Princeton University

Ph304 Problem Set 8
Electrodynamics

(Due in class, Wednesday Apr. 9, 2003)

Instructor: Kirk T. McDonald, Jadwin 309/361, x6608/4398
kirkmcd@princeton.edu
http://puhep1.princeton.edu/~mcdonald/examples/

AI: Matthew Sullivan, 303 Bowen Hall, x8-2123
mtsulliv@princeton.edu

Problem sessions: Sundays, 7 pm, Jadwin 303

Text: *Introduction to Electrodynamics, 3rd ed.*
by D.J. Griffiths (Prentice Hall, ISBN 0-13-805326-X, now in 6th printing)
Errata at http://academic.reed.edu/physics/faculty/griffiths.html

Reading: Griffiths chap 8.

Note that in working the extended version of Griffiths' prob. 5.42 (Set 5), you have already worked Griffiths' prob. 8.3.

1. Griffiths' prob. 7.57. Work the new part d) before parts a)-c), then do new part e).

   It is not useful to think of the wire as infinite (since there are no infinities in physics), but rather as having length $L \gg b$ so that we can ignore end effects over most of the length of the wire. For a current I to flow in a wire of nonzero resistivity ρ there must be a power source (battery) AND a return wire to complete the circuit. The question of surface charges is easiest to analyze if the return wire is a tube of very low resistivity that is coaxial with the resistive wire. The batteries then are at one or both ends of the wire.

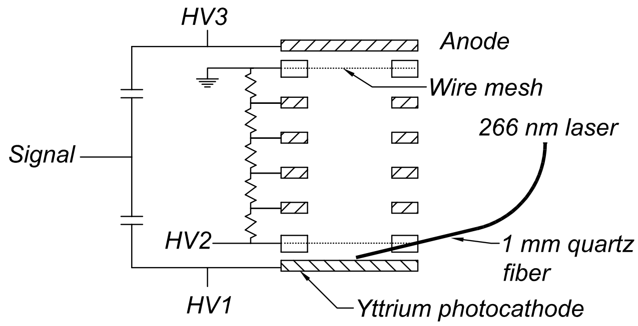
   The resistance per unit length of the inner wire is $R = \rho/\pi a^2$, assuming the current density is uniform. Then the voltage difference between the two ends of the inner wire is $\Delta V = IRL$ when current I is flowing. We take the return wire to be at zero potential.

   For the current to flow in the $+z$ direction, the potential within the inner conductor must vary as $V(s < a, z) = V_0 - IRz$. A general solution can be based on the convention that $V_0 = 0$, *i.e.*, that $V(s < a, z) = -IRz$. The actual wire has a battery at least one end, and possibly a load resistor or "short-circuit" termination at the other. The voltage at the left end, $V_{\rm left}$ and the voltage at the right end, $V_{\rm right}$ obey $V_{\rm left} - V_{\rm right} = IRL$. To be consistent with the convention that $V(z = 0) = 0$, the left end of the wire is at $z = -V_{\rm left}/IR$, and the right end is at $z_{\rm right} = -V_{\rm right}/IR$. A short circuit termination implies that the inner wire has zero potential there, so such a termination must be at $z = 0$. A load resistor $R_{\rm load}$ would require the inner wire to be at potential $\pm IR_{\rm load}$, and hence must be located at $z_{\rm load} = \pm R_{\rm load}/R$ (remember, R is resistance per unit length).

   For example, if the system includes only a single battery at one end and is shorted at the other, this could be realized in the above convention by having the shorted end at $z = 0$ and the battery at $z = L$ with voltage $-IRL$ (at the inner conductor), or the battery at $z = -L$ with voltage IRL. If we suppose instead that there is a battery at each end of the system, each with voltage $IRL/2$ (which is the unstated scenario of Griffiths prob. 7.57) we take the system to extend from $z = -L/2$ to $L/2$.

   For uniform current density $J = I/\pi a^2$ to flow down the inner conductor there must be a uniform electric field of strength $E = IR$ inside the tube of length L and radius a. We could start with conducting electrodes of radius a at the ends of the tube, and place potential difference $\Delta V = IRL$ between them. For the moment, imagine the rest of space is empty. Then, the fringe field of this "capacitor" would result in a very nonuniform electric field inside the tube. We desire the equipotential surfaces for radius $s < a$ to be planes perpendicular to z, and uniformly spaced in z. To achieve this, we might add a set of conducting rings of radius a uniformly spaced along z, setting the rings to potentials that vary linearly with position. The charge on a ring is related by $Q = CV$, where C is the capacitance of the ring. Hence, the charge on the rings, like the voltage, varies linearly with position.

Below is a sketch of a device that I'm building that uses such a set of field shaping rings:



A current-carrying wire has an aspect of the continuum limit of a discrete set of field-shaping rings. Namely, the wire takes on a surface charge that varies linearly with position to create equipotential planes perpendicular to z (inside the wire), and hence a uniform longitudinal electric field inside the wire.

d) Calculate the capacitance per unit length between the wire of radius a and the outer conductor of radius b. Then calculate the surface charge density σ by finding the charge $Q = CV$ per unit length needed to maintain the desired potential $V(a, z) = -IRz$ at the surface of the wire.

For a long wire, the charge $Q(z)$ per unit length varies slowly, and the electric field at a plane of constant z_0 is essentially the same at that for a wire of uniform charge per unit length with the value $Q(z_0)$. Use this assumption to find $V(a < s < b, z)$, and the electric field $\mathbf{E}(a < s < b, z)$.

At positions far from the ends of the wire, the capacitance C per unit length has a well-defined value. Depending on exactly how the ends of the wire are made, C will vary somewhat over distance a few times b from the ends. However, our solution for the potential and charge distribution near the center of the wire is clearly independent of such details.

Now do Griffiths' parts a)-c) to verify the results of part d). The proposed technique is a separation of variables solution to Laplace's equation, which holds in the charge-free region $a < s < b$.

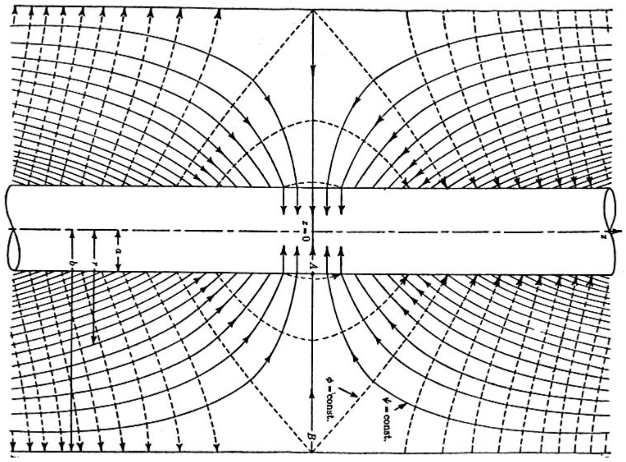
You can obtain a solution based on the convention that $V(s, 0) = 0$ without knowing the precise form of $V(a < s < b)$ at the ends of the wire. This is to be expected since the behavior in the central region of a long system should be insensitive to the details at the ends.

Note that your solution for $V(s, z)$ is consistent with the "natural" boundary condition that at the ends of the system the current flows radially from $s = a$ to b in a resistive material, where current conservation implies that $J_s \propto 1/s$, so Ohm's law requires that $E_s \propto 1/s$, which implies that $V(a < s < b) = A\ln(s/B)$. The boundary conditions that $V(b, z) = 0$ and $V(a, z) = -IRz$ determine constants A and B to be exactly your separation of variables solution at the endpoint z!

This means in practice that if you tried imposing some other boundary condition on the potential at the ends of the system, nature would do its best to ignore you, and after a few times distance b from the ends of the wire the potential would behave again according to the separation of variables solution, since this holds independent of the details of the "missing" boundary condition.

In the language of guided waves, which we will study in sec. 9.5, the zero-frequency "wave" $V(s, z)$ can propagate long distances down the waveguide structure only if it is properly set up at the boundary. Otherwise, it dies out in a few waveguide diameters. Not yet knowing about waveguides, we are solving the problem from the inside out, and by assuming that a potential exists in the center of the structure, we are able predict what it must be like at the distant boundary.

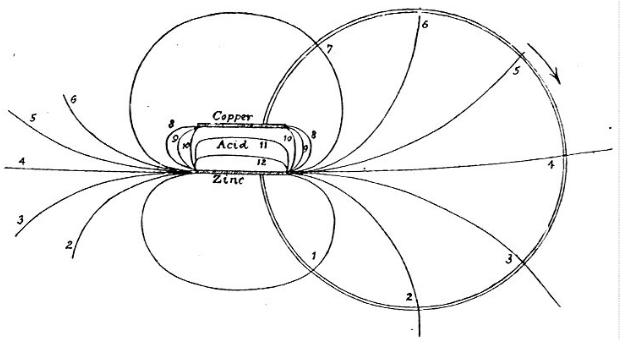
The figure below is from Sommerfeld. The dashed lines are along the electric field, while the solid lines are along the Poynting vector.



Because there is a positive longitudinal electric field inside the wire, and the tangential component of the electric field is continuous at a boundary, the field lines do not leave the surface $s = a$ at right angles. For $z < 0$, the surface charge is positive, so the field lines point outwards from $s = a$ and are bent somewhat towards positive z; for $z > 0$, $\sigma < 0$, so the field lines point inwards towards $s = a$ and again are bent somewhat towards positive z by the requirement of continuity of E_z at $s = a$. For $|z| \lesssim b$, the field lines do not reach the outer conductor, but run from positive charge at $z < 0$ to negative charge at $z > 0$. In a circuit where $b > L$, or in a wire bent into a loop, essentially all field lines outside the conductor behave this way.

e) Assuming that the current density is uniform in the center conductor ($s < a$), it consumes power I^2R per unit length. Calculate the Poynting vector $\mathbf{S}(s, z)$ for $0 < s < b$ and show that the Poynting flux per unit length across the surface $s = a$ is equal to the power consumption I^2R, and that the Poynting flux across a surface of constant $z_0 > 0$ for $a < s < b$ is equal to the power consumed by the center conductor for $0 < z < z_0$. The interpretation is that electromagnetic energy flows from the distant ends of the circuit where the "batteries" are located, down the vacuum (or air or other dielectric) gap and into the wire where it is transformed into heat.

Note that this interpretation holds for both the case of one and two batteries, and with either a short or a resistor at the other end in case of a single battery, once we have properly identified the coordinates $z_{\rm left}$ and $z_{\rm right}$ of the ends of the wire. The case of a circular loop of wire plus a battery was considered, as shown below, in the original paper by J.H. Poynting (1884), available in his *Collected Scientific Papers* (Cambridge U. Press, 1920).



We also realize that the details of the "batteries" cannot perturb our solution for more than a few wire diameters. This problem, though idealized, contains a robust core of truth.

As Sommerfeld has said: "Electromagnetic energy is transported without losses only in nonconductors. 'Conductors' are nonconductors of energy, which is dissipated in Joule heating".

2. Griffiths' prob. 8.4.

3. Griffiths' prob. 8.8.

4. Griffiths' prob. 8.9.

5. Griffiths' prob. 8.11.

6. Griffiths' prob. 8.12.