# Princeton University

# Ph205

# Mechanics

# Problem Set 8

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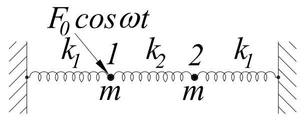
http://kirkmcd.princeton.edu/examples/

1. **Coupled, Damped, Forced Oscillation.**

A system of two equal masses connected by three collinear springs is subject to a collinear, external driving force $F_0 \cos \omega t$ on mass 1 only.




In addition, there are damping forces on the two masses, $F_1 = -b\dot{x}_1$ and $F_2 = -b\dot{x}_2$, which will prevent the amplitudes of the motion from diverging in case of resonance.

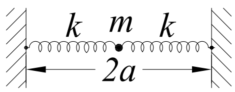
Demonstrate this by transformation the coupled equations of motion of $x_1$ and $x_2$ relative to the equilibrium positions of the masses into decoupled equations of motion of the normal coordinates $q_1$ and $q_2$ (for motion in the absence of the external force). Then, transform the solutions back to coordinates $x_1$ and $x_2$ to show that,

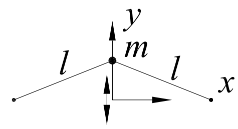
$$x_1 = \frac{F_0}{m} \frac{\omega_0^2 - \omega^2 + 2i\beta\omega}{\omega_1^2 - \omega^2 + 2i\beta\omega} \frac{e^{i\omega t}}{\omega_2^2 - \omega^2 + 2i\beta\omega}, \qquad x_2 = \frac{F_0}{2m} \frac{\omega_2^2 - \omega_1^2}{\omega_1^2 - \omega^2 + 2i\beta\omega} \frac{e^{i\omega t}}{\omega_2^2 - \omega^2 + 2i\beta\omega}, \tag{1}$$

$$\text{where} \qquad \frac{\beta = b}{2m}, \qquad \omega_0^2 = \frac{k_1 + k_2}{m}, \qquad \omega_1^2 = \frac{k_1}{m}, \qquad \omega_2^2 = \frac{k_1 + 2k_2}{m}. \tag{2}$$

*Hint: Guess the normal coordinates.*

2. A particle of mass $m$ is connected to two fixed points by springs of constant $k$ and rest length $l_0$. The fixed points are distance $2a$ apart.




Consider only transverse oscillations in the $x$-$y$ plane.

(a) Show that if $a < l_0$ there are three equilibrium points, and that the angular frequency of small oscillations about the one stable equilibrium is,

$$\omega = \sqrt{\frac{2k}{m}\left(1 - \frac{a^2}{l_0^2}\right)}. \tag{3}$$

(b) Show that if $a > l_0$ there is only one equilibrium point, and that the angular frequency of small oscillations about this is,

$$\omega = \sqrt{\frac{2k}{m}\left(1 - \frac{l_0}{a}\right)}. \tag{4}$$

(c) Show that if $a = l_0$ the potential is $V(y) \approx ky^4/4l_0^2$.

Then, use a method of successive approximations to show that the period is, to a first approximation,

$$T \approx \frac{4\pi\sqrt{3}}{3}\frac{l_0}{A}\sqrt{\frac{m}{k}}, \tag{5}$$

where $A$ is the amplitude.

It turns out that if the potential very precisely quartic then $T = 7.42(l_0/A)\sqrt{m/k}$.

3. A spring of constant $k$ is fixed at one end and attached to mass $m$ at the other. The mass is subject to friction such that the equation of (1-dimensional) motion is,

$$m\ddot{x} = -kx - \mu mg \cdot (\text{sign of } \dot{x}) \tag{6}$$

Let $\omega_0^2 = k/m$, and suppose that at $t = 0$, $x = a_0$ and $\dot{x} = 0$, where $a_0 \gg \mu g/\omega_0^2$.

(a) Consider the exact solution for half periods to calculate the loss of amplitude. Show that the amplitude will drop to zero in time $t \approx \pi a_0 \omega_0 / 2\mu g$.

(b) Use the method of averages[1] to show that the motion is approximately $x \approx (a_0 - 2\mu gt/\pi\omega_0)\cos\omega_0 t$.

[1] Pp. 161-162 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l15.pdf.

4. Consider a 1-dimensional oscillator subject to a damping force such that the equation of motion is,

$$\ddot{x} + \omega_0^2 = -\beta \dot{x} \left| \dot{x} \right| . \tag{7}$$

Suppose that at $t = 0$, $x = a_0$ and $\dot{x} = 0$.

(a) Show by a method of successive approximations that after one half period that amplitude has been reduced to (approximately) $a_0(1 - 4\beta a_0/3)$.

(b) Use the method of averages to show that the motion is approximately $x \approx a(t) \cos \omega_0 t$ where $1/a = 1/a_0 + (4\beta/3\pi) \cos \omega_0 t$, which agrees with the result of part (a) after one half period.

5. Consider a central force,[2]

$$F = -\frac{\alpha}{r^2} - \frac{\beta}{r^4}\,. \tag{8}$$

The "orbit equation" for a (small) mass $m$ subject to this force is,[3] with $u = 1/r$,

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}F(1/u) = \frac{\alpha m}{L^2} + \frac{\beta m u^2}{L^2}\,, \tag{9}$$

where $L$ is the angular momentum of mass $m$ about the force center.

If $\beta = 0$, the solution is the standard Newtonian ellipse, $u = (\alpha m/L^2)(1 + \epsilon\cos\theta)$.

For $\beta \neq 0$ consider a solution of the form,

$$u \approx k[1 + \epsilon(\theta)\cos\phi(\theta)]. \tag{10}$$

Supposing that $d\epsilon/d\theta$ is small, $d^2\epsilon/d\theta^2$ is of second order, and $d\phi/d\theta \approx 1$, show that,

$$u \approx \frac{\alpha m}{L^2}\left[1 + \epsilon\cos\left(1 - \frac{2\alpha\beta m^2}{L^2}\theta\right)\right]. \tag{11}$$

This is a slowly precessing ellipse, with angular velocity of precession,

$$\omega = \frac{2\alpha\beta m^2}{L^4}\Omega, \qquad \text{where} \qquad \Omega = \frac{2\pi}{T}\,, \tag{12}$$

is the average angular velocity of the orbital motion (of period $T$).

*It is not necessary here, but the approximation scheme could be cast into the form of a method of averages (that allows us to consider small oscillations about an elliptical orbit, rather than about a circular orbit as in our previous analyses).*

*For what it's worth,*

$$\frac{d\phi}{d\theta} = 1 - \frac{2}{\epsilon\pi}\int_0^{2\pi} f\cos\phi\,d\phi, \qquad \frac{d\epsilon}{d\theta} = -\frac{1}{\pi}\int_0^{2\pi} f\sin\phi\,d\phi, \tag{13}$$

*with,*

$$f = -\frac{\alpha\beta\epsilon m^2\cos\phi}{L^4}\,, \tag{14}$$

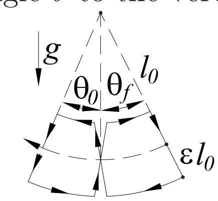
*for the present example.*

[2]Recall probs. 8 and 9 of http://kirkmcd.princeton.edu/examples/Ph205/ph205set8.pdf.
[3]See http://kirkmcd.princeton.edu/examples/Ph205/ph205l10.pdf.

6. **The Swing**

If one "pumps" a swing, the amplitude of the oscillation can be increased.

(a) As a simple model of the pumping action, consider a simple pendulum whose center of mass is (quickly) lowered by $2\epsilon l_0$ when the amplitude is maximum, and raised by $2\epsilon l_0$ when the angle $\theta$ to the vertical is 0.

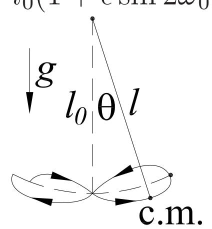



Use elementary methods to show that during one half period,

$$\theta_f^2 = \theta_0^2 \left(\frac{1+\epsilon}{1-\epsilon}\right)^3, \tag{15}$$

and that if $\epsilon$ is small this leads to $\theta_{\max} = \theta_0 \, e^{3\epsilon\omega_0 t/\pi}$, where $\omega_0 = \sqrt{\frac{g}{l_0}}$.

(b) In another model of pumping, suppose that the distance from the pivot to the center of mass varies as $l = l_0(1 + \epsilon \sin 2\omega_0 t)$.



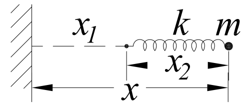

Deduce the equation of motion for angle $\theta$.

For small $\epsilon$ we suppose that solution will have the form $\theta(t) \approx a(t) \cos \omega_0 t$. Use a method of successive approximations, or a method of averages, to show that,

$$\theta_{\max} \approx \theta_0 \, e^{3\epsilon\omega_0 t/4}, \tag{16}$$

for this model.

7. (a) Mass $m$ is connected to one end of a spring of constant $k$ and rest length $l_0$. The other end of the spring is force to move according to $x_1 = a\cos\omega t$ in the inertial lab frame.




Go to the accelerated frame with origin at $x_1$ and solve for the steady-state motion $x(t) = x_1 + x_2$ with zero velocity at time $t = 0$.

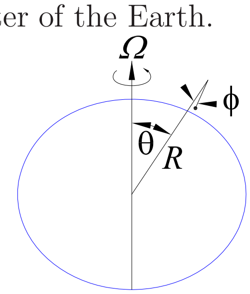
(b) A particle has velocity $\mathbf{v}$ on a smooth (frictionless) table.

Show that the particle will move in a circle, and find the radius of the circle and the angular velocity of the motion relative to the Earth.

8. (a) A plumb line does not point to the center of the Earth, due to the effect of the centrifugal force of the Earth's rotation. Show that the angle $\phi$ of deflection of the plumb line to the line from its support point to the center of the Earth is related by,

$$\tan\phi = \frac{\sin\theta\cos\theta}{g/\Omega^2 R - \sin^2\theta}\,, \tag{17}$$

where $\theta$ is the polar angle to the support point, $\Omega$ is the angular velocity of rotation of the Earth about its axis, and $R$ is the distance of the plumb bob from the center of the Earth. You may assume that the force of gravity at the Earth's surface points to the center of the Earth.




*Note that* $g/\Omega^2 R \approx 290$.

(b) **Centrifugal Bulge.**

We might expect the water surface of the Earth to be perpendicular to a plumb line, which implies a bulge at the equator.

The (perpendicular) surface will be an equipotential of the effective potential of gravity and the centrifugal force. Show that this leads to the equation of the surface as,

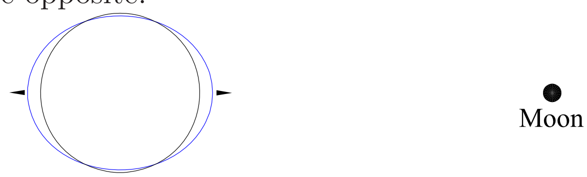
$$R^3\sin^2\theta = \frac{2GM}{\Omega^2}\left(\frac{R}{R_P} - 1\right), \tag{18}$$

where $R_P$ = radius at the pole and $\theta$ is as in part (a). You may assume that the gravitational potential of the bulging Earth is the same as that for a spherical Earth.

Write $R_E/R_P = 1 + \epsilon$, where $R_E$ is the radius at the equator, to show that $\epsilon \approx 1/580$. Experimentally, $\epsilon \approx 1/297$.

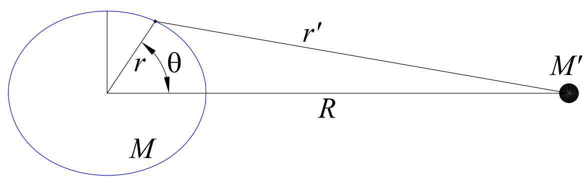
9. **Tidal Bulge.**

The shape of the Earth is also distorted by the pull of the Moon (and of the Sun). Roughly, the Moon pulls on the near side of the Earth more than on the center, and still more than on the far side. This leads to a bulge on the side facing the Moon, and another on the side opposite.




In this problem consider only the effect of the Moon, and suppose the Earth would be spherical in the absence of the Moon's pull.

Consider a frame with the center of the Earth at rest, and axes pointing to the "fixed stars". Construct the combined gravitational potential $V$ for a particle at the surface of the Earth.




This frame is accelerated, so there is a "fictitious" force needed to hold the Earth center of mass fixed. In the approximation of a rigid Earth, this force is the same on all particles (of mass $m$) throughout the Earth. Show that to order $(r/R)^2$,

$$\frac{V}{m} \approx -\frac{GM}{r} - \frac{GM'}{R} + \frac{3}{2}\frac{GM'r^2}{R^3}\left(\frac{1}{3} - \cos^2\theta\right), \tag{19}$$

where $r$ is the radius of the Earth of mass $M$, $R$ is the Earth-Moon distance, and $M'$ is the mass of the Moon, and $\theta$ is the angle between the radius to mass $m$ and the line of centers of the Earth and Moon).

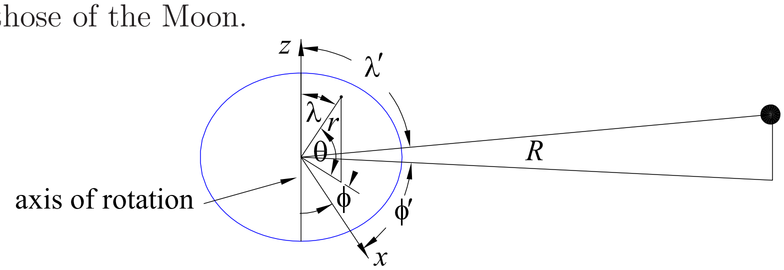
Evaluate $\epsilon$ in the expression for the equipotential at the surface of the Earth,

$$r \approx r_0\left[1 + \epsilon\left(\cos^2\theta - \frac{1}{3}\right)\right], \tag{20}$$

where $r_0$ is the mean radius of the Earth. This implies a tidal bulge of 1 foot.

Because the Earth turns around its axis, and the Moon orbits the Earth, the position of the bulge changes with time.

Taking the $z$-axis of the coordinate system considered above to be the $z$-axis of the Earth, let $(r, \lambda, \phi)$ be the spherical coordinates of a point on the Earth's surface, and $(R, \lambda', \phi')$ those of the Moon.



Show that,

$$\frac{r}{r_0} = 1 + \epsilon \left[ \frac{3}{2} \left( \cos^2 \lambda - \frac{1}{3} \right) \left( \cos^2 \lambda' - \frac{1}{3} \right) + \frac{1}{2} \sin^2 \lambda \sin^2 \lambda' \cos 2(\phi - \phi') + \frac{1}{2} \sin 2\lambda \sin 2\lambda' \cos(\phi - \phi') \right]. \tag{21}$$

Thus, while to total effect is given by eq. (20), an observer at a fixed location $(\lambda, \phi)$ on the Earth's surface can do a "Fourier analysis" of the bulge into 3 time-dependent tides (with 3 different periods).

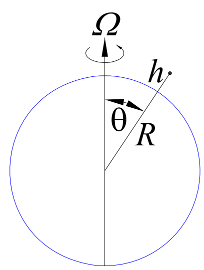
What is the period of each of these tides? Which one corresponds to the usual perception of "the" tide?

10. You may assume that the Earth is spherical in this problem, and ignore gravity except that due to the Earth.

    (a) A particle falls from rest at height $h \ll R$ above the Earth's surface, at polar angle $\theta$. To first order in $\Omega$, the angular velocity of the rotation of the Earth about its axis, show that the (horizontal) deflection during the fall is,

    $$d = \frac{2}{3}\sqrt{\frac{2h^3}{g}}\Omega \sin\theta, \tag{22}$$

    taking $g$ to be constant during the fall. In what direction is the deflection? Is this different in the southern hemisphere?




    (b) Suppose you jump up to height $h \ll R$ and then fall back. Show that the coriolis force increases the deflection/displacement to,

    $$d = \frac{8}{3}\sqrt{\frac{2h^3}{g}}\Omega \sin\theta. \tag{23}$$

    In what direction is this displacement?

    (c) Use conservation of angular momentum and a non-accelerated point of view to calculate the change in azimuthal angle $\phi$ during your jump. Show that this leads to the same displacement as in part (b).

11. You may assume that the Earth is spherical in this problem, and ignore gravity except that due to the Earth.

    A gun is located on the surface of the Earth at polar angle $\theta$. In the absence of the Coriolis force (and of the centrifugal force, and of air resistance), the shot would land distance $D$ away, having risen to height $h \ll R$ (such that you may also ignore effects of the curvature of the Earth, of radius $R$).

    (a) If the gun fired North, show that the shot is deflected by,

    $$d \approx 2\sqrt{\frac{2h}{g}}\Omega\left(\frac{4h}{3}\sin\theta - D\cos\theta\right), \tag{24}$$

    to first order in $\Omega$, the angular velocity of rotation of the Earth about its axis. In what direction?

    (b) If the gun fired East, show that the shot is deflected by,

    $$d \approx 2\sqrt{\frac{2h}{g}}\Omega D\cos\theta, \tag{25}$$

    to order $\Omega$. In what direction?
    Also show that it lands at distance,

    $$D'_E \approx D\left(1 + \frac{\Omega D\sin\theta}{\sqrt{2gh}}\right), \tag{26}$$

    to the East (which is orthogonal to the deflection (25)).

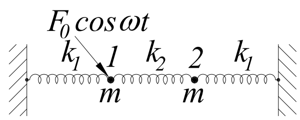
## Solutions

1. **Coupled, Damped, Forced Oscillation.**

The equations of motion of the two masses $m$ follow readily from $F = ma$,

$$m\ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) - b\dot{x}_1 + F_0 \cos\omega t, \qquad m\ddot{x}_2 = k_1(D - x_2) - k_2(x_2 - x_1) - b\dot{x}_2, \tag{27}$$

where the fixed ends of the springs of constant $k_1$ are at $x = 0$ and $D$, and the rest lengths of the springs is small compared to $D$.




In this problem, we are mainly concerned with motion relative to the equilibrium positions of the two masses, due the external force $F_0 \cos\omega t$ on mass 1. Then, if we define coordinates $x_1$ and $x_2$ relative to their equilibrium positions, the equations of motion (27) take the form,[4]

$$m\ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) - b\dot{x}_1 + F_0 \cos\omega t, \qquad m\ddot{x}_2 = -k_1 x_2 - k_2(x_2 - x_1) - b\dot{x}_2, \tag{31}$$

It is now straightforward to assume motion of the form $x_{1,2} = a_{1,2}\, e^{i\omega t}$ and solve for the complex constants $a_{1,2}$.

However, we take a slightly different approach, noting that in the absence of the external force there are two normal modes, one in which the two masses move with $x_1 = x_2$, and the other in which they move oppositely, $x_1 = -x_2$. This leads us to consider the normal (but not normalized) coordinates $q_1$ and $q_2$ related by,

$$q_1 = x_1 + x_2, \qquad q_2 = x_1 - x_2, \qquad x_1 = \frac{q_1 + q_2}{2}, \qquad x_2 = \frac{q_1 - q_2}{2} \tag{32}$$

The equations of motion for the normal coordinates can by found by adding and subtracting the equations of motion (27),

$$\ddot{x}_1 + \ddot{x}_2 + \frac{b}{m}(\dot{x}_1 + \dot{x}_2) + \frac{k_1}{m}(x_1 + x_2) = \frac{F_0}{m} e^{i\omega t}, \qquad \ddot{q}_1 + 2\beta\dot{q}_1 + \omega_1^2 q_1 = \frac{F_0}{m} e^{i\omega t}, \tag{33}$$

$$\ddot{x}_1 - \ddot{x}_2 + \frac{b}{m}(\dot{x}_1 - \dot{x}_2) + \frac{k_1 + 2k_2}{m}(x_1 - x_2) = \frac{F_0}{m} e^{i\omega t}, \qquad \ddot{q}_2 + 2\beta\dot{q}_2 + \omega_2^2 q_2 = \frac{F_0}{m} e^{i\omega t}, \tag{34}$$

$$\text{where} \qquad \beta = \frac{b}{2m}, \qquad \omega_1^2 = \frac{k_1}{m}, \qquad \omega_2^2 = \frac{k_1 + 2k_2}{m}, \qquad \omega_0^2 = \frac{k_1 + k_2}{m} = \frac{\omega_1^2 + \omega_2^2}{2}. \tag{35}$$

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[4]For what its worth, the equilibrium positions of the two masses can be found by setting $\ddot{x}_1 = \ddot{x}_2 = \dot{x}_1 = \dot{x}_2 = F_0 = 0$ in eq. (27) still regarding $x_1$ and $x_2$ as measured with respect to the left wall. Then,

$$0 = -k_1 x_{1,0} + k_2(x_{2,0} - x_{1,0}), \qquad x_{2,0} = \frac{k_1 + k_2}{k_2} x_{1,0}, \tag{28}$$

$$0 = k_1(D - x_{2,0}) - k_2(x_{2,0} - x_{1,0}) = k_1 D + k_2 x_{1,0} - (k_1 + k_2) x_{2,0}, \tag{29}$$

$$k_1 D = -k_2 x_{1,0} + \frac{(k_1 + k_2)^2 x_{1,0}}{k_2}, \qquad x_{1,0} = \frac{k_2}{k_1 + 2k_2} D, \qquad x_{2,0} = \frac{k_1 k_2}{k_1 + 2k_2} D. \tag{30}$$

The decoupled equations of motion for $q_1$ and $q_2$ imply that,

$$q_{1,2} = \frac{F_0}{m} \frac{e^{i\omega t}}{\omega_{1,2}^2 - \omega^2 + 2i\beta\omega} . \tag{36}$$

We transform these back to coordinates $x_1$ and $x_2$,

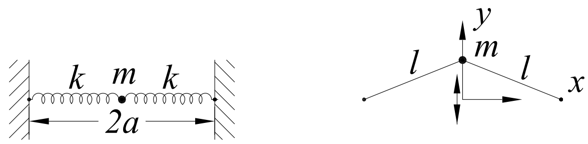
$$x_{1,2} = \frac{q_1 \pm q_2}{2} = \frac{F_0\, e^{i\omega t}}{2m} \left( \frac{1}{\omega_1^2 - \omega^2 + 2i\beta\omega} \pm \frac{1}{\omega_2^2 - \omega^2 + 2i\beta\omega} \right), \tag{37}$$

$$\begin{aligned} x_1 &= \frac{F_0\, e^{i\omega t}}{m} \frac{\frac{\omega_1^2+\omega_2^2}{2} - \omega^2 + 2i\beta\omega}{(\omega_1^2 - \omega^2 + 2i\beta\omega)\,(\omega_2^2 - \omega^2 + 2i\beta\omega)} \\ &= \frac{F_0\, e^{i\omega t}}{m} \frac{\omega_0^2 - \omega^2 + 2i\beta\omega}{(\omega_1^2 - \omega^2 + 2i\beta\omega)\,(\omega_2^2 - \omega^2 + 2i\beta\omega)} , \end{aligned} \tag{38}$$

$$x_2 = \frac{F_0\, e^{i\omega t}}{2m} \frac{\omega_2^2 - \omega_1^2}{(\omega_1^2 - \omega^2 + 2i\beta\omega)\,(\omega_2^2 - \omega^2 + 2i\beta\omega)} . \tag{39}$$

2. *Adapted from prob. 6.5.2, p. 159 of* W. Chester, *Mechanics* (Allen & Unwin, 1979), http://kirkmcd.princeton.edu/examples/mechanics/chester_mechanics_79.pdf

A particle of mass $m$ is connected to two fixed points by springs of constant $k$ and rest length $l_0$. The fixed points are distance $2a$ apart.




We consider only transverse oscillations in the $x$-$y$ plane, in which case the potential is (ignoring gravity),

$$V(y) = 2\frac{k(l-l_0)^2}{2} = k\left(\sqrt{a^2+y^2} - l_0\right)^2. \tag{40}$$

$$\frac{dV}{dy} = 2k\left(\sqrt{a^2+y^2} - l_0\right)\frac{y}{\sqrt{a^2+y^2}} = 2ky\left(1 - \frac{l_0}{\sqrt{a^2+y^2}}\right), \tag{41}$$

$$\frac{d^2V}{dy^2} = 2k\left(1 - \frac{l_0}{\sqrt{a^2+y^2}}\right) + \frac{2ky^2 l_0}{(a^2+y^2)^{3/2}}, \tag{42}$$

$$\frac{d^3V}{dy^3} = \frac{6kyl_0}{(a^2+y^2)^{3/2}} - \frac{6ky^2 l_0}{(a^2+y^2)^{5/2}}, \tag{43}$$

$$\frac{d^4V(y=0)}{dy^4} = \frac{6kl_0}{a^3}. \tag{44}$$

Possible equilibrium points, where $dV/dy = 0$, and $y = 0$, and $y = \pm\sqrt{l_0^2 - a^2}$ (where $\sqrt{a^2+y^2} = l_0$) if $a < l_0$.

(a) If $a < l_0$ there are three equilibrium points, $y_0 = 0, \ \pm\sqrt{l_0^2 - a^2}$.
As to their stability, we consider the second derivative of the potential,

$$\frac{d^2V(0)}{dy^2} = 0, \qquad \Rightarrow \text{unstable}, \tag{45}$$

$$\frac{d^2V(\pm\sqrt{l_0^2-a^2})}{dy^2} = \frac{2k(l_0^2-a^2)}{l_0^2} \equiv k_{\text{eff}}, \qquad \Rightarrow \text{stable}, \tag{46}$$

The angular frequency of small oscillations about the stable equilibria is,

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}} = \sqrt{\frac{2k}{m}\left(1 - \frac{a^2}{l_0^2}\right)}. \tag{47}$$

(b) If $a > l_0$ there is only one equilibrium point, $y_0 = 0$, for which

$$k_{\text{eff}} = \frac{d^2V(0)}{dy^2} = 2k\left(1 - \frac{l_0}{a}\right), \qquad \Rightarrow \text{stable}, \tag{48}$$

and the angular frequency of small oscillations is

$$\omega = \sqrt{\frac{k_{\rm eff}}{m}} = \sqrt{\frac{2k}{m}\left(1 - \frac{l_0}{a}\right)}. \tag{49}$$

(c) If $a = l_0$ the $0 = dV(0)/dy = d^2V(0)/dy^2 = d^3V(0)/dy^3$, and the leading nonzero term in the expansion of the potential is

$$V(y) \approx \frac{1}{4!}\frac{d^4V(0)}{dy^4}y^4 = \frac{1}{24}\frac{6kl_0}{a^3}y^4 = \frac{ky^4}{4l_0^2}\,. \tag{50}$$

For small oscillations, the equation of motion is,

$$m\ddot{y} = -\frac{dV}{dy} = -\frac{ky^3}{l_0^2} + \cdots. \tag{51}$$

A solution that starts from rest with nonzero amplitude at $t = 0$ can be a Fourier series of the form $y \sum_n A_n \cos n\omega t$ where $\omega = \sqrt{k/m}$. Since $\cos 2\omega t = 2\cos^2\omega t - 1$ while $\cos 3\omega t = 4\cos^3\omega t - 3\cos\omega t$, we anticipate that $A_2 = 0$, and the first two terms of the series have the form,

$$y = A\cos\omega t + B\cos 3\omega t. \tag{52}$$

We first consider just the form $y = A\cos\omega t$,

$$\begin{aligned}\ddot{y} = -\omega^2 A\cos\omega t = -\frac{ky^3}{ml_0^2} = -\frac{kA^3\cos^3\omega t}{ml_0^2} &= -\frac{kA^3(3\cos\omega t + \cos^3\omega t)}{4ml_0^2} \\ &\approx -\frac{3kA^3\cos\omega t}{4ml_0^2}\,,\end{aligned} \tag{53}$$

$$\omega^2 \approx \frac{3kA^2}{4ml_0^2}\,, \qquad T \approx \frac{4\pi\sqrt{3}}{3}\frac{l_0}{A}\sqrt{\frac{m}{k}} = 7.26\frac{l_0}{A}\sqrt{\frac{m}{k}}\,. \tag{54}$$

If the potential (50) were precisely quartic, and mechanical energy is conserved, then we have,

$$E = \frac{m\dot{y}^2}{2} + \frac{ky^4}{4l_0^2}\,. \tag{55}$$

If the amplitude is $A$ when $\dot{y} = 0$, we have that,

$$\dot{y}^2 = \frac{k(A^4 - y^4)}{2ml_0^2}\,, \qquad T = 4\int_0^A \frac{dy}{\dot{y}} = 4\sqrt{\frac{2ml_0^2}{k}}\int_0^A \frac{dy}{\sqrt{A^4 - y^4}}\,. \tag{56}$$

Then, with the substitution $y/A = \cos\phi$, the period of oscillation is,

$$T = \frac{4l_0}{A}\sqrt{\frac{m}{k}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}} = \frac{4l_0}{A}\sqrt{\frac{m}{k}}K(1/\sqrt{2}) = 7.42\frac{l_0}{A}\sqrt{\frac{m}{k}}\,, \tag{57}$$

where $K$ is the complete elliptic integral.

3. The equation of motion of the damped oscillator is,

$$\ddot{x} + \omega_0^2 x = -\mu g \cdot (\text{sign of } \dot{x}), \tag{58}$$

where $\omega_0^2 = k/m$. We suppose that at $t = 0$, $x = a_0$ and $\dot{x} = 0$, where $A \gg \mu g/\omega_0^2$.

(a) During odd-numbered half periods $\dot{x} < 0$ and the damping force is positive. At the beginning of such a half period, at time $t_n$, $x(t_n) > 0$ and $\dot{x}(t_n) = 0$. The equation of motion is,

$$\ddot{x} + \omega_0^2 x = \mu g \qquad \text{(odd half period)}. \tag{59}$$

The motion during this half period is,

$$x = \frac{\mu g}{\omega_0^2} + a_n \cos \omega_0 (t - t_n), \tag{60}$$

and the initial condition tells us that,

$$a_n = x(t_n) - \frac{\mu g}{\omega_0^2}\,. \tag{61}$$

At the end of the $n^{\text{th}}$ half period, which is the beginning of half period $n+1$, we have,

$$x(t_{n+1}) = \frac{\mu g}{\omega_0^2} - \left( x(t_n) - \frac{\mu g}{\omega_0^2} \right) = -x(t_n) + \frac{2\mu g}{\omega_0^2}\,. \tag{62}$$

During the half period $n+1$, the damping force is negative, so eqs. (60)-(62) take the forms,

$$x = -\frac{\mu g}{\omega_0^2} + a_{n+1} \cos \omega_0 (t - t_{n+1}), \tag{63}$$

and the initial condition tells us that,

$$a_{n+1} = x(t_{n+1}) + \frac{\mu g}{\omega_0^2}\,. \tag{64}$$

At the end of the $n+1^{\text{th}}$ half period, which is the beginning of half period $n+2$, we have,

$$x(t_{n+2}) = -\frac{\mu g}{\omega_0^2} - \left( x(t_{n+1}) + \frac{\mu g}{\omega_0^2} \right) = -x(t_{n+1}) - \frac{2\mu g}{\omega_0^2} = x(t_n) - \frac{4\mu g}{\omega_0^2}\,. \tag{65}$$

Thus, during each full period, of duration $2\pi/\omega_0$, the amplitude of the oscillation decreases by $4\mu g/\omega_0^2$. Hence, the oscillations damp to zero in time,

$$t = \frac{a_0}{4\mu g/\omega_0^2} \frac{2\pi}{\omega_0} = \frac{\pi a_0 \omega_0}{2\mu g}\,. \tag{66}$$

(b) According to the method of averages for a damped oscillator, $\ddot{x} + \omega^2 x = \epsilon f(x, \dot{x})$,

$$x \approx a(t)\cos\phi(t), \quad \dot{a} = -\frac{\epsilon}{2\pi\omega_0}\int_0^{2\pi} f\sin\phi\, d\phi, \quad \dot{\phi} = \omega_0 - \frac{\epsilon}{2\pi\omega_0}\int_0^{2\pi} f\cos\phi\, d\phi. \tag{67}$$

In the present example, $\epsilon = -\mu g$ and $f$ is $+1$ for the first half period and $-1$ for the second half period. Hence,

$$\dot{a} = -\frac{-\mu g}{2\pi\omega_0}2\int_0^{\pi}(-\sin\phi)\, d\phi = -\frac{2\mu g}{\pi\omega_0}, \qquad a = a_0 - \frac{2\mu g t}{\pi\omega_0}, \tag{68}$$

$$\dot{\phi} = \omega_0 - \frac{-\mu g}{2\pi\omega_0}2\int_0^{\pi}(-\cos\phi)\, d\phi = \omega_0, \qquad \phi = \omega_0 t. \tag{69}$$

The motion is approximately,

$$x \approx \left(a_0 - \frac{2\mu g t}{\pi\omega_0}\right)\cos\omega_0 t, \tag{70}$$

and amplitude drops to zero in time,

$$t \approx \frac{\pi a_0 \omega_0}{2\mu g}, \tag{71}$$

as found in part (a).

4. We consider a 1-dimensional oscillator subject to a damping force such that the equation of motion is,

$$\ddot{x} + \omega_0^2 x = -\beta \dot{x}\,|\dot{x}|\,. \tag{72}$$

Suppose that at $t = 0$, $x = a_0$ and $\dot{x} = 0$.

(a) During the first half period, during which $\dot{x} < 0$, the equation of motion is,

$$\ddot{x} + \omega_0^2 x = \beta \dot{x}^2. \tag{73}$$

If we ignore the damping, the solution would be just,

$$x_0 = a_0 \cos\omega_0 t. \tag{74}$$

As a next approximation, we consider the form,

$$x = a_1 \cos\omega_0 t + \beta x_1. \tag{75}$$

With this in eq (73), and ignoring higher-order terms in $\beta$, we have,

$$\beta\ddot{x}_1 + \beta\omega_0^2 x_1 \approx \beta a_1^2\omega_0^2 \sin^2\omega_0 t, \qquad \ddot{x}_1 + \omega_0^2 x_1 \approx \frac{a_1^2\omega_0^2}{2}(1 - \cos 2\omega_0 t) \tag{76}$$

This is solved by,

$$x_1 \approx \frac{a_1^2}{2}\left(1 + \frac{\cos 2\omega_0 t}{3}\right). \tag{77}$$

At $t = 0$, $x(0) = a_0 = a_1 + \beta 2a_1^2/3 \approx a_1 + 2\beta a_0^2/3$, such that $a_1 \approx a_0(1 - 2\beta a_0/3)$. After one half period, the amplitude is,

$$\begin{aligned} |x| = |x_0(\pi/\omega_0) + \beta x_1(\pi/\omega_0)| &= \left|-a_1 + \beta\frac{a_1^2}{2}\frac{4}{3}\right| = a_1\left(1 - \frac{2\beta a_1}{3}\right) \\ &\approx a_0\left(1 - \frac{2\beta a_0}{3}\right)^2 \approx a_0\left(1 - \frac{4\beta a_0}{3}\right). \end{aligned} \tag{78}$$

(b) According to the method of averages for a damped oscillator, $\ddot{x} + \omega^2 x = \epsilon f(x, \dot{x})$,

$$x \approx a(t)\cos\phi(t), \quad \dot{a} = -\frac{\epsilon}{2\pi\omega_0}\int_0^{2\pi} f\sin\phi\,d\phi, \quad \dot{\phi} = \omega_0 - \frac{\epsilon}{2\pi\omega_0}\int_0^{2\pi} f\cos\phi\,d\phi, \tag{79}$$

In the present example, $\epsilon = -\beta$ and $f = \dot{x}\,|\dot{x}| \approx -\left\langle a^2\dot{\phi}^2\right\rangle \sin\phi\,|\sin\phi|$, where the average $\left\langle a^2\dot{\phi}^2\right\rangle$ is approximately the value of $a^2\dot{\phi}^2$ at the beginning of the period of oscillation of interest. Hence,

$$\dot{\phi} = \omega_0 - \frac{-\beta a^2\dot{\phi}^2}{2\pi\omega_0}\int_0^{2\pi}\sin\phi\,|\sin\phi|\cos\phi\,d\phi = \omega_0, \qquad \phi = \omega_0 t, \tag{80}$$

$$\dot{a} = -\frac{-\beta a^2\omega_0^2}{2\pi\omega_0}\int_0^{2\pi}\sin^2\phi\,|\sin\phi|\;d\phi = \frac{\beta a^2\omega_0}{\pi}\int_0^{\pi}\sin^3\phi\,d\phi = -\frac{4\beta a^2\omega_0}{3\pi}\,. \tag{81}$$

We have an approximate differential equation for $a$,

$$\frac{da}{a^2} = -\frac{4\beta\omega_0}{3\pi}\,dt, \qquad \frac{1}{a_0} - \frac{1}{a} = -\frac{4\beta\omega_0 t}{3\pi}\,. \tag{82}$$

The motion is approximately,

$$x \approx a\cos\omega_0 t, \qquad \frac{1}{a} = \frac{1}{a_0} + \frac{4\beta\omega_0 t}{3\pi}\,. \tag{83}$$

After one half period, $t \approx \pi/\omega_0$, and,

$$\frac{1}{a} = \frac{1}{a_0} + \frac{4\beta}{3}\,, \qquad a = \frac{a_0}{1 + 4\beta a_0/3} \approx a_0\left(1 - \frac{4\beta a_0}{3}\right), \tag{84}$$

as found in part (a).

5. *Adapted from prob. 13.15, p. 425 of* W. Chester, *Mechanics* (Allen & Unwin, 1979), http://kirkmcd.princeton.edu/examples/mechanics/chester_mechanics_79.pdf

We consider a central force,

$$F = -\frac{\alpha}{r^2} - \frac{\beta}{r^4}\,. \tag{85}$$

The "orbit equation" for a (small) mass $m$ subject to this force is,[5] with $u = 1/r$,

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2u^2}F(1/u) = \frac{\alpha m}{L^2} + \frac{\beta m u^2}{L^2}\,, \tag{86}$$

where $L$ is the angular momentum of mass $m$ about the force center.

If $\beta = 0$, the solution is the standard Newtonian ellipse, $u = (\alpha m/L^2)(1 + \epsilon\cos\theta)$.

For $\beta \neq 0$ we consider a solution of the form,

$$u \approx k[1 + \epsilon(\theta)\cos\phi(\theta)], \tag{87}$$

where $\phi$ differs only slightly from $\theta$, and $d\epsilon/d\theta$ is small. Then (somewhat delicately),

$$\frac{du}{d\theta} \approx k\frac{d\epsilon}{d\theta}\cos\phi - k\epsilon\frac{d\phi}{d\theta}\sin\phi \approx k\frac{d\epsilon}{d\theta}\cos\phi - k\epsilon\sin\phi, \tag{88}$$

$$\frac{d^2u}{d\theta^2} \approx k\frac{d^2\epsilon}{d\theta^2}\cos\phi - k\frac{d\epsilon}{d\theta}\frac{d\phi}{d\theta}\sin\phi - k\frac{d\epsilon}{d\theta}\sin\phi - k\epsilon\frac{d\phi}{d\theta}\cos\phi$$

$$\approx -2k\frac{d\epsilon}{d\theta}\sin\phi - k\epsilon\frac{d\phi}{d\theta}\cos\phi, \tag{89}$$

supposing that $d^2\epsilon/d\theta^2$ is of second order of smallness. Together with eqs. (86)-(87) we have,

$$-2k\frac{d\epsilon}{d\theta}\sin\phi - k\epsilon\frac{d\phi}{d\theta}\cos\phi + k(1+\epsilon\cos\phi) \approx \frac{\alpha m}{L^2} + \frac{\beta m u^2}{L^2} \approx \frac{\alpha m}{L^2} + \frac{k^2\beta m}{L^2}(1 + 2\epsilon\cos\phi), \tag{90}$$

$$k \approx \frac{\alpha m}{L^2} + \frac{k^2\beta m}{L^2}\,, \qquad k \approx \frac{1 \pm \sqrt{1 - 4\alpha\beta m^2/L^4}}{2\beta m/L^2} \approx \frac{1 \pm (1 - 2\alpha\beta m^2/L^4)}{2\beta m/L^2} \approx \frac{\alpha m}{L^2}\,, \tag{91}$$

$$\frac{d\epsilon}{d\theta}\sin\phi + \frac{\epsilon}{2}\left(\frac{d\phi}{d\theta} - 1\right)\cos\phi \approx -\frac{\beta\epsilon k m\cos\phi}{L^2} \approx -\frac{\alpha\beta\epsilon m^2\cos\phi}{L^4}\,, \tag{92}$$

$$\frac{d\epsilon}{d\theta} \approx 0, \qquad \epsilon \approx \text{constant}, \tag{93}$$

$$\frac{d\phi}{d\theta} \approx 1 - \frac{2\alpha\beta m^2}{L^4}\,, \qquad \phi \approx \left(1 - \frac{2\alpha\beta m^2}{L^4}\right)\theta, \tag{94}$$

$$u \approx \frac{\alpha m}{L^2}\left[1 + \epsilon\cos\left(1 - \frac{2\alpha\beta m^2}{L^4}\right)\theta\right]. \tag{95}$$

This is a slowly precessing ellipse, with angular velocity of precession,

$$\omega = \frac{2\alpha\beta m^2}{L^4}\Omega, \qquad \text{where} \qquad \Omega = \frac{2\pi}{T}\,, \tag{96}$$

[5] See http://kirkmcd.princeton.edu/examples/Ph205/ph205l10.pdf.

is the average angular velocity of the orbital motion (of period $T$).

*We were able to find solutions for $\epsilon$ and $\phi$ directly from eq. (92), but if we had written this as,*

$$\frac{d\epsilon}{d\theta}\sin\phi + \frac{\epsilon}{2}\left(\frac{d\phi}{d\theta} - 1\right)\cos\phi \approx f, \tag{97}$$

$$f = -\frac{\alpha\beta\epsilon m^2\cos\phi}{L^4}, \tag{98}$$

*we could develop a method of averages, as recommended in the book by Chester.*

*For this, we multiply eq. (97) by $\sin\phi$ or $\cos\phi$ and integrate over one period in $\phi$, supposing that the prefactors of $\sin\phi$ and $\cos\phi$ on the lefthand side are essentially constant, leading to,*

$$\frac{d\epsilon}{d\theta}\int_0^{2\pi}\sin^2\phi\,d\phi + \frac{\epsilon}{2}\left(\frac{d\phi}{d\theta} - 1\right)\int_0^{2\pi}\cos\phi\sin\phi\,d\phi \approx \pi\frac{d\epsilon}{d\theta} \approx \int_0^{2\pi} f\sin\phi\,d\theta, \tag{99}$$

$$\frac{d\epsilon}{d\theta}\int_0^{2\pi}\sin\phi\cos\phi\,d\phi + \frac{\epsilon}{2}\left(\frac{d\phi}{d\theta} - 1\right)\int_0^{2\pi}\cos^2\phi\,d\phi \approx \frac{\pi\epsilon}{2}\left(\frac{d\phi}{d\theta} - 1\right) \approx \int_0^{2\pi} f\cos\phi\,d\phi, \tag{100}$$

$$\frac{d\epsilon}{d\theta} \approx \frac{1}{\pi}\int_0^{2\pi} f\sin\phi\,d\phi, \qquad \frac{d\phi}{d\theta} \approx 1 - \frac{2}{\epsilon\pi}\int_0^{2\pi} f\cos\phi\,d\phi. \tag{101}$$

*With $f$ as in eq. (98), we find from eq. (101),*

$$\frac{d\epsilon}{d\theta} \approx 0, \qquad \epsilon \approx \text{constant}, \tag{102}$$

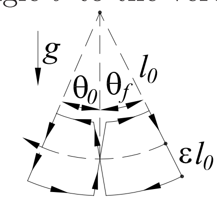
$$\frac{d\phi}{d\theta} \approx 1 - \frac{2}{\epsilon\pi}\int_0^{2\pi} f\cos\phi\,d\phi \approx 1 - \frac{2\alpha\beta m^2}{L^4}, \qquad \phi \approx \left(1 - \frac{2\alpha\beta m^2}{L^4}\right)\theta. \tag{103}$$

$$u \approx \frac{\alpha m}{L^2}\left[1 + \epsilon\cos\left(1 - \frac{2\alpha\beta m^2}{L^4}\right)\theta\right], \tag{104}$$

*as before.*

6. **The Swing**

(a) As a simple model of the pumping action, we consider a simple pendulum whose center of mass is (quickly) lowered by $2\epsilon l_0$ when the amplitude is maximum, and raised by $2\epsilon l_0$ when the angle $\theta$ to the vertical is 0.




If the force required to raise of lower the center of mass always points to the pivot of the swing, angular momentum is conserved about the pivot.

We consider a half period of the motion, beginning with the center of mass at rest at angle $\theta_0$ to the vertical an in the raised position. Then, the center of mass is quickly lowered, to be as rest while the angle is still $\theta_0$. At this time the mass is at distance $l_0(1+\epsilon)$ from the pivot, and its potential energy is,

$$V = -gml_0(1+\epsilon)\cos\theta_0. \tag{105}$$

When the center of mass reaches $\theta = 0$ its potential energy is $-gml_0(1+\epsilon)$ and, by conservation of mechanical energy during this phase of the motion, its velocity $v$ is related by

$$\frac{mv^2}{2} = \Delta V = gml_0(1+\epsilon)(1-\cos\theta_0) \approx gml_0(1+\epsilon)\frac{\theta_0^2}{2}, \tag{106}$$

for small $\theta_0$.

Next, the center of mass is quickly raised by $2\epsilon l_0$ (while the angle remains $\theta = 0$. Angular momentum is conserved during the raising, such that at the end of this action, the velocity is,

$$v' = v\left(\frac{1+\epsilon}{1-\epsilon}\right). \tag{107}$$

And, at the end of the raising, the mechanical energy is,

$$\begin{aligned} E' = \frac{mv'^2}{2} - gml_0(1-\epsilon) &= \frac{mv^2}{2}\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 - gml_0(1-\epsilon) \\ &\approx gml_0(1+\epsilon)\frac{\theta_0^2}{2}\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 - gml_0(1-\epsilon). \end{aligned} \tag{108}$$

This is larger than before the raising, so (of course) work had to be done to lift the center of mass.

Then, the mass swings upwards, coming to rest at angle $\theta_f$, where its mechanical energy is only $-gml_0(1-\epsilon)\cos\theta_f$. Mechanical energy is conserved during the upward swing, such that,

$$-\,gml_0(1-\epsilon)\cos\theta_f = E' \approx gml_0(1+\epsilon)\frac{\theta_0^2}{2}\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 - gml_0(1-\epsilon). \quad (109)$$

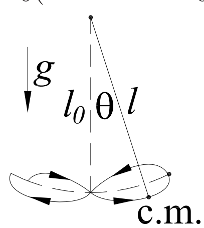
$$gml_0(1-\epsilon)(1-\cos\theta_f) \approx gml_0(1-\epsilon)\frac{\theta_f^2}{2} \approx gml_0(1+\epsilon)\frac{\theta_0^2}{2}\left(\frac{1+\epsilon}{1-\epsilon}\right)^2 \quad (110)$$

$$\theta_f^2 \approx \theta_0^2\left(\frac{1+\epsilon}{1-\epsilon}\right)^3, \qquad \theta_f \approx \theta_0\left(\frac{1+\epsilon}{1-\epsilon}\right)^{3/2} \approx \theta_0\frac{1+3\epsilon/2}{1-3\epsilon/2} \approx \theta_0\left(1+\frac{3\epsilon}{2}\right). \quad (111)$$

The increase from $\theta_0$ to $\theta_f$ occurs during one half period, $\Delta t = \pi/\omega_0$ where $\omega_0 = \sqrt{g/l_0}$. Hence, the amplitude of the swinging motion is related by,

$$\frac{\Delta\theta}{\Delta t} = \frac{\theta_f-\theta_0}{\pi/\omega_0} \approx \frac{3\epsilon\omega_0}{\pi}\theta_0, \qquad \frac{d\theta}{dt} \approx \frac{3\epsilon\omega_0}{\pi}\theta, \qquad \theta_{\max} \approx \theta_0\, e^{3\epsilon\omega_0 t/\pi}. \quad (112)$$

(b) In another model of pumping, we suppose that the distance from the pivot to the center of mass varies as $l = l_0(1+\epsilon\sin 2\omega_0 t)$ where $\omega_0 = \sqrt{g/l}$.




We can use Lagrange's method to find the equation of motion for coordinate $\theta$ (on which $l$ does not depend),

$$\mathcal{L} = T - V = \frac{m}{2}(\dot{l}^2 + l^2\dot{\theta}^2) + mgl\cos\theta, \quad (113)$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{d}{dt}ml^2\dot{\theta} = ml^2\ddot{\theta} + 2ml\dot{l}\dot{\theta} = \frac{\partial\mathcal{L}}{\partial\theta} = -mgl\sin\theta \quad (114)$$

$$\ddot{\theta} = -\frac{g\sin\theta}{l_0(1+\epsilon\sin 2\omega_0 t)} - \frac{4\epsilon\omega_0 l_0\dot{\theta}\cos 2\omega_0 t}{l_0(1+\epsilon\sin 2\omega_0 t)} \quad (115)$$

$$\ddot{\theta} + \omega_0^2\theta \approx \epsilon\omega_0^2\theta\sin 2\omega_0 t - 4\epsilon\omega_0\dot{\theta}\cos 2\omega_0 t \equiv \epsilon f. \quad (116)$$

For small $\epsilon$ we suppose that solution will have the form $\theta(t) \approx a(t)\cos\phi(t)$ where $\phi \approx \omega_0 t$, such that $\dot{\theta} \approx -\omega_0 a\sin\phi$. Using the method of averages with,

$$f \approx \omega_0^2\theta\sin 2\omega_0 t - 4\omega_0\dot{\theta}\cos 2\omega_0 t \approx \omega_0^2 a\cos\phi\sin 2\phi + 4\omega_0 a\sin\phi\cos 2\phi$$
$$= \frac{\omega_0^2 a}{2}\sin\phi + \frac{\omega_0^2 a}{2}\sin 3\phi - 2\omega_0 a\sin\phi + 2\omega_0^2 a\sin 3\phi, \quad (117)$$

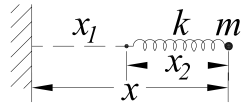
we have that,

$$\dot{\phi} \approx \omega_0 - \frac{\epsilon}{2\pi\omega_0}\int_0^{2\pi} f\cos\phi\, d\phi = \omega_0, \qquad \phi \approx \omega_0 t, \quad (118)$$

$$\dot{a} \approx -\frac{\epsilon}{2\pi\omega_0}\int_0^{2\pi} f\sin\phi\, d\phi = \frac{3\epsilon\omega_0 a}{4\pi}\int_0^{2\pi}\sin^2\phi\, d\phi = \frac{3\epsilon\omega_0 a}{4}, \quad a \approx a_0\, e^{3\epsilon\omega_0 t/4}. \quad (119)$$

That is, $\theta_{\max} \approx \theta_0\, e^{3\epsilon\omega_0 t/4}$ for this model.

7. (a) Mass $m$ is connected to one end of a spring of constant $k$ and rest length $l_0$. The other end of the spring is force to move according to $x_1 = a\cos\omega t$ in the inertial lab frame.




We go to the accelerated frame with origin at $x_1$, where $F = ma$ becomes,

$$m\ddot{x}_2 = F = -k(x_2 - l_0) - m\ddot{x}_1 = -k(x_2 - l_0) + ma\omega^2\cos\omega t, \tag{120}$$

$$\ddot{x}_2 + \omega_0^2 x_2 = \omega_0^2 l_0 + a\omega^2\cos\omega t, \qquad \omega_0^2 = \frac{k}{m}, \tag{121}$$

where we have included the "fictitious" (coordinate) force $-m\ddot{x}_1$.

The steady-state solution with zero velocity at time $t = 0$ is,

$$x_2 = l_0 + \frac{a\omega^2\cos\omega t}{\omega_0^2 - \omega^2}, \tag{122}$$

$$x = x_1 + x_2 = l_0 + a\cos\omega t\left(1 + \frac{\omega^2}{\omega_0^2 - \omega^2}\right) = l_0 + \frac{\omega_0^2}{\omega_0^2 - \omega^2}a\cos\omega t. \tag{123}$$

(b) A particle of mass $m$ has velocity $\mathbf{v}$ on a smooth horizontal table.

In the rotating frame of the Earth, whose angular velocity about its axis is $\mathbf{\Omega}$, the force is,

$$\mathbf{F} = m\mathbf{g} + \mathbf{N} - m\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) - 2m\mathbf{\Omega}\times\mathbf{v}, \tag{124}$$

where $\mathbf{N}$ is the normal force, $\mathbf{r}$ is the position of the mass with respect to the center of the Earth, and we include the centrifugal and Coriolis forces but neglect the coordinate force associated with the acceleration of the Earth with respect to the Sun, *etc.*

The first and third terms on the righthand side of eq. (124) define the "horizontal" (see also prob. 8(a) of this Set), in which plane the velocity $\mathbf{v}$ lies. That is, the only force with a component in the horizontal plane is the fourth term of eq. (124), which force is always perpendicular to the velocity $\mathbf{v}$, and also is constant in magnitude.

Hence, mass $m$ executes uniform circular motion with radius $R$ and angular velocity $\omega$ related by,

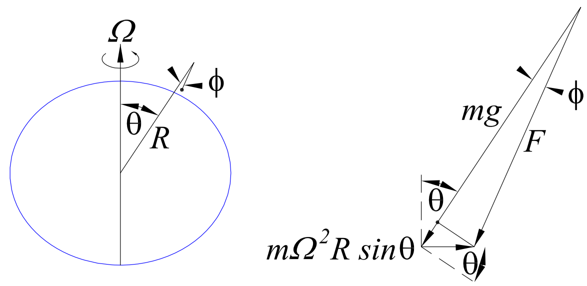
$$F_{\text{horiz}} = 2m\Omega v\cos\theta = \frac{mv^2}{R}, \qquad R = \frac{v}{2\Omega\cos\theta}, \qquad \omega = \frac{v}{R} = 2\Omega\cos\theta, \tag{125}$$

where $\theta$ is angle between the normal to the planet and the axis of the Earth (which is approximately the polar angle (colatitude) of the mass with respect to the axis).

The radius $R$ in kilometers is of order $10\ v$ for velocity in m/s. This behavior is an ingredient in the winds of hurricanes/cyclones.

8. (a) Assuming that the force of gravity at the Earth's surface points to the center of the Earth, the force on a plumb bob of mass $m$, at position $\mathbf{R} = (R, \theta, 0)$ in spherical coordinates with $\mathbf{z}$ along the Earth's axis, is,

$$\mathbf{F} = m\mathbf{g} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{R}) = -mg\,\hat{\mathbf{r}} + m\Omega^2 R \sin\theta\,\hat{\mathbf{x}}. \tag{126}$$




The components of $\mathbf{F}$ parallel and perpendicular to $\mathbf{g}$ are, referring to the right figure above,

$$F_{\parallel} = mg - m\Omega^2 R \sin\theta, \qquad F_{\perp} = m\Omega^2 R \sin\theta\cos\theta \tag{127}$$

Then,

$$\tan\phi = \frac{F_{\perp}}{F_{\parallel}} = \frac{\sin\theta\cos\theta}{g/\Omega^2 R - \sin^2\theta}\,, \tag{128}$$

*We note that* $g/\Omega^2 R \approx 290$.

(b) **Centrifugal Bulge.**

We can also write the force (126) as,

$$\mathbf{F} = -\frac{GMm}{R^2}\,\hat{\mathbf{r}} + m\Omega^2 x\,\hat{\mathbf{x}}, \tag{129}$$

where $M$ is the mass of the Earth. This force can be related to an effective potential $V$ as $\mathbf{F} = -\boldsymbol{\nabla} V$ where,

$$V = -\frac{GMm}{R} - \frac{m\Omega^2 x^2}{2} = -\frac{GMm}{R} - \frac{m\Omega^2 R^2 \sin^2\theta}{2}\,. \tag{130}$$

The liquid surface of the Earth will be an equipotential of $V$. Thus, with $R_P$ being the radius of the Earth along its axis, the equation of the surface is,

$$-\frac{GMm}{R_P} = -\frac{GMm}{R} - \frac{m\Omega^2 R^2 \sin^2\theta}{2}\,, \qquad R^3\sin^2\theta = \frac{2GM}{\Omega^2}\left(\frac{R}{R_P} - 1\right), \tag{131}$$

Writing the equatorial radius as $R_E/R_P = 1 + \epsilon$, we have (for $\theta = \pi/2$),

$$\frac{2GM\epsilon}{\Omega^2} = R_E^3 = R_P^3(1+\epsilon)^3 \approx R_P^3(1+3\epsilon), \tag{132}$$

$$\frac{2GM\epsilon}{\Omega^2 R_P^3} = \frac{2g\epsilon}{\Omega^2 R_P} \approx 1 + 3\epsilon, \qquad \epsilon \approx \frac{1}{\frac{2g}{\Omega^2 R_P} - 3} \approx \frac{1}{\frac{2g}{\Omega^2 R_P}} \approx \frac{1}{2\cdot 290} = \frac{1}{580}\,, \tag{133}$$

recalling that $g/\Omega^2 R \approx 290$.

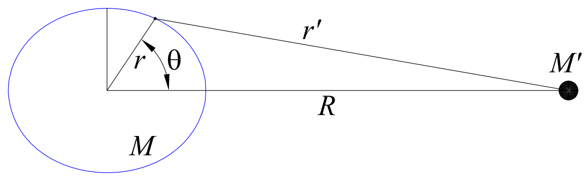
*The above analysis made the oversimplified assumption that the gravitational potential of the bulging Earth is that of a uniform sphere. A better approximation is to consider the potential due to an oblate spheroid of uniform density. This improves the estimate of* $\epsilon$ *to* $5\Omega^2 R_P/4g \approx 1/230$, *which is about 20% less than the observed value,* $\epsilon \approx 1/297$. *See,*

http://farside.ph.utexas.edu/teaching/336k/Newton/node109.html

9. **Tidal Bulge.**

In this problem we only consider the tides on the Earth due to the pull of the Moon.

In an inertial frame, the center of mass of the Earth experiences the acceleration $GM'\,\hat{\mathbf{R}}/R^2$, where $M$ is the mass of the Earth, $M'$ is the mass of the Moon, and $R$ is the Earth-Moon distance.




In a nonrotating frame with the center of the Earth at rest, the force on a particle of mass $m$ on the surface of the Earth as position $\mathbf{r}$ is,

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} - \frac{GM'm}{r'^2}\hat{\mathbf{r}}' - \frac{GM'm}{R^2}\hat{\mathbf{R}}\,, \tag{134}$$

where the third term is the "fictitious" force associated (in the accelerated frame) with the acceleration of the center of the Earth in an inertial frame.

The potential of this force is

$$\frac{V}{m} = -\frac{GM}{r} - \frac{GM'}{r'} + \frac{GM'}{R^2}z = -\frac{GM}{r} - \frac{GM'}{r'} + \frac{GM'}{R^2}r\cos\theta, \tag{135}$$

Where $z = r\cos\theta$ is the coordinate of mass $m$ along the $z$-axis ($=\mathbf{R}$-axis). Now,

$$\begin{aligned}\frac{1}{r'} = \frac{1}{(R^2 - 2Rr\cos\theta + r^2)^{1/2}} &= \frac{1}{R}\left(1 - 2\frac{r}{R}\cos\theta + \frac{r^2}{R^2}\right)^{-1/2} \\ &\approx \frac{1}{R}\left(1 + \frac{r}{R}\cos\theta - \frac{r^2}{2R^2} + \frac{1}{2}\frac{3}{2}\left(2\frac{r}{R}\cos\theta\right)^2\right),\end{aligned} \tag{136}$$

top order $r^2/R^2$, so,

$$\begin{aligned}\frac{V}{m} &\approx -\frac{GM}{r} - \frac{GM'}{R} - \frac{GM'r\cos\theta}{R^2} + \frac{GM'r^2}{2R^3} - \frac{3GM'r^2\cos^2\theta}{R^3} + \frac{GM'}{R^2}r\cos\theta \\ &= -\frac{GM}{r} - \frac{GM'}{R} + \frac{3}{2}\frac{GM'r^2}{R^3}\left(\frac{1}{3} - \cos^2\theta\right).\end{aligned} \tag{137}$$

We suppose that the (liquid) surface of the Earth is an equipotential of eq. (137),

$$k = -\frac{GM}{r} + \frac{3}{2}\frac{GM'r^2}{R^3}\left(\frac{1}{3} - \cos^2\theta\right) = -\frac{GM}{r_0}\,, \tag{138}$$

where $r_0$ is the radius for $\cos^2\theta = 1/3$, which is a kind of mean radius. Then,

$$r = r_0 - \frac{3}{2}\frac{M'r_0r^3}{MR^3}\left(\frac{1}{3} - \cos^2\theta\right) \approx r_0\left[1 - \frac{3}{2}\frac{M'r_0^3}{MR^3}\left(\frac{1}{3} - \cos^2\theta\right)\right]. \tag{139}$$

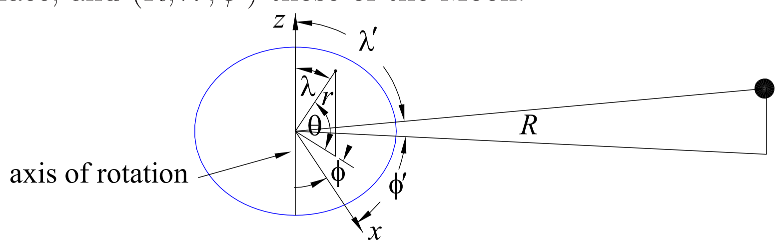
On the near side, $\theta = 0$,

$$\Delta r = r - r_0 \approx r_0 \frac{M'}{M} \frac{r_0^3}{R^3} \approx 6 \times 10^6 \frac{1}{80} \left(\frac{1}{60}\right)^3 = 0.33 \text{ m} \approx 1 \text{ foot}. \tag{140}$$

*For the Sun alone, the effect would be about 3 mm.*

Because the Earth turns around its axis, and the Moon orbits the Earth, the position of the bulge changes with time.

We now take the $z$-axis of the (accelerated, but not rotating) coordinate system to be that of the Earth, and let $(r, \lambda, \phi)$ be the spherical coordinates of a point on the Earth's surface, and $(R, \lambda', \phi')$ those of the Moon.




Then,

$$\cos\theta = \hat{\mathbf{r}} \cdot \hat{\mathbf{R}} = \cos\lambda \cos\lambda' + \sin\lambda \sin\lambda' \cos(\phi - \phi'), \tag{141}$$

$$\begin{aligned}
\cos^2\theta &= \cos^2\lambda \cos^2\lambda' + \sin^2\lambda \sin^2\lambda' \cos^2(\phi - \phi') + 2\cos\lambda\cos\lambda'\sin\lambda\sin\lambda'\cos(\phi - \phi') \\
&= \cos^2\lambda\cos^2\lambda' + \sin^2\lambda\sin^2\lambda' \frac{1 + \cos 2(\phi - \phi')}{2} + \frac{\sin 2\lambda \sin 2\lambda'}{2}\cos(\phi - \phi'), \\
&= \cos^2\lambda\cos^2\lambda' + \frac{1 - \cos^2\lambda - \cos^2\lambda' + \cos^2\lambda\cos^2\lambda'}{2} \\
&\quad + \frac{\sin^2\lambda\sin^2\lambda'}{2}\cos 2(\phi - \phi') + \frac{\sin 2\lambda\sin 2\lambda'}{2}\cos(\phi - \phi') \\
&= \frac{1}{3} + \frac{3}{2}\left(\cos^2\lambda - \frac{1}{3}\right)\left(\cos^2\lambda' - \frac{1}{3}\right) \\
&\quad + \frac{\sin^2\lambda\sin^2\lambda'}{2}\cos 2(\phi - \phi') + \frac{\sin 2\lambda\sin 2\lambda'}{2}\cos(\phi - \phi').
\end{aligned} \tag{142}$$

In eq. (139). we define $\epsilon = 3M'r_0^2/2MR^3$, and rewrite it as,

$$\begin{aligned}
\frac{r}{r_0} &= 1 + \epsilon\left(\cos^2\theta - \frac{1}{3}\right) = 1 + \epsilon\left[\frac{3}{2}\left(\cos^2\lambda - \frac{1}{3}\right)\left(\cos^2\lambda' - \frac{1}{3}\right)\right. \\
&\quad \left. + \frac{1}{2}\sin^2\lambda\sin^2\lambda'\cos 2(\phi - \phi') + \frac{1}{2}\sin 2\lambda\sin 2\lambda'\cos(\phi - \phi')\right].
\end{aligned} \tag{143}$$

Thus, while to total effect is given by eq. (139), an observer at a fixed location $(\lambda, \phi)$ on the Earth's surface can do a "Fourier analysis" of the bulge. In the accelerated coordinates system where eq. (143) holds, $\phi = \Omega_E t$ where $\Omega_E$ is the angular velocity of the rotation of the Earth about its axis, $\phi' = \omega' t$ where $\omega'$ is the angular velocity of the Moon in its orbit about the Earth, and $\lambda' = \lambda'_0 \cos\omega' t$ as the polar angle of the Moon with respect to the Earth's axis varies during a month. Then, $\sin\lambda' \propto \cos\omega' t$

while $\cos^2\lambda'$, $\sin^2\lambda'$ and $\sin 2\lambda$ have terms proportional to $\cos 2\omega' t$ (as well as constant terms).

Hence, the first term in eq. (143) has a component whose period is 1/2 month. When this term is maximal, one speaks of a "spring" tide (which is NOT related to the Spring season).

The second term in eq. (143) has components that vary as $\cos(2\Omega_E t - 2\omega' t)$ and as $\cos 2\omega' t \cos(2\Omega_E t - 2\omega' t)$, which latter varies as $\cos 2\Omega_E t$ and $\cos(2\Omega_E t - 4\omega' t)$. Since $\omega' \ll \Omega_E$, all of these have periods of roughly 12 hours, and correspond to the nominal twice-daily tides.

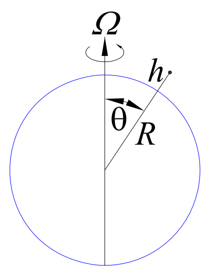
The third term in eq. (143) has components that vary as $\cos(\Omega_E t - \omega' t)$ and as $\cos 2\omega' t \cos(\Omega_E t - \omega' t)$, which latter varies as $\cos(\Omega_E t + \omega' t)$ and $\cos(\Omega_E t - 3\omega' t)$. All of these have periods of roughly 1 day, which implies that the two daily tides (of the second term) have somewhat different amplitudes.

10. In this problem, we assume the Earth is spherical, and ignore gravity except that due to the Earth.

(a) In the rotating frame of the (spherical) Earth, the force on a falling particle of mass $m$ at position $\mathbf{r}$ with velocity $\mathbf{v}$ is,

$$\mathbf{F} = -mg\,\hat{\mathbf{r}} - 2m\boldsymbol{\Omega}\times\mathbf{v} - m\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}), \tag{144}$$

where $\boldsymbol{\Omega}$ is the angular velocity of rotation of the Earth about its axis, and we take the acceleration $g$ due gravity to be constant during the fall (in which $r = R + h \approx R$, the radius of the Earth).




During a fall, which begins at time $t = 0$, $\mathbf{v} \approx -gt\,\hat{\mathbf{r}}$, so the horizontal force component is,

$$\mathbf{F}_\perp = -2m\boldsymbol{\Omega}\times\mathbf{v} \approx -2m\Omega v\sin\theta\,\hat{\mathbf{e}} \approx 2m\Omega gt\sin\theta\,\hat{\mathbf{e}}, \tag{145}$$

where $\hat{\mathbf{e}}$ points East in the both the Northern and Southern hemispheres.

For a fall from height $h \ll r$, the fall time is $T \approx \sqrt{2h/g}$, and the horizontal displacement during the fall is related by,

$$\ddot{x}_\perp = \frac{F_\perp}{m} \approx 2\Omega gt\sin\theta, \qquad \dot{x}_\perp \approx \Omega gt^2\sin\theta, \qquad x_\perp \approx \frac{\Omega gt^3\sin\theta}{3}. \tag{146}$$

Hence, the displacement $d$ during the fall (to the East in both the Northern and Southern hemispheres) is,

$$d \approx \frac{\Omega gT^3\sin\theta}{3} = \frac{2}{3}\sqrt{\frac{2h^3}{g}}\Omega\sin\theta. \tag{147}$$

(b) In case of a jump at $t = 0$ to maximum height $h \ll R$, the initial vertical velocity is $v_0 \approx \sqrt{2gh}$, $v \approx v_0 - gt$, and the total time of the jump is $T \approx 2\sqrt{2h/g}$. Then, eq. (146) becomes,

$$\ddot{x}_\perp \approx -2\Omega v_0\sin\theta + 2\omega gt\sin\theta, \qquad \dot{x}_\perp \approx -2\Omega v_0 t\sin\theta + \Omega gt^2\sin\theta, \tag{148}$$

$$x_\perp \approx -\Omega v_0 t^2\sin\theta + \frac{\Omega gt^3\sin\theta}{3}. \tag{149}$$

Hence, the displacement $d$ during the jump is,

$$d \approx -\Omega\sqrt{2gh}\frac{8h}{g}\sin\theta + \frac{\Omega g\sin\theta}{3}\frac{16h}{g}\sqrt{\frac{2h}{g}} = -\frac{8}{3}\sqrt{\frac{2h^3}{g}}\Omega\sin\theta. \tag{150}$$

The $-$ sign implies that the displacement is to the West (in both hemispheres).

(c) We now consider the jump of part (b) in an inertial frame in which the center of the Earth is at rest.

In this frame, the only force on the jumping mass $m$ is the central force of gravity. Hence, the angular momentum $\mathbf{L} = \mathbf{r} \times m\mathbf{v} = \mathbf{r} \times m\mathbf{v}_\perp$ of mass $m$ about the center of the Earth is conserved.

At the beginning of the jump, when the mass is at position $\mathbf{R} = (r, \theta, 0)$, the velocity in the inertial frame is $\mathbf{v} = v_0\,\hat{\mathbf{r}} + \mathbf{\Omega} \times \mathbf{R} = v_0\,\hat{\mathbf{r}} + \Omega R \sin\theta\,\hat{\boldsymbol{\phi}}$ in a spherical coordinate system with the $z$-axis along $\mathbf{\Omega}$. Hence, the conserved angular momentum is

$$\mathbf{L} = mr^2\dot{\phi}\sin\theta\,\hat{\mathbf{z}} = \Omega R^2 \sin\theta\,\hat{\mathbf{z}}, \tag{151}$$

supposing that the polar angle remains $\theta$ at all times during the jump. Hence,

$$\dot{\phi} = \frac{\Omega R^2}{r^2} = \frac{\Omega R^2}{(R + v_0 t - gt^2/2)^2} \approx \Omega_0 \left(1 - 2\frac{v_0 t}{R} + \frac{gt^2}{R}\right), \tag{152}$$

$$\phi = \Omega_0 \left(t - \frac{v_0 t^2}{R} + \frac{gt^3}{3R}\right), \tag{153}$$

$$\begin{aligned} d = R\sin\theta(\phi(T) - \Omega T) &\approx -\sqrt{2gh}\frac{2h}{g}\Omega\sin\theta + g\frac{2h}{g}\sqrt{\frac{2h}{g}}\frac{\Omega\sin\theta}{3} \\ &= -8\sqrt{\frac{2h^3}{g}}\Omega\sin\theta + \frac{16}{3}\sqrt{\frac{2h^3}{g}}\Omega\sin\theta = -\frac{8}{3}\sqrt{\frac{2h^3}{g}}\Omega\sin\theta, \end{aligned} \tag{154}$$

as found in part (b).

11. We assume that the Earth is spherical in this problem, and ignore gravity except that due to the Earth. We also neglect the centrifugal force and air resistance on the shot fired from a gun (located on the surface of the Earth at polar angle $\theta$)

    If the shot rises to $h \ll R$, its initial radial velocity is $v_{r0} = \sqrt{2gh}$. Subsequently, it radial velocity is $v_r(t) = v_{r0} - gt$, and its flight time is $T = 2\sqrt{2h/g}$.

    If the shot travels horizontal distance $D$ in the absence of the Coriolis for (and of air resistance), its horizontal velocity is $v_H = D/T = (D/4)\sqrt{2g/h}$.

    The Coriolis force on a shot of mass $m$ with velocity $\mathbf{v} = v_E\,\hat{\mathbf{E}} + v_N\,\hat{\mathbf{N}} + v_r\,\hat{\mathbf{r}}$ is,

$$\mathbf{F}_C = -2m\boldsymbol{\Omega}\times\mathbf{v} = -2m\Omega[(v_r\sin\theta - v_N\cos\theta)\,\hat{\mathbf{E}} + v_E\cos\theta\,\hat{\mathbf{N}} - v_E\sin\theta\,\hat{\mathbf{r}}, \qquad (155)$$

    noting that $\boldsymbol{\Omega} = \Omega(\sin\theta\,\hat{\mathbf{N}} + \cos\theta\,\hat{\mathbf{r}})$, where $\hat{\mathbf{E}}$ points East and $\hat{\mathbf{N}}$ points North.

    (a) If the gun fired North, the Coriolis force, $-2m\boldsymbol{\Omega}\times\mathbf{v}$ causes an initial Westward deflection.

    We ignore the effect of the Coriolis force on the resulting Westward velocity, as this effect is second order in the small quantity $\Omega$. In this approximation, the Northward velocity remains at its initial value, $v_N = D/T$, where the flight time is still $T = 2\sqrt{2h/g}$ (as we ignore the second-order radial component of the Coriolis force).

    Then, the Westward acceleration and deflection are,

$$a_W = -\frac{F_{C,E}}{m} = 2\Omega[v_r(t)\sin\theta - v_N\cos\theta] = 2\Omega\left[\left(\sqrt{2gh} - gt\right)\sin\theta - v_N\cos\theta\right], \qquad (156)$$

$$\begin{aligned} d_W(T) &= 2\Omega\left[\left(\sqrt{2gh}\frac{T^2}{2} - g\frac{T^3}{6}\right)\sin\theta - v_N\frac{T^2}{2}\cos\theta\right] \\ &= 2\Omega\left[\left(\sqrt{2gh}\frac{2h}{g} - g\frac{2}{3}\frac{2h}{g}\sqrt{\frac{2h}{g}}\right)\sin\theta - D\sqrt{\frac{2h}{g}}\cos\theta\right] \\ &= 2\Omega\sqrt{\frac{2h}{g}}\left(\frac{4h}{3}\sin\theta - D\cos\theta\right). \end{aligned} \qquad (157)$$

    *If the gun had pointed South, the sign of $D$ in eq. (157) would be $+$.*

    (b) If the gun fired East, the initial Coriolis force has an inward radial component and (in the Northern hemisphere) a Northward component. As in part (a), we ignore the (second-order) Coriolis force on the change in the velocity from its initial value. In this approximation the Eastward velocity component, $v_E$, is constant, with value $(D/4)\sqrt{2g/h}$.

    The radial force on the shot is,

$$F_r = -mg + 2m\Omega v_E\sin\theta \equiv -mg_{\rm eff}, \quad g_{\rm eff} = g - 2\Omega v_E\sin\theta = g\left(1 - \frac{\Omega D\sin\theta}{\sqrt{2gh}}\right) \qquad (158)$$

    The decreased effective gravity increases the time of flight of the shot to $T = 2v_{r0}/g_{\rm eff} = 2\sqrt{2gh}/g_{\rm eff}$.

The Northward acceleration and deflection are,

$$a_N = \frac{F_{C,N}}{m} = 2\Omega v_E \cos\theta, \tag{159}$$

$$d_N = \Omega v_E T^2 \cos\theta = \Omega \frac{D}{4}\sqrt{\frac{2g}{h}}\frac{8gh}{g_{\rm eff}^2}\cos\theta = 2D\sqrt{\frac{2h}{g}}\frac{\Omega\cos\theta}{(1 - \Omega D\sin\theta/\sqrt{2gh})^2}$$
$$\approx 2D\sqrt{\frac{2h}{g}}\Omega\cos\theta, \tag{160}$$

Meanwhile, the distance traveled East by the shot is not $D$, but approximately,

$$D'_E = V_E T = \frac{D}{4}\sqrt{\frac{2g}{h}}\frac{2\sqrt{2gh}}{g_{\rm eff}} = \frac{D}{1 - \Omega D\sin\theta/\sqrt{2gh}} \approx D\left(1 + \frac{\Omega D \sin\theta}{\sqrt{2gh}}\right). \tag{161}$$