Princeton University
Ph304 Problem Set 1
Electrodynamics

(Due 5 pm, Tuesday Feb. 11, 2003 in Sullivan's mailbox, Jadwin atrium)

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Problem sessions: Sundays, 7 pm, Jadwin 303

Text: *Introduction to Electrodynamics, 3rd ed.*
by D.J. Griffiths (Prentice Hall, ISBN 0-13-805326-X, now in 6th printing)
Errata at http://academic.reed.edu/physics/faculty/griffiths.html

Reading: Griffiths chap.1 as needed, secs. 2.1-2.3.

1. Griffiths' prob. 1.60.

2. Griffiths' prob. 1.62. In part a), comment on whether $\nabla \cdot r^n \hat{\mathbf{r}}$ is meaningful at the origin, by using the divergence integral theorem for a sphere of radius a, and for a spherical shell of inner radius b and outer radius a.

3. Variant of Griffiths' prob. 2.7. After working Griffiths' prob. 2.7 you are meant to be impressed at how effective Gauss' law (2.13-14) is for problems of high symmetry. But since you already know Gauss' law, it may be more instructive to work a variant: Find the electric potential $V(z)$ relative to infinity everywhere along the axis of symmetry (the z axis) of a HEMISPHERICAL shell of radius R with uniform charge density σ. In particular, what is $V(z = 0)$ at the center of curvature of the shell. Then use eq. (2.23) to find the electric field $E_z(z)$. Show that the value $E_z(z) - E_z(-z)$ based on a hemispherical shell corresponds to the field E_z at a distance z from the center of a uniform spherical shell of charge.

4. Griffiths' prob. 2.18.

5. Griffiths' prob. 2.47.

   The following **digression** is not part of the cirriculum of Ph304, but you might find it interesting.

   Electric potential problems in two dimensions can often be usefully related to functions of a complex variable, $z = x + iy$. In particular, any analytic function $f(z) = u + iv$ obeys

$$i\frac{\partial f}{\partial x} = if'\frac{\partial z}{\partial x} = if' = i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x},$$

$$\frac{\partial f}{\partial y} = f'\frac{\partial z}{\partial y} = if' = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

   Since both of the above lines are equal to if', we can equate their real and imaginary parts to find the so-called Cauchy-Reimann relations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

   Taking second derivates, we also find

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}.$$

   Thus, both functions $u(x, y)$ and $v(x, y)$ obeys Laplace's equation, $\nabla^2 V = 0$ for the electric potential in a charge-free region in two dimensions.

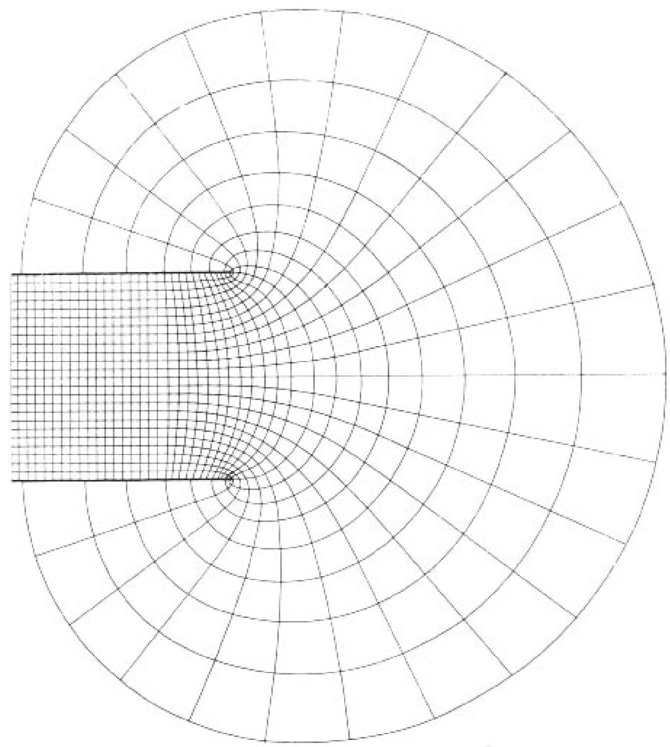
   Hence, any analytic function of a complex variable provides us with not one but two solutions to electrostatics problems. Mathematics hands us the solutions; the game is to figure out what the problem is.....

From the functions u and v we can, of course, deduce the associated electric fields $\mathbf{E}_u = -\nabla u$ and $\mathbf{E}_v = -\nabla v$. In general, lines of electric field are orthogonal to their corresponding equipotential surfaces.

Note that the Cauchy-Riemann equations imply that the lines of the field $\mathbf{E}_u$ are orthogonal to the lines of the field $\mathbf{E}_v$. Hence the equipotentials of field $\mathbf{E}_u$ (= lines of constant u lie along the lines of field $\mathbf{E}_v$, and vice versa.

So the use of complex functions for two dimensional problems can give us quick prescriptions for both equipotentials and field lines.

Example: The function defined by the inverse relation $z = f + e^f$ describes the eqiupotentials and electric fields of a (semi-infinite) parallel plate capacitor. Can you show that the plates are at $-\infty < x < -1$ and $y = \pm\pi$?



From sec. 202 of *A Treatise on Electricity and Magnetism* by J.C. Maxwell.

http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_04.pdf

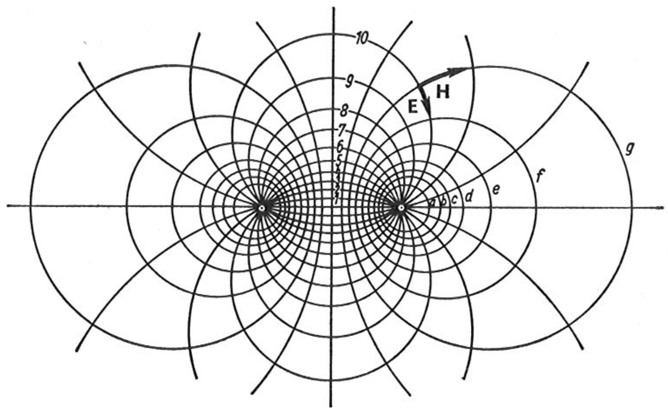
Example: The function $f = -2\lambda \ln(z - z_0)$ describes the potential and field due to a line charge λ located at (x_0, y_0), where, of course, $z_0 = x_0 + iy_0$. From this, we see that the situation of Griffiths' prob. 2.47 is described by the function

$$f(z) = -2\lambda \ln \frac{z-a}{z+a}\,.$$

Then, $Re(f) = V(x, y)$ can be used to show that the equipotentials are circles, AND by considering $Im(f) =$ constant, you can show that the electric fields lines are also circles (which always include the wires at $(\pm a, 0)$. See the figure on p. 3.

6. Griffiths' prob. 2.48. For an additional viewpoint on the Child-Langmuir law, see
   http://kirkmcd.princeton.edu/examples/vacdiode.pdf

It turns out the equipotentials of two wires carrying opposite line charges have the same form as the magnetic field lines of two wires carrying opposite currents:



From sec. 61.2 of *Electromagnetic Fields and Interactions* by R. Becker.

Griffiths' prob. 2.52 is not assigned, but summarizes a famous bit of lore. The derivation of eq. (2.57) given in the book of Smythe is elegantly algebraic.
http://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf
For a highly geometric derivation due to Lord Kelvin, see
http://kirkmcd.princeton.edu/examples/ellipsoid.pdf