Princeton University

Ph205

Mechanics

Problem Set 9

Kirk T. McDonald

(1988)

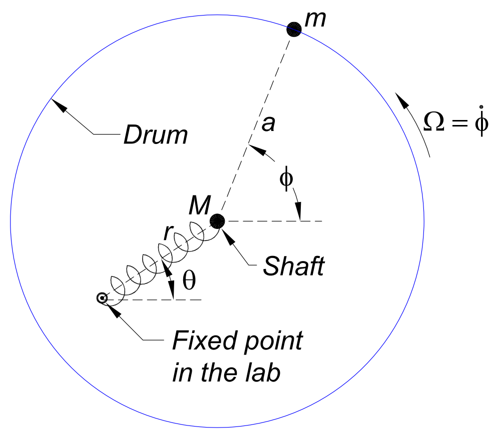
kirkmcd@princeton.edu

http://kirkmcd.princeton.edu/examples/

1. **Physics in the Laundromat.**

If a washing machine (with a vertical shaft) is unevenly loaded, violent motions can occur at the start of a spin cycle. Nonetheless, as the spin angular velocity increases a stable motion (usually) results.

The mechanism of the washing machine is roughly as follows. The center of the wash tube/drum, of mass M and radius a, is tied to a fixed point in the lab by spring of constant k and zero rest length. The line from the fixed point to the center of the drum has length r, and makes angle θ to a fixed direction in the lab. There is velocity-dependent damping $-\gamma\,\dot{\mathbf{r}}$ on the distance r.



A motor forces the drum to rotate about its center with constant angular velocity $\dot{\phi} = \Omega$.

The uneven load can be taken as a point mass m on the surface of the drum.

(a) Deduce the equations of motion for coordinates r and θ.

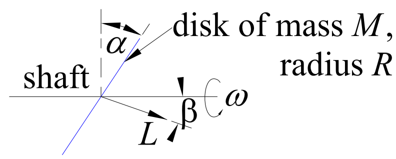
(b) Consider steady motion, to show that for large spin Ω, the center of mass of unevenly loaded drum approaches the fixed point in the lab frame (the origin of coordinates $(r, theta)$. That is, the motion of the washing machine is self-centering.

(c) Consider small perturbations about the steady motion, to show that they are stable. It suffices to consider underdamped motion.

This stability can be attributed to the effect of the Coriolis force.

2. **Wobble-Plate Engine** (also called a swash-plate engine).

A disk of mass M and radius R is mounted on a shaft through its center, making angle α as shown in the figure below. The disk rotates with constant angular velocity ω about the shaft.

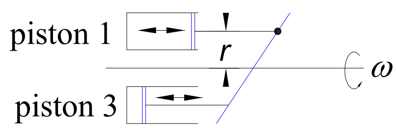


(a) Identify the principal axes, and the principal moments of inertia, to obtain the angular momentum $\mathbf{L}$ of the disc in the principal-axis frame.

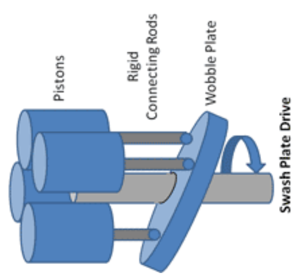
Show that $\mathbf{L}$ makes angle β to the shaft where,

$$\tan\beta = \frac{\tan\alpha}{2 + \tan^2\alpha}\,. \tag{1}$$

(b) Four pistons have rods that are parallel to the main shaft, and press against the rotating, tilted disk. Hence, the pistons are driven back and forth by the rotating disk (or *vice versa*) to make an engine.



The 4 pistons are mounted 90° apart in azimuth, two in the plane of the page as shown, and two perpendicular to the page.



The pistons each have mass m, with their axes at distance r from the main shaft. The piston cylinders are a rest with respect to the shaft.

Show that the pistons execute simple harmonic motion.

Calculate the angular momentum of the four pistons about the center of the tilted disk. Show that the total angular momentum of the disk plus pistons will be "balanced", *i.e.*, parallel to the main shaft, if,

$$m = \frac{MR^2\cos^2\alpha}{8r^2}\,. \tag{2}$$

A video of a swash-plate pump: https://www.youtube.com/watch?v=MdesnTOpkCM
A variant in which the plate is fixed and the piston cylinder block rotates: https://www.youtube.com/watch?v=_VSOgJn-3wo

3. (a) A coin initially in a horizontal plane is tossed vertically into the air with angular velocity components ω_1 about a diameter and ω_3 about its symmetry axis. If ω_3 were zero, the coin would simply spin about a (horizontal) diameter. For nonzero ω_3, the coin will precess (about what axis?) What is the minimum value of ω_3/ω_1 for which the wobble is such that the same face of the coin is always visible to an observer looking from above (and the coin lands without "flipping"?

(b) A space station in the form of a ring is rotating about its symmetry axis with initial angular velocity $\boldsymbol{\omega}_0$. It is struck at a point on its rim by a meteor that imparts impulse $\mathbf{P}$ parallel to $\boldsymbol{\omega}_0$. What is the angle between $\boldsymbol{\omega}_0$ and the new instantaneous axis of rotation after the collision?

What is the motion of the space station after the collision, ignoring gravity?

(c) A body with a symmetry axis, $\hat{\mathbf{3}}$, is rotating subject to an external drag torque due to air resistance,

$$\boldsymbol{\tau} = -C\boldsymbol{\omega}, \tag{3}$$

where $\boldsymbol{\omega}$ is the total angular velocity vector. Use Euler's equations to show that ω_3 decreases exponentially with time. Combine the equations for ω_1 and ω_2 to show that the angle between $\boldsymbol{\omega}$ and $\hat{\mathbf{3}}$ varies as,

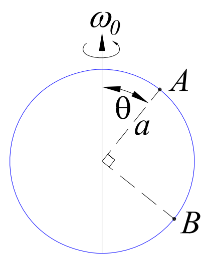
$$\tan\theta = \frac{\sqrt{\omega_1^2 + \omega_2^2}}{\omega_3} = \tan\theta_0\, e^{-C(\mathsf{I}_3 - \mathsf{I}_1)t/\mathsf{I}_1\mathsf{I}_3}, \tag{4}$$

and so, $\theta \to 0$ or 90° as $t \to \infty$.

4. **ωhoops!**

A uniform sphere of mass m and radius a is spinning about a diameter with angular velocity $\boldsymbol{\omega}_0$. Ignore gravity, and work in a frame in which the center of the sphere is initially at rest.

Suddenly, a point A on the surface of the sphere, at angle θ to the axis of rotation, is fixed.



(a) What is the instantaneous axis of rotation of the sphere just after the moment when point A becomes fixed?

It is useful to express your answer with respect to axes that emphasize point A.

(b) What is the speed of point B at the same azimuth as P and at polar angle $\theta + 90°$, just after A becomes fixed?

Answer: $v_B = a\omega_0\sqrt{\frac{\sin^2\theta}{9} + \cos^2\theta}$.

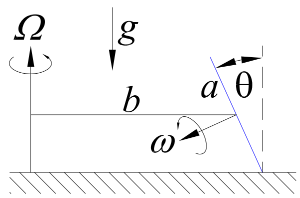
(c) What is the (vector) impulse $\mathbf{P}$ required to fix point A?

(d) What are the (vector) force $\mathbf{F}$ and torque $\boldsymbol{\tau}$ about point A that would be required to maintain steady motion after point A becomes fixed?

Answer: $\tau = \frac{1}{3}ma^2\omega_0^2 \sin\theta\cos\theta$.

5. **Steady Motion of a Rolling Wheel.**

A wheel in the form of a hoop of mass m and radius a rolls without slipping on a horizontal surface. The plane of the hoop makes angle θ to the vertical. Its center moves in a (horizontal) circle of radius b, with angular velocity Ω. The wheel rotates about its axle with angular velocity ω.



Consider steady motion, for which ω, Ω and θ are constants.

(a) What is the relation (rolling constraint) between ω and Ω for rolling without slipping?

You must get this right before proceeding.

(b) What is the (vector) torque $\boldsymbol{\tau}$ about the center of the hoop?

In the rest of the problem, use parts (a) and (b) to find a relation between Ω, a and b.

(c) Use Euler's equations.

That is, use a set of body/principal axes, rotating with the wheel. Identify the principal moments of inertia.

Note that in the lab frame, the total angular velocity is $\boldsymbol{\omega}_{\rm tot} = \boldsymbol{\omega} + \boldsymbol{\Omega}$. Calculate $\boldsymbol{\omega}_{\rm tot}$ in both the lab frame and the principal-axis frame.

To plug into Euler's equations, it is convenient to consider a time when one of the principal axes is horizontal, and the other two are in the vertical plane. Then, only one of Euler's equations is nontrivial, leading to,

$$\Omega^2 = \frac{2g \tan\theta}{4b + a\sin\theta} \,. \tag{5}$$

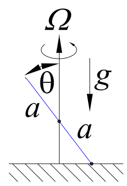
(d) Instead of using Euler's equations, consider a set of principal axes that don't roll with the hoop, but rather rotate about the vertical $\boldsymbol{\Omega}$. Again, choose one principal axis horizontal for simplicity.

Compare the time derivative of the angular momentum $\mathbf{L} = \mathsf{I} \cdot \boldsymbol{\omega}_{\rm tot}$ in the rotating frame defined above to that in the lab frame, which should lead to the result of part (c).

6. **Spinning Coin.**

A special case of the rolling wheel is when the center of mass does not move ($b = 0$ in Prob. 5).

An example is a coin spinning steadily on a table (ignoring friction, which eventually brings the coin to rest on the table). Try it!



Notice that the coin wobbles rapidly, but the figure on the coin precesses slowly.

(a) Show that $\Omega^2(\theta) = 4g/a\cos\theta$, where $\mathbf{\Omega}$ is the angular velocity about the vertical axis through the center of the coin, by an analysis in the lab frame for a thin disc of mass m and radius a whose plane makes angle θ to the vertical.

What is the angular momentum $\mathbf{L}(\theta, \Omega)$.

(b) As the coin rolls without slipping, the point of contact moves. By what angle does the orientation of the coin's face change during one revolution of the point of contact (in a circle on the table)?

Show that the orientation of the figure precesses (slowly) at angular velocity $\omega_{\text{figure}} = \Omega(1 - \sin\theta)$, such that $\omega_{\text{figure}} \to 0$ as $\theta \to 90°$ (and $\Omega \to \infty$).

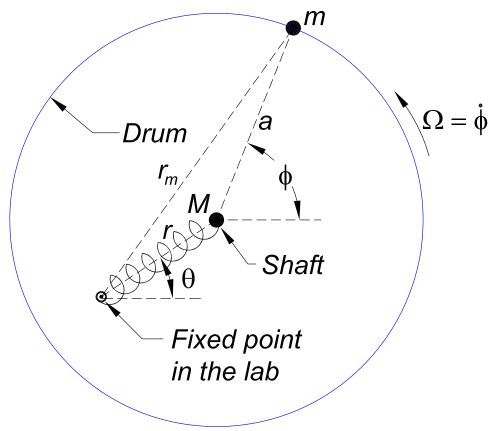
Can the speed of the point of contact of the coin exceed Mach 1?

Solutions

1. **Physics in the Laundromat.**

We consider the following model of a washing machine.

The center of the washtub/drum is tied to a fixed point in the lab by spring of constant k and zero rest length. The line from the fixed point to the center of the drum has length r, and makes angle θ to a fixed direction in the lab. There is a velocity-dependent damping $-\gamma\,\dot{\mathbf{r}}$ on the distance r.



A motor forces the drum to rotate about its center with constant angular velocity $\dot\phi = \Omega$.

Because of the uneven load, the center of mass of the drum (of mass M) plus load (of mass m) is not at the center of the tub, but distance a away.

(a) The equations of motion of this system of two degrees of freedom, r and θ, are readily deduced from a Newtonian approach,

$$m_{\rm cm}\ddot{\mathbf{r}}_{\rm cm} = M\,\ddot{\mathbf{r}} + m\,\ddot{\mathbf{r}}_m = -k\,\mathbf{r} - \gamma\,\dot{\mathbf{r}}, \qquad (6)$$

where the position $\mathbf{r}_m$ of mass m is,

$$\mathbf{r}_m = \mathbf{r} + \mathbf{a} = r\,\hat{\mathbf{r}} + a\cos(\phi-\theta)\,\hat{\mathbf{r}} + a\sin(\phi-\theta)\,\hat{\boldsymbol{\theta}}, \qquad (7)$$

$$\begin{aligned}\dot{\mathbf{r}}_m &= \dot{\mathbf{r}} + \dot{\mathbf{a}} = \dot r\,\hat{\mathbf{r}} + r\dot\theta\,\hat{\boldsymbol{\theta}} + a\Omega\,\hat{\boldsymbol{\phi}} \\ &= \dot r\,\hat{\mathbf{r}} + r\dot\theta\,\hat{\boldsymbol{\theta}} + a\Omega[-\sin(\phi-\theta)\,\hat{\mathbf{r}} + \cos(\phi-\theta\,\hat{\boldsymbol{\theta}}] \qquad (8)\end{aligned}$$

$$\begin{aligned}\ddot{\mathbf{r}}_m &= \ddot{\mathbf{r}} + \ddot{\mathbf{a}} = (\ddot r - r\dot\theta^2)\,\hat{\mathbf{r}} + (r\ddot\theta + 2\dot r\dot\theta)\,\hat{\boldsymbol{\theta}} \\ &\quad -a\Omega(\Omega-\dot\theta)[\cos(\phi-\theta)\,\hat{\mathbf{r}} + \sin(\phi-\theta)\,\hat{\boldsymbol{\theta}}] - a\Omega\dot\theta[\sin(\phi-\theta)\,\hat{\boldsymbol{\theta}} + \cos(\phi-\theta)\,\hat{\mathbf{r}}] \\ &= [\ddot r - r\dot\theta^2 - a\Omega^2\cos(\phi-\theta)]\,\hat{\mathbf{r}} + [r\ddot\theta + 2\dot r\dot\theta - a\Omega^2\sin(\phi-\theta)]\,\hat{\boldsymbol{\theta}}, \qquad (9)\end{aligned}$$

where $\hat{\boldsymbol{\theta}}$ is perpendicular to $\hat{\mathbf{r}}$, $\dot{\hat{\mathbf{r}}} = \dot\theta\,\hat{\boldsymbol{\theta}}$ and $\dot{\hat{\boldsymbol{\theta}}} = -\dot\theta\,\hat{\mathbf{r}}$.

Then, the equation of motion associated with coordinate r is the $\hat{\mathbf{r}}$ component of eq. (6),

$$\ddot r = r\,\dot\theta^2 + b\,\Omega^2\cos(\phi-\theta) - \omega_0^2\,r - \Gamma\,\dot r, \qquad (10)$$

and that with coordinate θ is the $\hat{\boldsymbol{\theta}}$ component of eq. (6),

$$0 = r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta} - b\,\Omega^2 \sin(\phi - \theta) + \Gamma\, r\,\dot{\theta}, \qquad (11)$$

where we have introduced the notation,

$$\omega_0 = \sqrt{\frac{k}{m+M}}. \qquad (12)$$

for the natural frequency of vibration of the washing machine,

$$b = \frac{m}{m+M}a, \qquad (13)$$

for the distance of the center of mass from the shaft and,

$$\Gamma = \frac{\gamma}{m+M}. \qquad (14)$$

These equations can be interpreted in a frame rotating with angular velocity $\dot{\theta}$. Equation (10) tells us that the total mass times the radial acceleration of mass M equals the spring force plus the radial component of the centrifugal force and friction. Equation (11) indicates that the azimuthal coordinate forces plus friction sum to zero; the term $2\dot{r}\dot{\theta}$ is the Coriolis acceleration.

The equations of motion with the neglect of friction are also readily deduced from the Lagrangian,

$$L = \frac{1}{2}(m+M)(\dot{r}^2+r^2\,\dot{\theta}^2) - ma\dot{r}\,\Omega \sin(\phi-\theta) + mar\,\dot{\theta}\Omega\cos(\phi-\theta) + \frac{1}{2}(I+ma^2)\Omega^2 - \frac{1}{2}kr^2, \qquad (15)$$

where I is the moment of inertia of the drum plus symmetric part of the load. The rotational kinetic energy is constant by assumption, so the moment of inertia does not appear further in the analysis.

(b) We first discuss steady motion in which $\dot{r} = 0$, $\ddot{r} = 0$ and $\ddot{\theta} = 0$. The shaft of the drum moves in a circle of radius r_0 and the mass m is at constant azimuth $\phi_0 = \phi - \theta$ relative to the azimuth of the shaft. Then, eq. (10) tells us that,

$$r_0 = \frac{b\,\Omega^2 \cos\phi_0}{\omega_0^2 - \Omega^2}, \qquad (16)$$

while eq. (11) indicates,

$$r_0 = \frac{b\,\Omega \sin\phi_0}{\Gamma}. \qquad (17)$$

Together,

$$\cos\phi_0 = \frac{\omega_0^2 - \Omega^2}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \Gamma^2\Omega^2}}, \qquad \sin\phi_0 = \frac{\Gamma\,\Omega}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \Gamma^2\Omega^2}}, \qquad (18)$$

and,

$$r_0 = \frac{b\,\Omega^2}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \Gamma^2\Omega^2}}. \tag{19}$$

For a balanced load ($m = 0$) distance b is zero, so the equilibrium displacement is zero also.

For low spin ($\Omega \ll \omega_0$) an unbalanced load finds itself at relative azimuth $\phi_0 \approx 0$, while near resonance ($\Omega \approx \omega_0$) the azimuth is $\approx \pi/2$, and for high spin ($\Omega \gg \omega$) the azimuth approaches π. In the latter case the system is a kind of inverted pendulum.

The center of mass of the system is at distance,

$$r_{\rm cm} = \frac{b\,\omega_0^2}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \Gamma^2\Omega^2}}. \tag{20}$$

Thus the center of mass approaches the origin as the spin Ω becomes large, even though the shaft is at radius $r_0 \approx b$. The system can be called self-centering as the spin Ω increases, once it successfully passes through the resonance region.

(c) Is the desirable self-centering motion found above stable against small perturbations?

If angle θ(t) were locked at $\phi - \phi_0 = \Omega\,t - \phi_0$, *i.e.*, if only radial oscillations of the axis of the drum were permitted, and $\Omega > \omega_0$ answer would be no!

To see this we refer to eq. (10), which for the locked hypothesis reads,

$$\ddot{r} = (\Omega^2 - \omega_0^2)r + b\,\Omega^2\cos\phi_0 - \Gamma\,\dot{r}. \tag{21}$$

For oscillatory radial motion the coefficient of the term in r must be negative. Hence, the locked motion would be stable only for low spin, $\Omega < \omega_0$.

However, we will find that the motion is stable when both radial and azimuthal oscillations are considered. The linked system of masses m and M forms a kind of double pendulum. The motion in which $\phi \approx \theta + \pi$ that arises when the drive frequency Ω exceeds the resonant frequency ω_0 is an example of a stable inverted pendulum.

To demonstrate this we perform a perturbation analysis, seeking solutions of the form,

$$r = r_0(1 + \epsilon), \qquad \theta = \phi - \phi_0 + \delta, \tag{22}$$

where the perturbations are desired to be small and oscillatory with angular frequency ω,

$$\epsilon = \epsilon_0\,e^{i\omega t}, \qquad \text{and} \qquad \delta = \delta_0\,e^{i\omega t} \qquad \text{with} \qquad \epsilon_0,\ \delta_0 \ll 1. \tag{23}$$

The constants ϵ_0, δ_0 and ω are complex in general, and, of course, the physical motion is described by the real parts of eq. (22). Both the real and imaginary parts of ω should be positive; the real part is the frequency of oscillation and the imaginary part is the damping decay constant.

In the first approximation we now have,

$$\cos(\phi - \theta) = \cos\phi_0 + \delta \sin\phi_0, \qquad \text{and} \qquad \sin(\phi - \theta) = \sin\phi_0 - \delta\cos\phi_0. \quad (24)$$

Then, using (21)-(24) in (10) and keeping terms only of first order of smallness, we find,

$$-\omega^2\epsilon_0 = \Omega^2 + 2i\Omega\delta_0 + \frac{b\,\Omega^2 \sin\phi_0}{r_0}\delta_0 - \omega_0^2\epsilon_0 - i\omega\Gamma\epsilon_0. \quad (25)$$

With eq. (17) this tells us that,

$$\epsilon_0 = -\frac{\Gamma\omega + 2i\omega\Omega}{\omega^2 - \omega_0^2 + \Omega^2 - i\omega\Gamma}\delta_0. \quad (26)$$

Similarly, eq. (11) leads to,

$$0 = -\omega^2\delta_0 + 2i\Omega\omega\epsilon_0 + \frac{b\,\Omega^2\cos\phi_0}{r_0}\delta_0 + \Gamma\Omega_0\epsilon_0 + i\omega\Gamma\delta_0, \quad (27)$$

which together with eq. (16) tells us that,

$$\delta_0 = \frac{\Gamma\Omega + 2i\Omega\omega}{\omega^2 - \omega_0^2 + \Omega^2 - i\omega\Gamma}\epsilon_0. \quad (28)$$

Equations (26) and (28) are consistent only if,

$$\frac{\Gamma\Omega + 2i\Omega\omega}{\omega^2 - \omega_0^2 + \Omega^2 - i\omega\Gamma} = \pm i, \quad (29)$$

which leads to the quadratic equation,

$$\omega^2 - 2\omega(\pm\Omega - i\Gamma/2) - \omega_0^2 + \Omega^2 \pm i\Gamma\Omega = 0. \quad (30)$$

The roots of this with positive real parts are,

$$\omega = \begin{cases} \sqrt{\omega_0^2 - (\Gamma/2)^2} \pm \Omega + i\Gamma/2, & \Omega < \sqrt{\omega_0^2 - (\Gamma/2)^2}, \\ \Omega \pm \sqrt{\omega_0^2 - (\Gamma/2)^2} + i\Gamma/2, & \Omega > \sqrt{\omega_0^2 - (\Gamma/2)^2}. \end{cases} \quad (31)$$

In the above we have assumed that the damping is weak enough that $\omega_0 > \Gamma/2$. Then, perturbations die out with characteristic time $2/\Gamma$ which is greater than the natural period of oscillation, $1/\omega_0$.

Thus stable motion exists for all values of the spin Ω. Referring to eq. (27), we note that the key coupling between the radial and azimuthal perturbations (ϵ and δ) is provided by the Coriolis force.

As the spin frequency Ω approaches the resonant frequency ω_0 the lower frequency of the perturbed motion goes to zero. If the amplitude of the perturbation is large it will be noticeable throughout the laundromat.

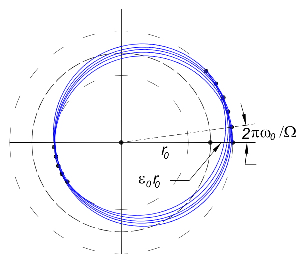
For high spin, eqs. (28) and (31) yield the relation,

$$\delta_0 = \frac{i\epsilon_0}{1 - \frac{i\Gamma}{2\Omega} - \frac{\Gamma^2}{8\Omega\left(\Omega \pm \sqrt{\omega_0^2 - (\Gamma/2)^2}\right)}} \approx i\epsilon_0, \tag{32}$$

which indicates that the radial and azimuthal perturbations are 90° out of phase. The angular velocity of the motion of the center of the drum is,

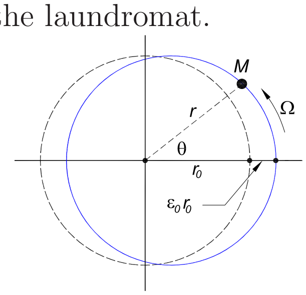
$$\dot{\theta} = \Omega + \epsilon_0 \sin\left(\Omega \pm \sqrt{\omega_0^2 - (\Gamma/2)^2}\right), \tag{33}$$

which is Ω on average. This implies that the perturbed motion of the shaft of the drum is still a circle of radius r_0 to first approximation. However, since the frequency ω of the perturbation differs from the average rotation frequency Ω, the orbit of the center of the drum is not closed but precesses at angular frequency ω_0, as sketched below.



The steady motion of the axis of the drum is at angular velocity Ω in a circle of radius r_0 about the origin. The perturbed orbit is nearly circular, but precesses with angular velocity ω_0 and lies in the annulus $r_0(1 - \epsilon_0) < r < r_0(1 + \epsilon_0)$.

In the limit that $\Omega \gg \omega_0$ the motion of the center of the drum is essentially a circle of radius r_0 displaced by distance $\epsilon_0 r_0$ from the origin, as shown below. In practice this displacement can be quite noticeable, as the reader can confirm on his or her next trip to the laundromat.



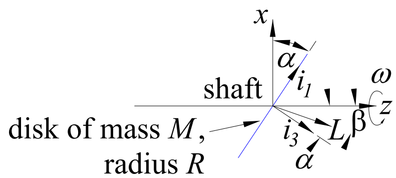
If the drive frequency Ω is large compared to the resonant frequency ω_0 the perturbed motion is a circle of radius r_0 whose center is displaced by ϵr_0.

The motion of the axis of the drum of the washer is an example of the motion of unbalanced shafts of large rotating machines, as has been well described by Landau and Kitaigorodsky in a popular book, pp. 176-180 of
http://kirkmcd.princeton.edu/examples/mechanics/landau_motion_74.pdf

2. **Wobble-Plate Engine.**

This example is from Probs. 324-325, p. 437 of J.P. den Hartog, *Mechanics* (McGraw-Hill, 1948), http://kirkmcd.princeton.edu/examples/mechanics/denhartog_48.pdf

A disk of mass M and radius R is mounted on a shaft through its center, making angle α as shown in the figure below. The disk rotates with constant angular velocity ω about the shaft.



(a) The principal axes of a uniform disk are its axis, $\hat{\mathbf{3}}_1$, and any two orthogonal axes in the plane of the disk. We take axis $\hat{\mathbf{i}}_1$ to lie in the plane of the shaft and the axis of the disk, and $\hat{\mathbf{i}}_2 = \hat{\mathbf{i}}_3 \times \hat{\mathbf{i}}_1$ perpendicular to this plane (*i.e.*, out of the page). The moment of inertia of the disk about its axis $\hat{\mathbf{i}}_3$ is,

$$I_{\rm pa,3} = \frac{MR^2}{2}\,. \tag{34}$$

Then, by the perpendicular-axis theorem, the (equal) moments of inertia about principal axes 1 and 2 are,

$$I_{\rm pa,1} = I_{\rm pa,2} = \frac{MR^2}{4}\,. \tag{35}$$

The angular velocity of the disk in the frame of the principal axes is,

$$\boldsymbol{\omega}_{\rm pa} = \omega(\sin\alpha, 0, \cos\alpha,). \tag{36}$$

The inertia tensor $\mathsf{I}_{\rm pa}$ of the disk is diagonal, so its angular momentum $\mathbf{L}_{\rm disk,pa}$ has the form,

$$\mathbf{L}_{\rm disk,pa} = \mathsf{I}_{\rm pa}\cdot\boldsymbol{\omega}_{\rm pa} = (I_{\rm pa,1}\,\omega_{\rm pa,1}, I_{\rm pa,2}\,\omega_{\rm pa,2}, I_{\rm pa,3}\,\omega_{\rm pa,3}) = \frac{MR^2\omega}{4}(\sin\alpha, 0, 2\cos\alpha). \tag{37}$$

Hence, the angle β of $\mathbf{L}_{\rm disk,pa}$ to the shaft is related by,

$$\tan(\alpha-\beta) = \frac{\sin\alpha}{2\cos\alpha} = \frac{\tan\alpha}{2}\,, \tag{38}$$

$$\tan\beta = \tan([\alpha-(\alpha-\beta)] = \frac{\tan\alpha - \tan(\alpha-\beta)}{1+\tan\alpha\tan(\alpha-\beta)} = \frac{\tan\alpha}{2(1+\frac{1}{2}\tan^2\alpha)} = \frac{\tan\alpha}{2+\tan^2\alpha}\,. \tag{39}$$

The principal-axis frame rotates with respect the lab frame, so the angular momentum is not constant in the lab frame, and the wobble plate would be described as "unbalanced" in the lab frame.

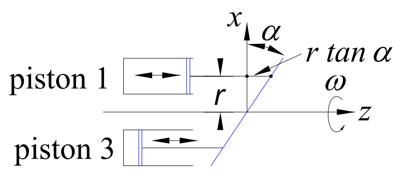
In the lab frame, we consider a (fixed) coordinate system (x, y, z) with $\hat{\mathbf{z}}$ along the shaft, $\hat{\mathbf{x}}$ in the plane of the shaft and the axis of the disk, and $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$

perpendicular to this plane (*i.e.*, out of the page). Then, the z-component of $\mathbf{L}_{\text{disk,pa}}$ is "balanced", as it requires no forces/torques on the shaft to maintain it. In contrast, the component $\mathbf{L}_{\text{disk,pa},\perp}$ perpendicular to the shaft, with magnitude $L_{\text{disk,pa}} \sin\beta$, is "unbalanced" in the sense of requiring forces/torques on the shaft to be maintained. This "unbalanced" component of the angular momentum of the disk rotates with angular velocity ω about the shaft, in the (fixed) (x, y, z) frame.

(b) Four pistons of mass m each have rods that are parallel to the main shaft, at distance r. The piston rods press against the rotating tilted disk at,

$$\mathbf{r}_{\text{rod},n} = (r \sin n\pi/2, -r\cos n\pi/2, r\tan\alpha \sin(n\pi/2 - \omega t)), \tag{40}$$

in a fixed rectangular coordinate system with z along the shaft and z in the page, for $n = $ 1-4. Hence, the pistons are driven back and forth by the rotating disk (or *vice versa*) in simple harmonic motion, to make an engine.



The piston rods have length l, so the centers of the pistons are at,

$$\mathbf{r}_n = (r \sin n\pi/2, -r\cos n\pi/2, r\tan\alpha \sin(n\pi/2 - \omega t) - l). \tag{41}$$

The pistons have velocity (in the z-direction),

$$v_{n,z} = -\omega r \tan\alpha \cos(n\pi/2 - \omega t), \tag{42}$$

and their total angular momentum about the origin (in the fixed (x, y, z) coordinate system) is,

$$\begin{aligned} \mathbf{L}_p &= \sum_n \mathbf{r}_n \times m\mathbf{v}_n \\ &= mr^2\omega\tan\alpha \sum_n (\cos(n\pi/2)\cos(n\pi/2 - \omega t), \sin(n\pi/2)\cos(n\pi/2 - \omega t), 0) \\ &\frac{mr^2\omega\tan\alpha}{2} \sum_n (\cos\omega t + \cos(n\pi - \omega t), \sin\omega t - \sin(n\pi - \omega t), 0) \\ &= 2mr^2\omega\tan\alpha\,(\cos\omega t, \sin\omega t, 0). \end{aligned} \tag{43}$$

Thus, the total angular momentum of the piston rotates about the z axis with angular velocity ω, and points in the x-direction at $t = 0$.

If the angular momentum (43) of the pistons cancels the component $\mathbf{L}_{\text{disk,pa},\perp}$ of the angular momentum of the disk, found in part (a), then the total angular momentum of the system would be "balanced" (*i.e.*, about the z-axis of the shaft). For this, we need,

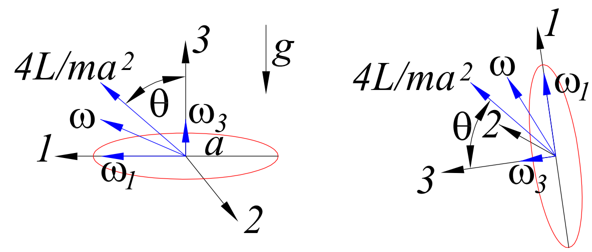
$$L_{p,\perp} = 2mr^2\omega\tan\alpha = L_{\text{disk,pa},\perp}\sin\beta = \frac{MR^2\omega}{4}\sqrt{\sin^2\alpha + 4\cos^2\alpha}\,\sin\beta$$
$$= \frac{MR^2\omega}{4}\cos\alpha\sqrt{\tan^2\alpha+4}\frac{\tan\alpha}{\sqrt{\tan^2\alpha + (2+\tan^2\alpha)^2}}$$
$$= \frac{MR^2\omega}{4}\sin\alpha\frac{\sqrt{\tan^2\alpha+4}}{\sqrt{4+5\tan^2\alpha+\tan^4\alpha}} = \frac{MR^2\omega}{4}\sin\alpha\frac{\sqrt{\tan^2\alpha+4}}{\sqrt{(4+\tan^2\alpha)(1+\tan^2\alpha)}}$$
$$= \frac{MR^2\omega}{4}\sin\alpha\cos\alpha, \qquad (44)$$
$$m = \frac{MR^2\cos^2\alpha}{8r^2}. \qquad (45)$$

Although the wobble-plate engine is technically clever, it was not a commercial success. See, for example,
http://kirkmcd.princeton.edu/examples/EM/anning_per_248_11.pdf

3. (a) *This is Prob. 13, p. 259 of* V. Barger and M. Olsson, *Classical Mechanics A Modern Perspective* (McGraw-Hill), 1973),
http://kirkmcd.princeton.edu/examples/EM/barger_73.pdf

A coin initially in a horizontal plane is tossed vertically into the air with angular velocity components ω_1 about a diameter and ω_3 about its symmetry axis. If ω_3 were zero, the coin would simply spin about a (horizontal) diameter. If ω_3 is nonzero, the coin will precess about the conserved angular momentum $\mathbf{L} = \mathsf{I} \cdot \boldsymbol{\omega}$.

We adopt a principal-axis coordinate system, with $\hat{\mathbf{i}}_3$ along the symmetry axis of the coin, and $\hat{\mathbf{i}}_1$ and $\hat{\mathbf{i}}_2$ in the plane of the coin, which is initially horizontal, as shown on the left below.



The inertia tensor I is diagonal in this system, with $\mathsf{I}_{11} = \mathsf{I}_{22} = ma^2/4$ and $\mathsf{I}_{33} = ma^2/2$, where m is the mass of the coin, and a is its radius. The (conserved) angular momentum $\mathbf{L}$ of the coin is, in the initial principal-axis coordinate system,

$$\mathbf{L} = (\mathsf{I}_{11}\omega_1, \mathsf{I}_{22}\omega_2, \mathsf{I}_{33}\omega_3) = \frac{ma^2}{4}(\omega_1, 0, 2\omega_3), \qquad \tan\theta = \frac{\omega_1}{2\omega_3}. \tag{46}$$

After the coin has precessed by 180° about $\mathbf{L}$, it is in the configuration shown on the right above. For the initial bottom face of the coin to be visible from above at this time (such that the coin can "flip"), we must have $\theta > 45°$, where θ is the angle between $\mathbf{L}$ and $\hat{\mathbf{i}}_3$.

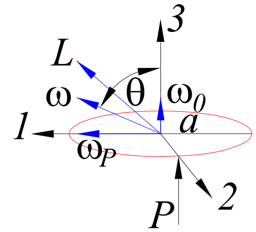
For the same face of the coin to be always visible to an observer looking from above (such that the coin lands without "flipping"), we must have $\theta < 45°$, *i.e.*,

$$\frac{\omega_1}{2\omega_3} < 1, \qquad \frac{\omega_3}{\omega_1} > \frac{1}{2}. \tag{47}$$

(b) *This is Prob. 9.14, p. 414 of* G.R. Fowles and G.L. Cassiday, *Analytical Mechanics*, 7th ed. (Thomson Brooks/Cole, 2004).

The instantaneous axis of rotation of the space station is along the total angular velocity vector $\boldsymbol{\omega}$.

Just after the collision with a meteor, the space station still has angular velocity component $\omega_0\,\hat{\mathbf{3}}$ in the principal-axis system shown in the figure below.



The impulse $\mathbf{P}$ of the meteor, which struck the station of mass m at radius $a\,\hat{\mathbf{2}}$ gives it angular momentum component,

$$L_1 = aP = \mathsf{I}_{11}\omega_P = \frac{ma^2}{2}\omega_P, \qquad \omega_P = \frac{2aP}{m}\,, \tag{48}$$

recalling that for a ring, $\mathsf{I}_{11} = \mathsf{I}_{22} = ma^2/2$, $\mathsf{I}_{33} = ma^2$.
The angle θ of the total angular velocity $\boldsymbol{\omega} = (\omega_P, 0, \omega_0)$ to the axis of the space station (after the collision) is related by,

$$\tan\theta = \frac{\omega_P}{\omega_0} = \frac{2aP}{\omega_0\, m}\,. \tag{49}$$

After the collision, the space station takes on center-of mass velocity $\mathbf{P}/m$, and precesses about the conserved angular momentum,

$$\mathbf{L} = (\mathsf{I}_{11}\omega_P, 0, \mathsf{I}_{33}\omega_0) = (aP, 0, ma^2\omega_0). \tag{50}$$

Euler's equations for free precession tell us that in the principal-axis frame,

$$\dot{\boldsymbol{\omega}} = \frac{\mathsf{I}_{33} - \mathsf{I}_{11}}{\mathsf{I}_{11}}\omega_0\,\hat{\mathbf{3}} \times \boldsymbol{\omega} = \omega_0\,\hat{\mathbf{3}} \times \boldsymbol{\omega} \equiv \boldsymbol{\Omega} \times \boldsymbol{\omega}, \tag{51}$$

so the angular velocity of precession is,

$$\Omega = \omega_0. \tag{52}$$

In the nonrotating frame of the space station before the collision, the principal axes (and $\boldsymbol{\omega}$) precess about $\mathbf{L}$ with angular velocity $\Omega = \omega_0$ (such that the precession angular velocity vector is $\boldsymbol{\Omega} = \omega_0\,\hat{\mathbf{L}}$ in this frame).

(c) *From sec. 39, p. 71 of* R.F. Deimel, *Mechanics of the Gyroscope* (Macmillan, 1929), http://kirkmcd.princeton.edu/examples/mechanics/deimel_gyro_52.pdf

A body with a symmetry axis, $\hat{\mathbf{3}}$, is rotating subject to an external drag torque due to air resistance,

$$\boldsymbol{\tau} = -C\boldsymbol{\omega}, \tag{53}$$

where $\boldsymbol{\omega}$ is the total angular velocity vector.
By the symmetry, $\mathsf{I}_{11} = \mathsf{I}_1 = \mathsf{I}_2$, so the third Euler equation simplifies to,

$$-C\omega_3 = \mathsf{I}_3\dot{\omega}_3 + \omega_1\omega_2(\mathsf{I}_2 - \mathsf{I}_1) = \mathsf{I}_3\dot{\omega}_3, \qquad \omega_3 = \omega_{3,0}\, e^{-Ct/\mathsf{I}_3}. \tag{54}$$

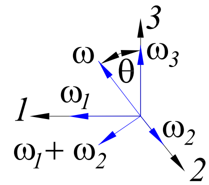
The other two Euler equations are, again noting the symmetry,

$$-C\omega_1 = \mathsf{I}_1\dot{\omega}_1 + \omega_2\omega_3(\mathsf{I}_3 - \mathsf{I}_1), \qquad -C\omega_2 = \mathsf{I}_1\dot{\omega}_2 + \omega_3\omega_1(\mathsf{I}_1 - \mathsf{I}_3). \tag{55}$$

Multiplying these by ω_1 and ω_2 and adding, we find,

$$-C(\omega_1^2 + \omega_2^2) = \mathsf{I}_1(\omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2) = \frac{\mathsf{I}_1}{2}\frac{d}{dt}(\omega_1^2 + \omega_2^2), \tag{56}$$

$$\omega_1^2 + \omega_2^2 = (\omega_1^2 + \omega_2^2)_0\, e^{-2Ct/\mathsf{I}_1}. \tag{57}$$



Hence, the angle between $\boldsymbol{\omega}$ and $\hat{\mathbf{3}}$ varies as,

$$\tan\theta = \frac{\sqrt{\omega_1^2+\omega_2^2}}{\omega_3} = \frac{\sqrt{\omega_1^2+\omega_2^2}_0\, e^{-Ct/I_1}}{\omega_{3,0}\, e^{-Ct/I_3}} = \tan\theta_0\, e^{-C(1/I_1-1/I_3)t}$$
$$= \tan\theta_0\, e^{-C(I_3-I_1)t/I_1I_3}, \qquad (58)$$

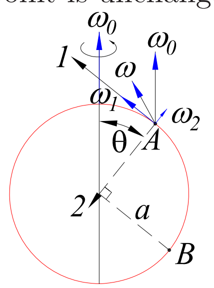
For an oblate spheroid (such as a Frisbee), $I_3 > I_1$, so $\theta \to 0$ as $t \to \infty$, while for a prolate spheroid (such as a pencil), $I_3 < I_1$, so $\theta \to 90^\circ$ as $t \to \infty$.

See also, http://kirkmcd.princeton.edu/examples/mechanics/wainwright_pm_3_641_27.pdf

4. **$\boldsymbol{\omega}$hoops!**

Aspects of this problem appear in Art. 290, p. 240 of E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7th ed. (Macmillan, 1905),
http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf
and in Ex. 3, p. 169 of E.T. Whittaker, *A Treatise on Analytical Dynamics of Particles and Rigid Bodies* (Cambridge U. Press, 1904, 1917, 1927, 1937),
http://kirkmcd.princeton.edu/examples/mechanics/whittaker_dynamics_17.pdf

A hoop of mass m and radius a is spinning about a diameter with angular velocity $\boldsymbol{\omega}_0$. If a point A, at angle θ to the axis of rotation, on the hoop is suddenly fixed, the angular momentum about this point is unchanged



In a frame where the center of the hoop is initially at rest, the angular momentum about point A is only that due to the motion relative to the center (of mass of) the hoop, namely,

$$\mathbf{L} = I_d\,\boldsymbol{\omega}_0 = \frac{ma^2}{2}\boldsymbol{\omega}_0. \tag{59}$$

recalling that the moment of inertia I_d of a hoop about a diameter is 1/2 of that (ma^2) about its symmetry axis.

(a) After point A becomes fixed, we consider principal axes centered on point A, as shown in the figure above.

Just after point A becomes fixed, the motion does not include rotation about axis $\hat{\mathbf{3}}$, perpendicular to the plane of the hoop, so it is convenient to consider principal axes centered on point A, as shown in the figure above. Then $\omega_3 = 0$ just after the sudden fixture, and the angular momentum $\mathbf{L}$, eq. (59), about A, and the angular velocity $\boldsymbol{\omega}$, have components (in the principal-axis system),

$$L_1 = L\sin\theta = I_1\,\omega_1 = (I_d + ma^2)\,\omega_1, \qquad \omega_1 = \frac{I_d}{I_d + ma^2}\omega_0 \sin\theta \tag{60}$$

$$L_2 = -L\cos\theta = I_2\,\omega_2 = I_d\,\omega_2, \qquad \omega_2 = -\omega_0\cos\theta, \tag{61}$$

$$L_3 = 0 = I_3\,\omega_3, \qquad \omega_3 = 0. \tag{62}$$

For a hoop, $I_2 = ma^2/2$, so the instantaneous axis, along $\boldsymbol{\omega}$, just after the sudden fixture is,

$$\boldsymbol{\omega} = \omega_0\left(\frac{\sin\theta}{3}, -\cos\theta, 0\right). \tag{63}$$

For a uniform disk, $I_d = ma^2/4$, and for a uniform sphere $I_d = 2ma^2/5$, so the instantaneous axes just after the sudden fixture those spinning bodies would be,

$$\boldsymbol{\omega}_{\rm disk} = \omega_0 \left(\frac{\sin\theta}{5}, -\cos\theta, 0 \right), \qquad \boldsymbol{\omega}_{\rm disk} = \omega_0 \left(\frac{2\sin\theta}{7}, -\cos\theta, 0 \right). \tag{64}$$

(b) Point B has coordinates $\mathbf{r}_B = (-a, a, 0)$ in the principal-axis system, so its velocity just after the sudden fixture is,

$$\mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{r}_B = a\omega_0 \left(0, 0, \frac{\sin\theta}{3} - \cos\theta \right). \tag{65}$$

(c) Just after the sudden fixture the center (of mass of) the hoop, at $\mathbf{r}_{\rm cm} = (0, a, 0)$ in the principal-axis system, has velocity,

$$\mathbf{v}_{\rm cm} = \boldsymbol{\omega} \times \mathbf{r}_{\rm cm} = a\omega_0 \left(0, 0, \frac{\sin\theta}{3} \right), \tag{66}$$

which is out of the page (and $-1/3$ of the velocity of point A just before the sudden fixture).

The impulse $\mathbf{P}$ at point A provides the center-of-mass momentum just after the sudden fixture, so,

$$\mathbf{P} = m\mathbf{v}_{\rm cm} = ma\omega_0 \left(0, 0, \frac{\sin\theta}{3} \right), \tag{67}$$

in the principal-axis system.

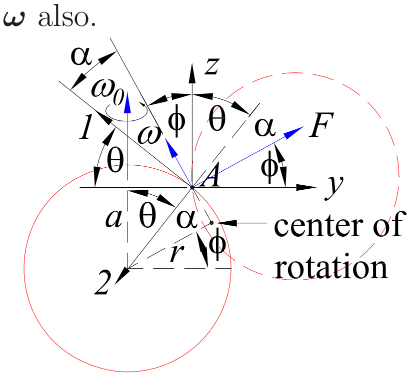
(d) If the hoop were to rotate steadily about point A with constant angular velocity (63), the angular momentum $\mathbf{L}$ would have to precess around $\boldsymbol{\omega}$, requiring a torque,

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} = \boldsymbol{\omega} \times \mathbf{L}, \tag{68}$$

about point A. Just after the sudden fixture, when $\mathbf{L} = L(\sin\theta, -\cos\theta, 0) = I_d\omega_0(\sin\theta, -\cos\theta, 0)$ in the principal-axis system, the required torque is,

$$\boldsymbol{\tau} = \frac{2}{3} I_d \omega_0^2 \sin\theta\cos\theta\, \hat{\mathbf{3}} = \frac{ma^2\,\omega_0^2 \sin\theta\cos\theta}{3}\, \hat{\mathbf{3}}. \tag{69}$$

As the hoop rotates about the constant vector $\boldsymbol{\omega}$, the torque vector must rotate/precess about $\boldsymbol{\omega}$ also.



For steady rotation about the fixed point A, the center of rotation of the hoop lies along vector $\boldsymbol{\omega}$ through A, at distance $a \sin\alpha$ from it, where α is the angle between $\boldsymbol{\omega}$ and the 1-axis. We also define ϕ as the angle between $\boldsymbol{\omega}$ and the z-axis (which is parallel to $\boldsymbol{\omega}_0$). Then,

$$\alpha + \theta + \phi = \frac{\pi}{2}, \qquad \cos\alpha = \frac{\omega_1}{\omega} = \frac{\sin\theta}{3\sqrt{\frac{1}{9}\sin^2\theta + \cos^2\theta}}\,. \tag{70}$$

The center of the hoop moves in a circle of radius $r = a\cos\alpha$ with angular velocity $\boldsymbol{\omega}$, which requires a force $\mathbf{F}$ on point A of magnitude,

$$F = m\omega^2 r = ma\,\omega_0^2 \cos\alpha \left(\frac{\sin^2\theta}{9} + \cos^2\theta\right) = \frac{ma\,\omega_0^2 \sin^2\theta}{9}\sqrt{1 + 9\cot^2\theta}. \tag{71}$$

Vector $\mathbf{F}$ makes angle ϕ to the y-axis (perpendicular to $\boldsymbol{\omega}_0$) just after the sudden fixture, and as the hoop rotates, vector $\mathbf{F}$ also precesses about $\boldsymbol{\omega}$.

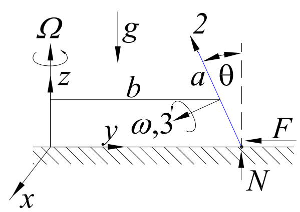
For the trivial case of $\theta = 0$, point A is already fixed in the initial state, so $\mathbf{P}$, $\boldsymbol{\tau}$ and $\mathbf{F}$ are all zero, as consistent with eqs. (67), (69) and (71).

For $\theta = 90^\circ$, $\mathbf{P} = ma\omega_0(0, 0, 1/3)$, $\boldsymbol{\tau} = 0$, and $F = ma\omega_0^2/9$, with $\alpha = \phi = 0$.

5. **Steady Motion of a Rolling Wheel.**

This problem is discussed, for example, in sec. 29, p. 54 of R.F. Deimel, *Mechanics of the Gyroscope* (Macmillan, 1929), http://kirkmcd.princeton.edu/examples/mechanics/deimel_gyro_52.pdf *and in prob. 3.7, p. 38 of* D.F. Lawden, *Analytical Mechanics* (Allen & Unwin, 1972), http://kirkmcd.princeton.edu/examples/mechanics/lawden_72.pdf

A wheel in the form of a hoop of mass m and radius a rolls without slipping on a horizontal surface. The plane of the hoop makes angle θ to the vertical. Its center moves in a (horizontal) circle of radius b, with angular velocity Ω. The wheel rotates about its axle with angular velocity ω.



We consider steady motion, for which ω, Ω and θ are constants.

(a) As the hoop rolls around the vertical axis $\mathbf{\Omega}$, the velocity of the point of contact of the hoop with the horizontal plane is,

$$v = \Omega r = \Omega(b + a\sin\theta). \tag{72}$$

Also, this velocity is related to the angular velocity $\boldsymbol{\omega}$ by $v = \omega a$, such that,

$$\omega = \Omega\left(\frac{b}{a} + \sin\theta\right). \tag{73}$$

(b) In the lab frame, we first consider a fixed, rectangular coordinate system, (x, y, z), with at the center of the circle on the horizontal plane in which the wheel rolls. For steady motion of the wheel, the forces on it are a centripetal force $\mathbf{F} = -m\Omega^2 b\,\mathbf{y}$, and a normal force $\mathbf{N} = mg\,\hat{\mathbf{z}}$.

We also consider body axes $(1, 2, 3)$ with $\hat{\mathbf{1}}$ and $\hat{2}$ in the plane of the wheel, and $\hat{\mathbf{3}}$ along its axis (and hence along $\boldsymbol{\omega}$).

The torque about the center (of mass of) the wheel at a moment when $\hat{\mathbf{2}}$ is in the y-z plane (with $\hat{\mathbf{1}}$ out of the page), and $\mathbf{y} = -\sin\theta\,\hat{\mathbf{2}} - \cos\theta\,\hat{\mathbf{3}}$, $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{3}} - \sin\theta\,\hat{\mathbf{3}}$, as shown in the figure above, is,

$$\begin{aligned}\boldsymbol{\tau} &= -a\,\hat{\mathbf{2}} \times (\mathbf{F} + \mathbf{N}) = -a\,\hat{\mathbf{2}} \times (-m\Omega^2 b\,\hat{\mathbf{y}} + mg\,\hat{\mathbf{z}}) \\ &= -a\,\hat{\mathbf{2}} \times [(m\Omega^2 b\sin\theta + mg\cos\theta)\,\hat{\mathbf{2}} + (m\Omega^2 b\cos\theta - mg\sin\theta)\,\hat{\mathbf{3}}] \\ &= ma(g\sin\theta - \Omega^2 b\cos\theta)\,\hat{\mathbf{1}}.\end{aligned} \tag{74}$$

(c) To use Euler's equations, we consider the body frame $(1, 2, 3)$, in which $\boldsymbol{\omega} = \omega\,\hat{\mathbf{3}}$ is constant, but $\mathbf{\Omega}$ is rotating. Here, the principal moments of inertia are $I_1 =$

$I_2 = I_3/2 = kma^2/2$, where $k = 1$ for a hoop and $k = 1/2$ for a uniform disk, so Euler's equations are, at the moment shown in the figure,

$$\tau_1 = ma(g\sin\theta - \Omega^2 b\cos\theta) = I_1\,\dot\omega_{\text{tot},1} + (I_3 - I_2)\omega_{\text{tot},2}\omega_{\text{tot},3}$$
$$= (kma^2/2)(\dot\omega_{\text{tot},1} + \Omega\cos\theta(\omega - \Omega\sin\theta), \qquad (75)$$
$$\tau_2 = 0 = I_2\,\dot\omega_{\text{tot},2} + (I_1 - I_3)\omega_{\text{tot},1}\omega_{\text{tot},3} = (kma^2/2)(\dot\omega_{\text{tot},2} - \omega_{\text{tot},1}\omega_{\text{tot},3}), \qquad (76)$$
$$\tau_3 = 0 = I_3\,\dot\omega_{\text{tot},3} + (I_1 - I_2)\omega_{\text{tot},2}\omega_{\text{tot},3} = kma^2\dot\omega_{\text{tot},3}. \qquad (77)$$

To complete the solution using the only nontrivial Euler equation, (75), we need $\dot\omega_{\text{tot},1}$ at the moment shown in the figure above. For this, we note that in the (x, y, z) lab frame, the total angular velocity precesses around $\mathbf{\Omega}$,

$$\left.\frac{d\boldsymbol{\omega}_{\text{tot}}}{dt}\right|_{(x,y,z)} = \mathbf{\Omega}\times\boldsymbol{\omega}_{\text{tot}} = \mathbf{\Omega}\times\boldsymbol{\omega} = \Omega(\cos\theta\,\hat{\mathbf{2}} - \sin\theta\,\hat{\mathbf{3}})\times\omega\,\hat{\mathbf{3}} = \omega\Omega\cos\theta\,\hat{\mathbf{1}}. \qquad (78)$$

Also, the relation between the rates of change in the (x, y, z) lab frame and the $(1, 2, 3)$ body frame that rotates with angular velocity $\boldsymbol{\omega}_{\text{tot}}$ is,

$$\left.\frac{d\boldsymbol{\omega}_{\text{tot}}}{dt}\right|_{(x,y,z)} = \left.\frac{d\boldsymbol{\omega}_{\text{tot}}}{dt}\right|_{(1,2,3)} + \boldsymbol{\omega}_{\text{tot}}\times\boldsymbol{\omega}_{\text{tot}} = \left.\frac{d\boldsymbol{\omega}_{\text{tot}}}{dt}\right|_{(1,2,3)} = \omega\Omega\cos\theta\,\hat{\mathbf{1}}. \qquad (79)$$

Using this in eq. (75), we find, recalling eq. (73),

$$2(g\sin\theta - \Omega^2 b\cos\theta) = ka[\omega\Omega\cos\theta + \Omega\cos\theta(\omega - \Omega\sin\theta)]$$
$$= k\Omega^2\cos\theta(b + a\sin\theta) + k\Omega^2\cos\theta(b + a\sin\theta) - ka\Omega^2\cos\theta\sin\theta)$$
$$= 2k\Omega^2 b\cos\theta + k\Omega^2 a\cos\theta\sin\theta), \qquad (80)$$
$$\Omega^2 = \frac{2g\tan\theta}{2(1+k)b + ka\sin\theta}. \qquad (81)$$

For a wheel/hoop with k = 1, recalling that $r = b + a\sin\theta$ is the radius of the circle of contact of the wheel with the ground,

$$\Omega^2 = \frac{2g\tan\theta}{4b + a\sin\theta} = \frac{2g\tan\theta}{4r - 3\sin\theta}, \qquad (82)$$

while for a uniform disk with $k = 1/2$,

$$\Omega^2 = \frac{4g\tan\theta}{6b + a\sin\theta} = \frac{4g\tan\theta}{6r - 5a\sin\theta}. \qquad (83)$$

The result of eq. (83) agrees with that in Prob. 9.23 of D. Morin, Introduction to Classical Mechanics *(Harvard U. Press, 2012), noting that our r is his R, our a is his r, and our θ is his $\pi/2 - \theta$.*
http://kirkmcd.princeton.edu/examples/mechanics/morin_9.23.pdf

(d) Instead of using Euler's equations, we can consider a set of principal axes $(1', 2', 3' = 3)$ that don't roll with the hoop, but rather rotate about the vertical $\mathbf{\Omega}$. The $1'$ axis is always horizontal, and $2'$ axis always lies in a vertical plane. Hence, at the moment shown in the figure above, $1' = 1$, $2' = 2$ and $3' = 3$.

The total angular velocity in the lab frame is,

$$\begin{aligned}\boldsymbol{\omega}_{\text{tot}} = \boldsymbol{\omega} + \mathbf{\Omega} = \omega(-\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}) + \Omega\,\hat{\mathbf{z}} = \omega\,\hat{\mathbf{3}}' + \Omega(\cos\theta\,\hat{\mathbf{2}}' - \sin\theta\,\hat{\mathbf{3}}') \\ = \Omega\cos\theta\,\hat{\mathbf{2}}' + (\omega - \Omega\sin\theta)\,\hat{\mathbf{3}}', \end{aligned} \tag{84}$$

while the principal moments of inertia are $I_{1'} = I_{2'} = I_{3'}/2 = kma^2/2$, so angular momentum of the rolling wheel is,

$$\mathbf{L} = \mathsf{I} \cdot \boldsymbol{\omega}_{\text{tot}} = \frac{ma^2}{2}[\Omega\cos\theta\,\hat{\mathbf{2}}' + 2(\omega - \Omega\sin\theta)\,\hat{\mathbf{3}}']. \tag{85}$$

In the (x, y, z) lab frame, the $(1', 2', 3')$ axes precess about the z axis with angular velocity $\mathbf{\Omega}$, while the angular momentum $\mathbf{L}$ is constant in the latter frame. Hence,

$$\boldsymbol{\tau}_{(x,y,z)} = \left.\frac{d\mathbf{L}}{dt}\right|_{(x,y,z)} = \left.\frac{d\mathbf{L}}{dt}\right|_{(1',2',3')} + \mathbf{\Omega} \times \mathbf{L} = \mathbf{\Omega} \times \mathbf{L}. \tag{86}$$

At the moment shown in the figure above we can write the angular momentum (85) as,

$$\begin{aligned}\mathbf{L} &= \frac{kma^2}{2}[\Omega\cos\theta(-\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}) + 2(\omega - \Omega\sin\theta)(-\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})] \\ &= \frac{kma^2}{2}[(\Omega\cos\theta\sin\theta - 2\omega\cos\theta)\,\hat{\mathbf{y}} + (\Omega(\cos^2\theta + 2\sin^2\theta) - 2\omega\cos\theta)\,\hat{\mathbf{z}})], \end{aligned} \tag{87}$$

Then, recalling eq. (74),

$$\boldsymbol{\tau} = ma(g\sin\theta - \Omega^2 b\cos\theta)\,\hat{\mathbf{x}} = \mathbf{\Omega} \times \mathbf{L} = \frac{kma^2\Omega}{2}(2\omega\cos\theta - \Omega\cos\theta\sin\theta)\,\hat{\mathbf{x}}. \tag{88}$$

Finally, we use eq. (73) to obtain,

$$2(g\sin\theta - \Omega^2 b\cos\theta) = 2k\Omega^2(b + a\sin\theta)\cos\theta - ka\Omega^2\cos\theta\sin\theta, \tag{89}$$

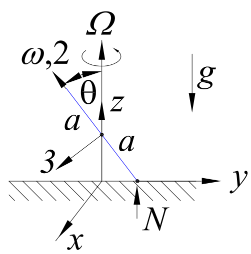
$$\Omega^2 = \frac{2g\tan\theta}{2(1+k)b + ka\sin\theta}, \tag{90}$$

as found in part (c).

(e) A solution without the use of vectors is discussed in sec. 68, p. 162 of H. Lamb, *Higher Mechanics* (Cambridge U. Press, 1920),
http://kirkmcd.princeton.edu/examples/mechanics/lamb_higher_mechanics.pdf

6. **Spinning Coin.**

A thin coin of mass m and radius a spins without slipping on a table, with its center at rest (in the absence of friction) and the plane of the coin at angle θ to the vertical. The point of contact moves in a circle of radius $a \sin\theta$ with angular velocity $\mathbf{\Omega}$ about the vertical.



This is a special case of Prob. 5, with $b = 0$, so from eqs. (81) and (90) we know that $\Omega^2 = 2g/a\cos\theta$ for a hoop (rather than a coin/solid disk).

However, the rolling constraint, eq. (73), which is the same for a hoop and a solid disk, says that the angular velocity about the symmetry axis of the coin is $\Omega\sin\theta$ for $b = 0$, which goes to Ω as $\theta \to 90°$. This does not correspond to the observed slow precession of the figure on the face of the coin.

So, we start a new analysis of the problem, in the lab frame, building on the method of Prob. 5, part (c).

(a) We introduce principal axes $(1, 2, 3)$, with axis 1 always horizontal (out of the page in the figure above), axes 2 and 3 in a vertical plane, and axis 3 along the symmetry axis of the disk. These axes rotate about the vertical through the center of the disk (which is fixed in the lab frame) at constant angular velocity,

$$\mathbf{\Omega} = \Omega\cos\theta\,\hat{\mathbf{2}} - \Omega\sin\theta\,\hat{\mathbf{3}}. \tag{91}$$

In addition, the coin rotates relative to axes $(1, 2, 3)$ at angular velocity $\boldsymbol{\omega}_{\rm rel} = \omega_{\rm rel}\,\hat{\mathbf{3}}$ about axis 3. Thus, the total angular velocity is,

$$\boldsymbol{\omega} = \mathbf{\Omega} + \boldsymbol{\omega}_{\rm rel} = \Omega\cos\theta\,\hat{\mathbf{2}} + (\omega_{\rm rel} - \Omega\sin\theta)\,\hat{\mathbf{3}}. \tag{92}$$

Furthermore, the instantaneous axis (in the direction of the total angular velocity $\boldsymbol{\omega}$) is along axis 2, which passes through the (fixed) center of the disk as well as the point on the disk (instantaneously at rest) in contact with the horizontal plane. Hence,

$$\boldsymbol{\omega} = \omega\,\hat{\mathbf{2}} = \Omega\cos\theta\,\hat{\mathbf{2}}, \qquad \omega_{\rm rel} = \Omega\sin\theta. \tag{93}$$

The angular momentum of the spinning coin (of uniform mass density) is,

$$\mathbf{L} = \mathsf{I}\cdot\boldsymbol{\omega} = I_2\omega\,\hat{\mathbf{2}} = \frac{ma^2}{4}\Omega\cos\theta\,\hat{\mathbf{2}}. \tag{94}$$

The angular momentum precesses about the vertical at angular velocity $\mathbf{\Omega}$, and so the torque equation about the center of mass is,

$$\frac{d\mathbf{L}}{dt} = \mathbf{\Omega} \times \mathbf{L} = \frac{ma^2}{4}\Omega^2 \cos\theta \sin\theta \, \hat{\mathbf{1}} = \boldsymbol{\tau} = -a\,\hat{\mathbf{2}} \times \mathbf{N} = mga \sin\theta \, \hat{\mathbf{1}}, \qquad (95)$$

noting that the force at the point of contact is only the normal force,

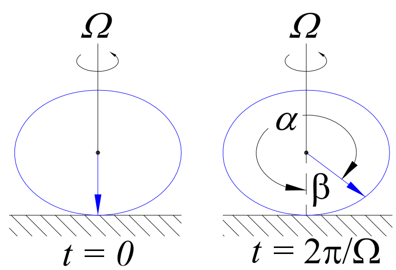
$$\mathbf{N} = mg\,\hat{\mathbf{z}} = mg(\cos\theta\,\hat{\mathbf{2}} - \sin\theta\,\hat{\mathbf{3}}), \qquad (96)$$

since the center of mass of the coin is at rest.

Thus, we have that,

$$\Omega^2(\theta) = \frac{4g}{a\cos\theta}\,. \qquad (97)$$

(b) As the coin rolls without slipping during one revolution, the point of contact on the horizontal surface moves in a circle through distance $2\pi a \sin\theta$.



Meanwhile, the original point of contact on the coin moves the same distance around the edge of the coin, which is less than the circumference $2\pi a$ of the coin. That is, a figure on the coin rotates by angle $\alpha = 2\pi\sin\theta$ in one revolution. Equivalently, an observer could say that the figure has rotated by angle $\beta = 2\pi - \alpha = 2\pi(1 - \sin\theta)$.

The angular velocity of rotation of the figure is,

$$\omega_{\rm figure} = \frac{\beta}{T} = \frac{\beta}{2\pi/\Omega} = \Omega(1 - \sin\theta) = \Omega - \omega_{\rm rel}, \qquad (98)$$

which goes to zero as $\theta \to 90^\circ$ (and $\Omega \to \infty$).

This problem is discussed in Art. 244, p. 196 of E.J. Routh, *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies*, $6^{\rm th}$ ed. (Macmillan, 1905), http://kirkmcd.princeton.edu/examples/mechanics/routh_advanced_rigid_dynamics.pdf
The effect of air resistance is discussed in http://kirkmcd.princeton.edu/examples/rollingdisk.pdf
Air resistance (and rolling friction) cause the spinning coin to fall over, increasing θ and Ω with time. The normal force also becomes time dependent, and eventually makes the disk lose contact with the horizontal surface – before the point of contact has reached the speed of sound.