency mode, the oscillations in θ and ϕ have opposite signs.

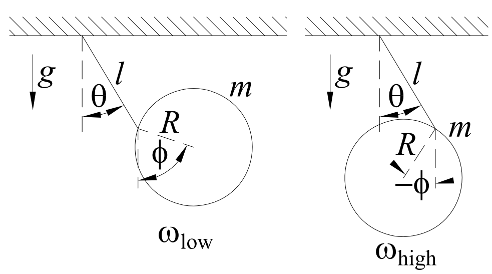
The lower frequency mode has $\omega^2 = (g/l)(2 - \sqrt{2})$, and eq. (80) becomes,

$$\begin{pmatrix} 1-\sqrt{2} & 1-\sqrt{2}/2 \\ 2-\sqrt{2} & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{84}$$

$$(1-\sqrt{2})\alpha + (1-\sqrt{2}/2)\beta = 0, \tag{85}$$

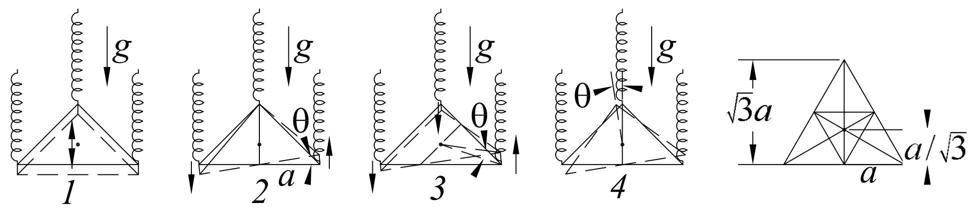
$$\frac{\beta}{\alpha} = -2\frac{1-\sqrt{2}}{2-\sqrt{2}} = -2\frac{1-\sqrt{2}}{2-\sqrt{2}}\frac{2+\sqrt{2}}{2+\sqrt{2}} = \sqrt{2}. \tag{86}$$

In the lower-frequency mode, the oscillations in θ and ϕ have the same signs.



8. This problem is Ex. 5, p. 381 of Routh, *Elementary Rigid Dynamics.*

An equilateral thin plate suspended by three springs has 4 modes of oscillation in which the center of mass moves only vertically (or not at all).



(1) In mode 1, the plate remains horizontal, while its center of mass oscillates in the vertical coordinate y. With three equal springs, each of constant k, the equation of motion is $m\ddot{y} = -3ky$, and the angular frequency of oscillation is,

$$\omega_1 = \sqrt{\frac{3k}{m}}\,, \tag{87}$$

where m is the mass of the triangular plate.

(2) In mode 2, a bisector/altitude of the triangle remains fixed (and horizontal) while one corner moves up and the other moves down.

There are 3 variants of this mode, for the 3 such bisectors.

If the plane of the triangle has rotated by small angle θ from the horizontal equilibrium position, the torque about the fixed bisector is $\tau = -2ka^2\theta$, where the edge of the equilateral triangle has length $2a$. The bisector has length $\sqrt{3}\,a$, and the area of the triangle is $A = \sqrt{3}\,a^2$. The moment of inertia of the triangle about a bisector is,

$$I_2 = \int_{-a}^{a} \frac{m}{\sqrt{3}\,a^2} x^2\,dx\,\sqrt{3}(a - x) = \frac{m}{a^2}\left(\frac{2a^4}{3} - \frac{2a^4}{4}\right) = \frac{ma^2}{6}. \tag{88}$$

The equation of motion of mode 2 is $I_2\,\ddot{\theta} = -2a^2k\theta$, and hence,

$$\omega_2 = 2\sqrt{\frac{3k}{m}} = 2\omega_1. \tag{89}$$

(3) In mode 3, the line (of length $4a/3$) through the center of mass of the triangle and parallel to one of its sides remains fixed, while one vertex moves up and the other two move down (or *vice versa*).

As for mode 2, there are 3 variants of mode 3.

If the plane of the triangle has rotated by small angle θ from the horizontal equilibrium position, the torque about the fixed line is,

$$\tau = -\frac{2a}{\sqrt{3}}k\frac{2a}{\sqrt{3}}\theta - 2\frac{a}{\sqrt{3}}k\frac{a}{\sqrt{3}}\theta = -2ak\theta. \tag{90}$$

The moment of inertia I_3 of the triangle about the fixed line through the center of mass is,

$$I_3 = \int_{-a/\sqrt{3}}^{2a/\sqrt{3}} \frac{m}{\sqrt{3}\,a^2} x^2\,dx\,\frac{4a}{3}\left(1 - \frac{x\sqrt{3}}{2a}\right)$$
$$= \frac{4m}{3\sqrt{3}\,a}\left[\frac{1}{3}\left(\frac{2a}{\sqrt{3}}\right)^3 + \frac{1}{3}\left(\frac{a}{\sqrt{3}}\right)^3 - \frac{\sqrt{3}}{8a}\left(\frac{2a}{\sqrt{3}}\right)^4 + \frac{\sqrt{3}}{8a}\left(\frac{a}{\sqrt{3}}\right)^4\right] = \frac{ma^2}{6}. \qquad (91)$$

The equation of motion of mode 3 is $I_3\,\ddot{\theta} = -2a^2k\theta$, and hence,

$$\omega_3 = 2\sqrt{\frac{3k}{m}} = \omega_2 = 2\omega_1. \qquad (92)$$

(4) In mode 4, the triangle rotates about the vertical axis through the center of mass, which remains fixed to a first approximation.

We suppose that the equilibrium stretch, $mg/3k$, of each spring is small compared to the rest length l of the springs. Then, for a small angle of rotation is θ, of the (horizontal) triangle about the vertical axis through the center of mass, the horizontal force of each spring on the triangle is $(mg/3)(2a\theta/\sqrt{3}\,l)$, and the torque is,

$$\tau = -3\frac{mg}{3}\frac{2a\theta}{\sqrt{3}\,l}\frac{2a}{\sqrt{3}} = -\frac{4mga^2}{3l}\theta. \qquad (93)$$

The moment of inertia I_4 of the equilateral triangle about the axis through the center of mass and perpendicular to the plane of the triangle is related by the perpendicular axis theorem to the sum of the moments of inertial about two perpendicular axes in the plane of the triangle. In particular,

$$I_4 = I_2 + I_3 = \frac{ma^2}{3}\,. \qquad (94)$$

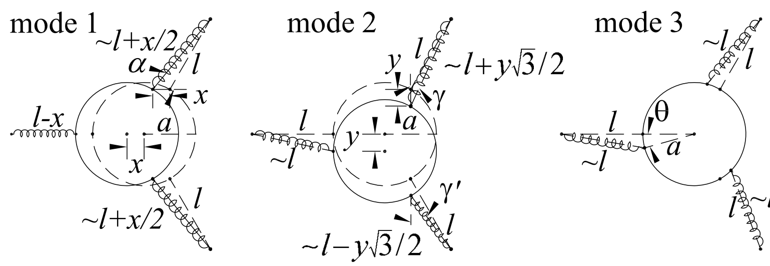
The equation of motion of mode 4 is $I_4\,\ddot{\theta} = -4mga^2\theta/3l$, and hence,

$$\omega_4 = 2\sqrt{\frac{g}{l}}\,, \qquad (95)$$

independent of the spring constant k and the mass m.

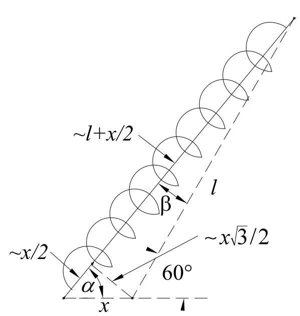
9. This problem is Ex. 10, p. 382 of Routh, *Elementary Rigid Dynamics.*

   A uniform disk of mass m and radius a connected by stretched springs to three fixed vertices of an equilateral triangle has three mode of motion in the plane of the triangle.rests on a frictionless, horizontal table.



   (1) In mode 1, the center of the disk moves along a bisector of the triangle, called the x-direction the left figure above. For displacement $x \ll l$, where l is the equilibrium stretched length of the springs, whose rest length is $l_0 < l$, the length of the other two springs is $l + x\cos 60^\circ = l + x/2$ to a first approximation. Each of these springs makes angle α to the x-axis, which is related by,

$$\begin{aligned}\cos\alpha = \cos(\pi - \beta - (\pi - 60^\circ)) &= \cos(60^\circ - \beta) \approx \cos 60^\circ + \sin 60^\circ \sin\beta \\ &\approx \frac{1}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}x}{2l} = \frac{1}{2} + \frac{3x}{4l}\,. \end{aligned} \tag{96}$$



   Then, the equation of motion of the oscillating disk is, to the first approximation,

$$m\ddot{x} = k(l - x - l_0) - 2k(l + x/2 - l_0)\cos\alpha \tag{97}$$

$$= k(l - x - l_0) - k(l + x/2 - l_0)\left(1 + \frac{3x}{2l}\right) = -\frac{3kx}{2} - 3kx\frac{l - l_0}{2l} = -3kx\frac{2l - l_0}{2l}\,.$$

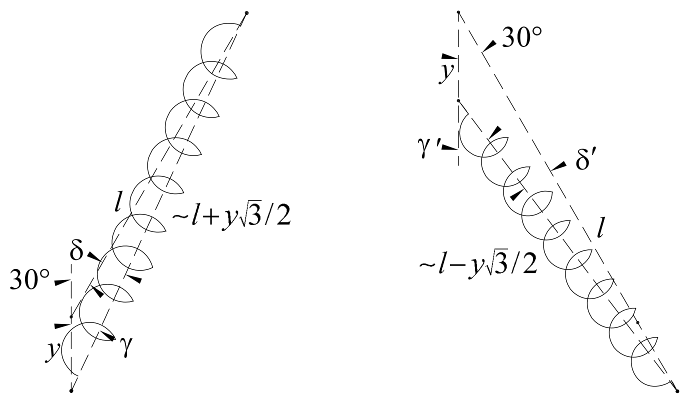
   Hence, the angular frequency of the small oscillations of mode 1 is,

$$\omega_1 = \sqrt{\frac{3k}{2m}\frac{2l - l_0}{l}}\,. \tag{98}$$

   (2) In mode 2, the center of the disk moves along a line perpendicular to a bisector of the triangle, called the y-direction the middle figure above. For displacement $x \ll l$, the length of the left springs remains l to a first approximation, while the other two springs take on lengths $l \pm y\sqrt{3}/2$. The y-component of the force

of the left spring on the disk is $-k(l-l_0)(y/l)$ to the first approximation. The y-component of the force of the second spring on the disk is $-k(l+y\sqrt{3}/3)\cos\gamma$ where,

$$\cos\gamma = \cos(\pi-\delta-(\pi-30^\circ)) = \cos(30^\circ-\delta) \approx \cos 30^\circ + \sin 60^\circ \sin\delta$$
$$\approx \frac{\sqrt{3}}{2} + \frac{1}{2}\frac{y}{2l}, \tag{99}$$
$$F_{2,y} = -k\left(l + \frac{y\sqrt{3}}{2} - l_0\right)\left(\frac{\sqrt{3}}{2} + \frac{y}{4}\right). \tag{100}$$



The y-component of the force of the third spring on the disk is $k(l'y\sqrt{3}/3)\cos\gamma'$ where,

$$\cos\gamma' = -\cos(\pi-\delta'-30^\circ) = \cos(30^\circ+\delta) \approx \cos 30^\circ - \sin 60^\circ \sin\delta,$$
$$\approx \frac{\sqrt{3}}{2} - \frac{1}{2}\frac{y}{2l}, \tag{101}$$
$$F_{3,y} = k\left(l - \frac{y\sqrt{3}}{2} - l_0\right)\left(\frac{\sqrt{3}}{2} - \frac{y}{4}\right). \tag{102}$$

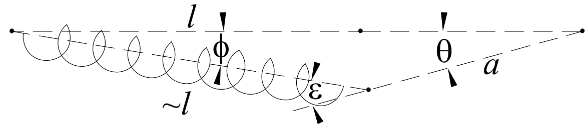
Then, the equation of motion of the oscillating disk is, to the first approximation,

$$m\ddot{y} = -k(l-l_0)\frac{y}{l}$$
$$-k\left(l+\frac{y\sqrt{3}}{2}-l_0\right)\left(\frac{\sqrt{3}}{2}\frac{y}{4}\right) + k\left(l-\frac{y\sqrt{3}}{2}-l_0\right)\left(\frac{\sqrt{3}}{2}-\frac{y}{4}\right) \tag{103}$$
$$= -k(l-l_0)\frac{y}{l} - ky\left(\frac{3}{4}+\frac{l-l_0}{4l}\right) - ky\left(\frac{3}{4}+\frac{l-l_0}{4l}\right) = -\frac{3ky}{2}\left(1+\frac{l-l_0}{l}\right).$$

Hence, the angular frequency of the small oscillations of mode 2 is,

$$\omega_2 = \sqrt{\frac{3k}{2m}\frac{2l-l_0}{l}} = \omega_1. \tag{104}$$

(1) In mode 3, the center of the disk remains fixed while the disk oscillates by angle θ about this point, as shown in the right figure on the previous page. For small rotations, the length of the springs remains l to a first approximation.



The torque exerted by each spring is $-ak(l - l_0)\sin\epsilon$ where,

$$\sin\epsilon \approx \epsilon = \theta + \phi \approx \theta + \frac{a\theta}{l}\,. \tag{105}$$

The moment of inertia of the disk is $I = ma^2/2$, so the equation of motion of the oscillating disk is, to the first approximation,

$$I\ddot{\theta} = \frac{ma^2}{2}\ddot{\theta} = \tau = -3ak\theta(l - l_0)\frac{a+l}{l}\,, \tag{106}$$

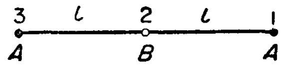
and the angular frequency of the small oscillations of mode 3 is,

$$\omega_3 = \sqrt{\frac{6k}{m}\frac{(l - l_0)(a + l)}{al}}\,. \tag{107}$$

For a solution via Lagrange's method, see
http://kirkmcd.princeton.edu/examples/Ph205/ph205sol7.pdf.

10. (a) Considering motion only along the axis of a linear triatomic molecule ABA with equilibrium spacing L between atoms A and B, the kinetic and potential energies are,

$$T = \frac{m_A(\dot{x}_1^2 + \dot{x}_3^2)}{2} + \frac{m_B \dot{x}_2^2}{2}, \qquad V = \frac{k}{2}\left[(x_2 - x_1 - L)^2 + (x_3 - x_2 - L)^2\right]. \tag{108}$$



Lagrange's equation of motion for this system, with $\mathcal{L} = T - V$, are,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m_A \ddot{x}_1 = \frac{\partial \mathcal{L}}{\partial x_1} = k(x_2 - x_1 - L), \tag{109}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m_B \ddot{x}_2 = \frac{\partial \mathcal{L}}{\partial x_2} = k(-2x_2 + x_1 + x_3), \tag{110}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_3} = m_A \ddot{x}_3 = \frac{\partial \mathcal{L}}{\partial x_3} = -k(x_3 - x_2 - L). \tag{111}$$

Taking the origin at the equilibrium position of atom B, small oscillations of the atoms about equilibrium have the forms,

$$x_1 = a_1\, e^{i\omega t} - L, \qquad x_2 = a_2\, e^{ei\omega t}, \qquad x_3 = a_3\, e^{i\omega t} + L, \tag{112}$$

for complex constants a_j. Using these forms in the equations of motion (109)-(111), we find,

$$-\,m_A\omega^2 a_1 = k(a_2 - a_1), \qquad (k - m_A\omega^2)a_1 - ka_2 + 0a_3 = 0, \tag{113}$$

$$-m_B\omega^2 a_2 = k(2a_2 - a_1 - a_3), \qquad ka_1 + (2k - m_B\omega^2)a_2 + ka_3 = 0, \tag{114}$$

$$-m_A\omega^2 a_3 = -k(a_3 - a_2), \qquad 0a_1 - ka_2 + (k - m_A\omega^2)a_3 = 0. \tag{115}$$

For there to be a solution, the determinant of the coefficient matrix must vanish,

$$(k - m_A\omega^2)^2(2k - m_B\omega^2) - 2k^2(k - m_A\omega^2) = 0. \tag{116}$$

The existence of the common factor $k - m_A\omega^2$ implies that,

$$\omega^2 = \frac{k}{m_A}, \tag{117}$$

is one solution.
After dividing out this common factor in eq. (116), we have

$$m_A m_B \omega^4 - k(2m_A + m_B)\omega^2 + 2k^2 - 2k^2 = 0, \tag{118}$$

such that two other solutions are,

$$\omega^2 = 0, \qquad \text{and} \qquad \omega^2 = k\frac{2m_A + m_b}{m_a M - B}. \tag{119}$$

The case of $\omega = 0$ means that there is a nonoscillatory motion possible for this system, which is just translation of the entire system, without internal motion, along the x-axis.

(b) If the middle atom B is tied to the origin by a spring, also of constant k, then the potential energy is that in eq. (108) plus an additional term $kx_2^2/2$. The equations of motion associated with coordinates x_1 and x_2 are again given be eqs. (109) and (111), while the equation of motion associated with coordinate x_2 is now,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}_2} = m_B\ddot{x}_2 = \frac{\partial\mathcal{L}}{\partial x_2} = k(-3x_2 + x_1 + x_3), \tag{120}$$

For small oscillations about equilibrium as in eq. (112), this implies,

$$-m_B\omega^1 a_2 = k(-3a_2 + a_1 + a_3). \tag{121}$$

The determinant of the coefficient matrix must again vanish, which leads to eq. (116) with the 2 replaced by 3,

$$(k - m_A\omega^2)^2(3k - m_B\omega^2) - 2k^2(k - m_A\omega^2) = 0. \tag{122}$$

Again, the existence of the common factor $k - m_A\omega^2$ implies that,

$$\omega^2 = \frac{k}{m_A}\,, \tag{123}$$

is one solution.

After dividing out the common factor in eq. (122), we have,

$$m_A m_B\omega^4 - k(3m_A + m_B)\omega^2 + k^2 = 0, \tag{124}$$

$$\begin{aligned}\omega^2 &= \frac{k(3m_A + m_B) \pm \sqrt{k^2(3m_A + m_B)^2 - 4k^2 m_A m_B}}{2m_A m_B} \\ &= k\frac{3m_A + m_B \pm \sqrt{9m_A^2 + 2m_A m_B + m_B^2}}{2m_A m_B}\,. \end{aligned}\tag{125}$$