Princeton University

Ph304 Problem Set 3
Electrodynamics

(Due 5 pm, Thursday Feb. 27, 2003 in Matt Sullivan's mailbox, Jadwin atrium)

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Problem sessions: Sundays, 7 pm, Jadwin 303

Text: *Introduction to Electrodynamics, 3rd ed.*
by D.J. Griffiths (Prentice Hall, ISBN 0-13-805326-X, now in 6th printing)
Errata at http://academic.reed.edu/physics/faculty/griffiths.html

Reading: Griffiths secs. 3.3-3.4, 4.1-4.2.

1. Griffiths' prob. 3.15.

2. Extended version of Griffiths' prob. 3.23. Show that the general form of solutions to Laplace's equation, $\nabla^2 V = 0$, in cylindrical coordinates (r, ϕ, z) when there is no z dependence is

$$V(r,\phi) = (a_0 + b_0 \ln r)(c_0 + d_0\phi) + \sum_k \left[\left(a_k r^k + \frac{b_k}{r^k} \right) \cos k\phi + \left(c_k r^k + \frac{d_k}{r^k} \right) \sin k\phi \right].$$

If the region of interest includes $0 \le \phi \le 2\pi$, show that the indices k are the positive integers.

Use this expansion to motivate an image method for the problem of a line charge λ per meter at position $(r, \phi) = (b, 0)$ in the presence of a grounded conducting cylinder of radius $a < b$, whose axis is the z axis. We hope that a solution exists in which the effect of the conducting cylinder on the region $r > a$ is the same as that of an imaginary line charge λ' per meter placed at some suitable radius $c < a$.

While you are welcome to carry out a solution in which all the Fourier coefficients are explicitly evaluated, this is not necessary to achieve the goal of deducing the image method. First, write down the form of the potential for the wire alone, for both $r < b$ and $r > b$, noting that some Fourier coefficients must by zero by symmetry arguments. It is helpful to write the radial dependence in terms of the scaled variable r/b rather than r alone. Likewise, write down the form of the potential due to the conducting cylinder for $r > a$, which will have the same symmetries at the potential of the wire, since the potential of the cylinder is induced by that of the wire. Then, since the total potential vanishes at $r = a$, one finds a simple relation between the potential due to the cylinder and that of the wire – namely, an image prescription.

The potential of a 2-dimensional charge distribution not including the z axis must be finite at this axis, but can have a logarithmic divergence as $r \to \infty$. With care you can include terms in $\ln r$ in appropriate places in your expansions of the potentials of the wire and the cylinder, to find that the coefficients of these terms obey the same transformation as do the other terms of the expansions.

Not for credit: you might wish to construct the image method directly, in the spirit of Griffiths' Ex. 3.2. Experts will also note that 2-D solutions to Laplace's equation are expressible in terms of a function of a complex variable, which have useful application to problems of line charges and cylinders. See, for example, Chap. 4 of *Static and Dynamic Electricity* by W.R. Smythe (McGraw-Hill, 1968).
http://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf

3. Griffiths' prob. 3.37. Without loss of generality, set the potential V_0 of the conducting sphere to zero. Griffiths recommends this problem be solved by separation of variables in spherical coordinates. Since the free charge distribution varies as $P_1(\cos\theta)$, the only terms in the Legendre polynomial expansion will be those in P_1. Further, if you write

the potential with normalized radial coordinates, then continuity of the potential at $r = b$ tells us that there are only two unknown expansion coefficients, A_1 and B_1:

$$V(a < r < b) = \left[A_1 \frac{r}{b} + B_1 \left(\frac{b}{r} \right)^2 \right] P_1(\cos\theta),$$

$$V(r > b) = C_1 \left(\frac{b}{r} \right)^2 P_1(\cos\theta) = (A_1 + B_1) \left(\frac{b}{r} \right)^2 P_1(\cos\theta).$$

Then, since $V(a) = 0$, there is really only one unknown....

However, this problem is also susceptible to treatment by the image method! Carefully deduce the image charge density σ' on the appropriate image sphere. As both charge densities σ and σ' vary as $\cos\theta = P_1(\cos\theta)$, both of these charge distributions have only dipole moments. Hence you should be able to quickly write down the potential both for $r > b$, and for $a < r < b$, recalling from Ex. 3.9 that the field inside a $\cos\theta$ charge distribution is uniform.

This last remark offers a third way to solve this problem. Since the electric field must vanish inside the conductor for $r < a$, a surface charge σ'' must be induced at $r = a$ whose interior field exactly cancels the interior field from the given charge distribution at $r = b$. Show that σ'' deduced this way is the same as that obtained from the potential that you have previously calculated.

4. Griffiths' prob. 3.40.

5. Griffiths' prob. 3.41.

6. Griffiths' prob. 3.44. Since conductors are equipotentials, the integrals mentioned in the statement of Green's reciprocity theorem in prob. 3.43 lead to the total charges on the conductors. Hence, this theorem is more often stated as: if a set $\{i\}$ of conductors is at potentials V_i when charges Q_i are placed on them, and instead charges Q_i' would result in potentials V_i', then

$$\sum_i Q_i V_i' = \sum_i Q_i' V_i.$$

A useful trick is to consider a point charge as residing on a tiny conductor, so that in a second scenario with the same conductors, the amount of the charge on this tiny conductor can be set to zero, and this conductor will take on the potential that would exist at the position of the absent charge.

Green's reciprocity theorem can be used to deduce charge distributions on conductors by imagining those conductors to be suitably subdivided. For an example of such a procedure, see prob. 4, Ph501 Set 4, which is presented separately at
http://kirkmcd.princeton.edu/examples/straw.pdf