Ph 406: Elementary Particle Physics
Problem Set 11

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1. Estimate the decay rate $\Gamma_{t\to bW^+}$ of the top quark, and give the order-of-magnitude of the lifetime in seconds.

   If you wish to pursue a more detailed calculation (not required), note that a top quark in a given spin state can decay to a W^+ in two different spin (helicity) states, so the total rate is the sum of that to these two W-states. A good approximation is that $m_b \ll m_W$, but m_W is comparable to m_t. It seems favorable to evaluate the matrix elements directly using the appropriate Dirac spinors, rather than resorting to Feynman's trace tricks.

2. In Set 8, Prob. 5 you considered the cross section for the reaction $e^+e^- \to \nu\overline{\nu}$ in the V$-$A Fermi theory of the weak interaction, which theory is significantly modified at high energy by the existence of the Z^0 boson. Compare the amplitudes for the inverse reaction, $\nu\overline{\nu} \to e^+e^-$, near threshold in the Weinberg-Salam model to those in the Fermi theory to deduce the ratio of the cross sections in these two models. Also, give an expression for the cross section as a function of center-of-mass energy $\sqrt{s} \approx m_Z$ supposing that the reaction is only $\nu\overline{\nu} \to Z^0 \to e^+e^-$.

   You may ignore lepton masses, and consider only lefthanded neutrinos (righthanded antineutrinos). The discussion in the Notes on p. 399, Lecture 22 has some unfortunate typos, which you can correct by noting that the result on p. 216, Lecture 12 should agree with that on p. 210, Lecture 9 when $\sqrt{s} = E_R = m_c$.

3. Now that the Higgs boson, h, has been discovered we optimistically contemplate measurement of the reaction $e + h \to e + Z^0$. Estimate the cross section for this reaction in the center-of-mass frame, and the form of its angular dependence assuming unpolarized initial electrons.

   Since the Higgs particle couples to mass, hee vertices are negligible here, although you might wish to draw the simplest diagrams that include them.

Solutions

1. A simple estimate of the decay rate $\Gamma_{t\to Wb}$ (whose dimension is energy/mass)

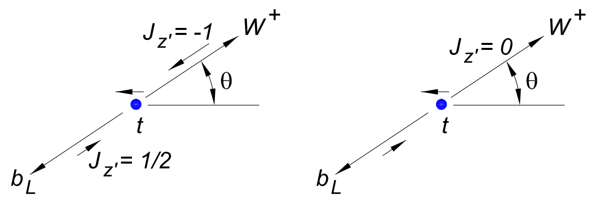
$$\Gamma_{t\to Wb} \approx C^2 m_t \approx g^2 m_t \approx G m_W^2 m_t \approx 10^4 m_p \approx \frac{10^4}{10^{-24}\ \mathrm{s}} = 10^{28}\ \mathrm{s}^{-1}, \tag{1}$$

where the coupling constant at the Wtb_θ vertex is $C = g/2\sqrt{2} = \sqrt{G m_W^2/\sqrt{2}}$, and we recall Set 1, probe. 1. The corresponding lifetime of the top quark, $\approx 10^{-28}$ s is too short for any top-quark hadrons to form.

In more detail, the top quark decays via $t_L \to W^+ b_{\theta,L}$ where the amplitude that the b_θ is a b quark is the C-K-M matrix element $V_{tb} \approx 1$.

A t quark at rest with spin down along the z-axis can be said to have negative helicity (to be lefthanded) if we regard this as the 4-spinor $u_-(p = 0, \theta = 0)$ in the notation of p. 114, Lecture 7 of the Notes. Likewise, a t quark at rest with spin up along the z-axis can be said to be lefthanded) if we regard this as the 4-spinor $u_-(p = 0, \theta = 180°)$. That is, top quarks at rest with either spin projection can decay weakly.

It suffices to consider one case, which we take to have spin down along the z-axis, as shown below.



Taking the z'-axis along the direction of the W, at angle θ to the z=axis, the W has $J_{z'} = -1$ or 0 while the b_L has $J_{z'} = 1/2$, so the final state has $J_{z'} = \mp 1/2$. The angular factor for the decay amplitude is therefore $d^{1/2}_{-1/2,\pm 1/2} = \cos\theta/2$ or $\sin\theta/2$ (while for the case that the top quark has $J_z = 1/2$ the angular factor is $d^{1/2}_{1/2,\pm 1/2} = \sin\theta/2$ or $\cos\theta/2$). The decay of a top quark for a particular spin configuration is parity violating with angular distribution $(1 \pm \cos\theta)/2$, while the decay of an unpolarized top quark is isotropic (in the rest frame of the t).

Turning now to a detailed calculation, we invoke the general formula for a 2-body decay for the process $t \to W + b$,

$$\frac{d\Gamma}{d\Omega} = \frac{P_f \sum_{W\ \mathrm{spins}} |\mathcal{M}|^2}{32\pi^2 m_t^2}, \tag{2}$$

where the square of the 4-vector relation $W = t - b$ yields $m_W^2 = m_t^2 + m_b^2 - 2m_t E_b$, and in the good approximation that $m_b \ll m_W < m_t$, we have that

$$P_f \approx E_b \approx \frac{m_t}{2}\left(1 - \frac{m_W^2}{m_t^2}\right). \tag{3}$$

The matrix element is (see p. 387, Lecture 21 of the Notes)

$$\mathcal{M} = \frac{g V_{tb}}{2}\langle \overline{u}_{b_\theta,L}|\gamma_\mu|u_{t_L}\rangle \epsilon^\mu = \sqrt{G m_W^2 \sqrt{2}}\, V_{tb}\langle \overline{u}_{b_\theta,L}|\gamma_\mu|u_{t_L}\rangle \epsilon^\mu, \tag{4}$$

where ϵ^μ is the polarization 4-vector of the W, for which we need the two cases ϵ_0 (W spin $J_{z'} = 0$) and ϵ_{-1} ($J_{z'} = -1$).

It seems more convenient to work in the (x', y', z') system than in (x, y, z). Recalling p. 367, Lecture 20 of the Notes, the W-polarization 4-vectors are

$$\epsilon_0 = \epsilon_{z'} = (\gamma_W \beta_W, 0, 0, \gamma_W) = \frac{1}{m_W}(P_W, 0, 0, E_W) \approx \frac{m_t}{2m_W}\left(1 - \frac{m_W^2}{m_t^2}, 0, 0, 1 + \frac{m_W^2}{m_t^2}\right) \quad (5)$$

noting that $P_W = P_f$ and the square of the 4-vector relation $b = t - W$ yields $m_b^2 \approx 0 = m_t^2 + m_W^2 - 2m_t E_W$; and ϵ_{-1} is like a negative helicity (right-circularly polarized) photon,[1]

$$\epsilon_{-1} = \frac{\epsilon_{x'} - i\epsilon_{y'}}{\sqrt{2}} = \frac{1}{\sqrt{2}}(0, 1, -i, 0). \quad (6)$$

With respect to the z'-axis, the top-quark spin is at angle $-\theta$, so its spinor is (again recalling p. 114, Lecture 7 of the Notes)

$$|u_{t_L}\rangle = \sqrt{2m_t}\begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \\ 0 \\ 0 \end{pmatrix}. \quad (7)$$

Similarly, the bottom quark has negative helicity with respect to the $-z'$ axis ($\theta' = \pi$), so its spinor is (for $E_b \approx P_f \gg m_b$)

$$|u_{b_{\theta,L}}\rangle = \sqrt{P_f}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \langle \overline{u}_{b_{\theta,L}}| = \langle u^\dagger_{b_{\theta,L}}|\gamma_0 = \sqrt{P_f}\,(0, 1, 0, -1)\,. \quad (8)$$

Recalling that the Dirac matrices are

$$\gamma_0 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ -\boldsymbol{\sigma}_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

[1] See, for example, sec. 7.2 of J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf.

$$\gamma_2 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_2 \\ -\boldsymbol{\sigma}_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_3 \\ -\boldsymbol{\sigma}_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

we find that the matrix element is (see p. 387, Lecture 22 of the Notes)

$$\langle \overline{u}_{b_\theta,L} | \gamma_\mu | u_{t_L} \rangle = \sqrt{2m_t P_f} \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}, -i\sin\frac{\theta}{2}, \cos\frac{\theta}{2} \right). \tag{9}$$

Hence, the matrix element for W-spin $J_{z'} = -1$ is

$$\mathcal{M}_{-1} = \sqrt{Gm_W^2\sqrt{2}}\, V_{tb} \langle \overline{u}_{b_\theta,L} | \gamma_\mu | u_{t_L} \rangle \epsilon_{-1}^\mu = \sqrt{Gm_W^2\sqrt{2}}\, V_{tb} \frac{2\sqrt{2m_t P_f}}{\sqrt{2}} \sin\frac{\theta}{2}, \tag{10}$$

with angular factor $\sin\theta/2$ as anticipated via the rotation matrix. The corresponding differential decay rate is

$$\frac{d\Gamma_{-1}}{d\Omega} = \frac{P_f \left|\mathcal{M}_{-1}\right|^2}{32\pi^2 m_t^2} = \frac{\sqrt{2} G m_W^2 \left|V_{tb}\right|^2 P_f^2}{8\pi^2 m_t} \sin^2\frac{\theta}{2}, \qquad \Gamma_{-1} = \frac{\sqrt{2} G m_W^2 \left|V_{tb}\right|^2 P_f^2}{4\pi m_t}. \tag{11}$$

Similarly, the matrix element for W-spin $J_{z'} = 0$ is

$$\mathcal{M}_0 = \sqrt{Gm_W^2\sqrt{2}}\, V_{tb} \langle \overline{u}_{b_\theta,L} | \gamma_\mu | u_{t_L} \rangle \epsilon_0^\mu = \sqrt{Gm_W^2\sqrt{2}}\, V_{tb} \frac{2\sqrt{2m_t P_f} m_W}{m_t} \cos\frac{\theta}{2}, \tag{12}$$

with angular factor $\sin\theta/2$ as anticipated via the rotation matrix. The corresponding differential decay rate is

$$\frac{d\Gamma_0}{d\Omega} = \frac{P_f \left|\mathcal{M}_0\right|^2}{32\pi^2 m_t^2} = \frac{2\sqrt{2} G m_W^4 \left|V_{tb}\right|^2 P_f^2}{8\pi^2 m_t^3} \cos^2\frac{\theta}{2}, \qquad \Gamma_0 = \frac{2\sqrt{2} G m_W^4 \left|V_{tb}\right|^2 P_f^2}{4\pi m_t^3}. \tag{13}$$

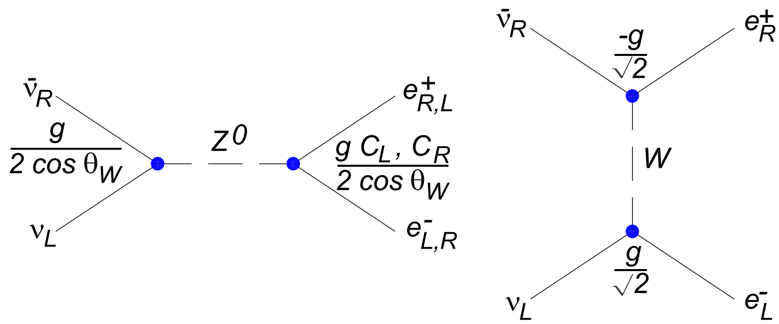
The total decay rate is

$$\begin{aligned} \Gamma_{t\to bW^+} &= \Gamma_{-1} + \Gamma_0 = \frac{\sqrt{2} G m_W^2 \left|V_{tb}\right|^2 P_f^2}{4\pi m_t} \left(1 + \frac{2m_W^2}{m_t^2}\right) \\ &= \frac{\sqrt{2} G m_W^2 m_t \left|V_{tb}\right|^2}{16\pi} \left(1 - \frac{m_W^2}{m_t^2}\right)^2 \left(1 + \frac{2m_W^2}{m_t^2}\right). \end{aligned} \tag{14}$$

The decay rate to d and s quarks has the same form on substitution of $\left|V_{td}\right|^2$ or $\left|V_{ts}\right|^2$ for $\left|V_{tb}\right|^2$; hence the corresponding branching fractions are tiny.

This result was first deduced by I Bigi *et al.*, *Production and Decay Properties of Ultra-Heavy Quarks*, Phys. Lett. **B181**, 157 (1986),
http://kirkmcd.princeton.edu/examples/EP/Big_pl_b181_157_86.pdf.

2. The two "tree-level" graphs for the reaction $\nu\overline{\nu} \to e^+e^-$ are shown below.



The two diagrams for the final state $e^-_L e^+_R$ interfere, so the amplitudes are proportional to

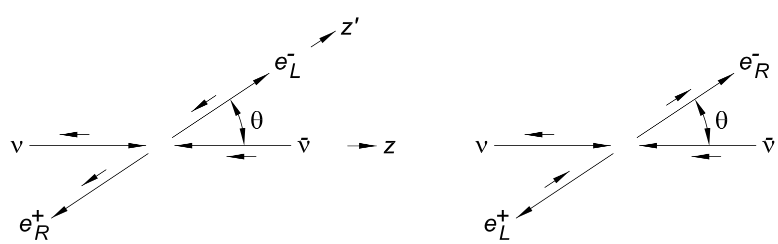
$$A_{\nu\overline{\nu}\to e^-_L e^+_R} \propto \frac{g^2 C_L}{4\cos^2\theta_W}\frac{1}{s-m_z^2} - \frac{g^2}{2}\frac{1}{t-m_W^2}\,, \qquad A_{\nu\overline{\nu}\to e^-_R e^+_L} \propto \frac{g^2 C_R}{4\cos^2\theta_W}\frac{1}{s-m_z^2}\,, \quad (15)$$

where $C_L = 2\sin^2\theta_W - 1 \approx -1/2$, $C_R = 2\sin^2\theta_W \approx 1/2$, referring to p. 388, Lecture 22 of the Notes.

For low center-of-mass energies, $\sqrt{s} \ll m_Z$ and $|t| \ll m_W^2$, we note that $m_Z \cos\theta_W = m_w$, so the amplitudes are approximately

$$A_{\nu\overline{\nu}\to e^-_L e^+_R} \propto -\frac{g^2}{2m_W^2}\left(1+\frac{C_L}{2}\right), \qquad A_{\nu\overline{\nu}\to e^-_R e^+_L} \propto -\frac{g^2 C_R}{4m_w^2}\,. \quad (16)$$

The angular dependence of the amplitudes can be inferred from spinology, as shown in the figure below:



$$A_{\nu\overline{\nu}\to e^-_L e^+_R} \propto d^1_{-1,-1}(\theta) = \frac{1+\cos\theta}{2}\,, \qquad A_{\nu\overline{\nu}\to e^-_R e^+_L} \propto d^1_{-1,1}(\theta) = \frac{1-\cos\theta}{2}\,, \quad (17)$$

so the integrals of the angular distributions are the same for the two cases (although each angular distribution violates parity). Hence, the low-energy cross section is proportional to

$$\left|A_{\nu\overline{\nu}\to e^-_L e^+_R}\right|^2 + \left|A_{\nu\overline{\nu}\to e^-_R e^+_L}\right|^2 \propto G^2\left(1 + C_L + \frac{C_L^2 + C_R^2}{4}\right) \approx \frac{5G^2}{8}\,. \quad (18)$$

In the V−A Fermi theory, $C_L = 0 = C_R$, so the low-energy cross-section in the Weinberg-Salam model is $\approx 5/8$ of that in the Fermi theory (compare Set 8, Prob. 5).

There is no divergence in the cross section $\sigma_{\nu\overline{\nu}\to e^-e^+}$ at low energies, since $E_{\nu,\min} = m_e/2$, for which $v_\nu \approx c$ even for massive neutrinos.

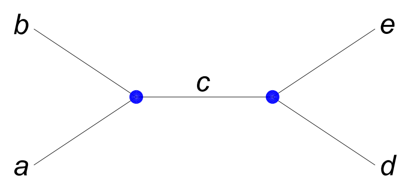
Once the center of mass energy becomes comparable to m_W, the t-channel, W-exchange diagram is negligible compared to the s-channel, Z-exchange diagram. Then, the latter should include the effect of the width of the Z^0, which implies that the (relativistic) Breit-Wigner cross section should be used. Since the discussion of this on p. 399, Lecture 22 of Notes contains various unfortunate typos, I attempt to (re)present the argument here.

The nonrelativistic argument, summarized on p. 210, Lecture 11 of the Notes, is that the cross section for the reaction $a + b \to c \to d + e$ is

$$\sigma_{a+b\to c\to d+e} = \frac{\pi}{P_i^2} \frac{2S_c + 1}{(2S_a + 1)(2S_b + 1)} \frac{\Gamma_{c\to ab}\Gamma_{c\to cd}}{(E_{\rm cm} - m_c)^2 + \Gamma_{c\to \rm all}^2/4}\,, \tag{19}$$

where $E_{\rm cm} = E_a + E_b = \sqrt{s}$.

On p. 216, Lecture 12 of the Notes, we remarked (without "proof") that in quantum field theory we are considering an s-channel process whose diagram is



for which the propagator of particle c, whose total decay rate is $\Gamma_t = \Gamma_{c\to\rm all}$, can be written as $1/[s - (m_c + i\Gamma_t/2)^2] \approx 1/(s - m_c^2 + im_c\Gamma_t)$. This propagator appears in the matrix element, so the cross section is proportional to its absolute square, $1/[(s - m_c^2)^2 + m_c^2\Gamma_t^2]$. The cross section when $E_{\rm cm} = \sqrt{s} = m_c$ should remain the same, so the relativistic form of the Breit-Wigner cross section is

$$\sigma_{a+b\to c\to d+e} = \frac{4\pi s}{P_i^2} \frac{2S_c + 1}{(2S_a + 1)(2S_b + 1)} \frac{\Gamma_{c\to ab}\Gamma_{c\to cd}}{(s - m_c^2)^2 + m_c^2\,\Gamma_t^2}\,, \tag{20}$$

where we insert a factor s in the numerator on dimensional grounds.

Then, the reaction $e^-e^+ \to Z^0 \to$ all has the Breit-Wigner cross section

$$\sigma_{e^-e^+\to Z^0\to\rm all} = 12\pi \frac{\Gamma_{Z^0\to e^-e^+}\Gamma_{Z^0\to\rm all}}{(s - m_Z^2)^2 + m_Z^2\,\Gamma_{Z^0\to\rm all}^2}\,, \qquad \sigma_{\rm peak} = 12\pi \frac{\Gamma_{Z^0\to e^-e^+}}{m_z^2\,\Gamma_{Z^0\to\rm all}}\,. \tag{21}$$

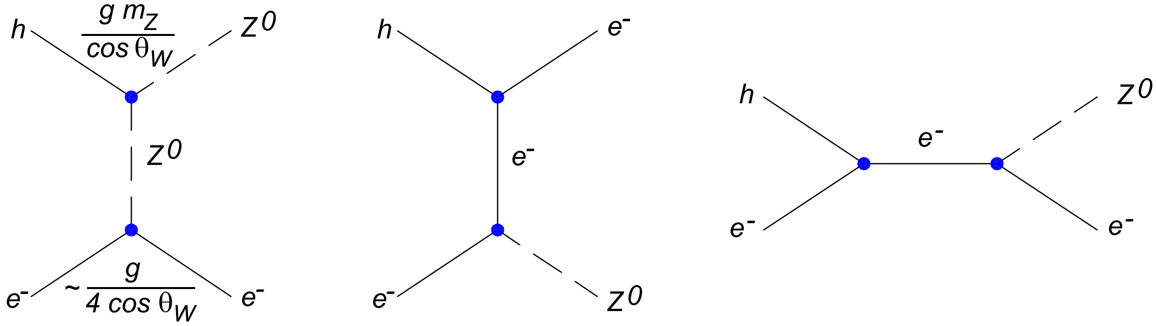
noting that $P_i \approx \sqrt{s}/2$, as quoted on p. 399, Lecture 22 of the Notes.

Returning to the case of $\nu\overline{\nu} \to e^-e^+$, we consider that the spin factor $2S_\nu + 1$ for massless neutrinos is 1, and again $P_i = \sqrt{s}/2$, such that

$$\sigma_{\nu\overline{\nu}\to e^-e^+} \approx \frac{4\pi s}{P_i^2} \frac{2S_Z + 1}{(2S_\nu + 1)^2} \frac{\Gamma_{Z^0\to\nu\overline{\nu}}\Gamma_{Z^0\to e^-e^+}}{(s - m_Z^2)^2 + m_z^2\Gamma_{Z^0\to\rm all}^2} \approx \frac{48\pi\Gamma_{Z^0\to\nu\overline{\nu}}\Gamma_{Z^0\to e^-e^+}}{(s - m_Z^2)^2 + m_Z^2\Gamma_{Z^0\to\rm all}^2}\,, \tag{22}$$

where $m_Z = 91.188$ GeV, $\Gamma_{Z^0\to\nu\overline{\nu}} \approx 0.152$ GeV, $\Gamma_{Z^0\to\nu\overline{\nu}} = 0.084$ GeV and $\Gamma_{Z^0\to\rm all} = 2.495$ GeV.

3. The reaction $e^- + h \to e^- + Z_0$ could proceed via three "tree level" diagrams, as shown below.



Because the Higgs particle h couples to mass, the two diagrams with hee vertices are negligible (at least at low energies) compared to the left one with a hZZ vertex. The vertex factors for the left diagram are taken from pp. 388 and 404, Lecture 22 of the Notes, such that the matrix element is approximately

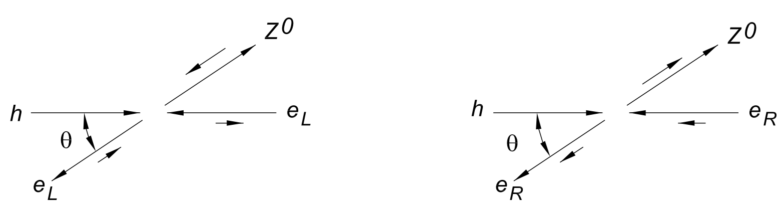
$$\mathcal{M} \approx \frac{g^2 m_Z}{\cos^2\theta_W m_Z^2} \approx \frac{g^2}{m_Z} \approx G\,, \tag{23}$$

where the factor of m_Z^2 in the denominator is from the propagator for Z-exchange at low q^2. Since G has dimensions of $1/E^2$, the cross section must vary as

$$\sigma_{eh\to eZ} \propto |\mathcal{M}|^2 \approx G^2 s. \tag{24}$$

Presumably, the contribution of the two diagrams with electron exchange keeps the cross section finite at high energies.

As to the angular dependence of the cross section, we note that the Z_0 couples to either left- or right-handed electrons, as sketched in the figure below. For coupling to e_L, the spin along the axis of the initial state is $S_z = -1/2$ and that along the axis of the final state is $S_{z'} = 1/2$, taking the axis in the direction of motion of the electron. Hence the amplitude for this case is proportional to $d^{1/2}_{-1/2,1/2} = -\sin\theta/2$. Similarly, for coupling to e_R, the spin along the axis of the initial state is $S_z = 1/2$ and that along the axis of the final state is $S_{z'} = -1/2$, taking the axis in the direction of motion of the electron. Hence the amplitude for this case is proportional to $d^{1/2}_{1/2,-1/2}(\theta) = \sin\theta/2$.



In both cases, the cross section varies as $\sin^2\theta/2 = (1-\cos\theta)/2$ (since the effect of the propagator, $1/(q^2 - m_Z^2) \approx -1/m_Z^2$ on the angular distribution is negligible, and for all t-channel weak-interaction diagrams).