Ph 406: Elementary Particle Physics
Problem Set 12

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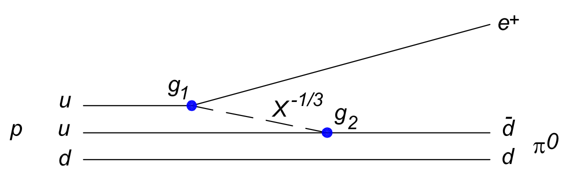
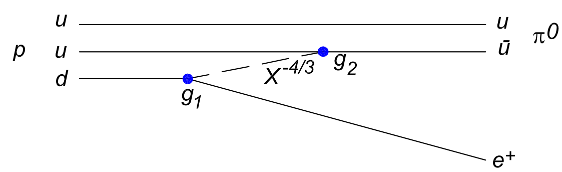
1. **Proton and Neutron Decay.** The emerging understanding that the strong force becomes weaker at high energies while the electromagnetic force grows stronger, such that $\alpha_s \approx \alpha_{\rm EM}$ at $E \approx 10^{15}$ GeV, led[1] to the conjecture of a "grand unification" of strong, electromagnetic (and weak) forces, associated with various new bosons of mass $\approx 10^{15}$ GeV that couple quarks and leptons. Such bosons (with fractional electric charge $1/3e$ and $-4/3e$, *etc.*) might mediate baryon-number- and lepton-number-violating transitions, as sketched below, such that a proton could decay via $p \to e^+\pi^0$.





The coupling constants g_1 and g_2 might have strengths at energy ≈ 1 GeV such that $g_1^2 \approx \alpha_{\rm EM}$ and $g_2^2 \approx \alpha_s$.

Make a simple estimate of the proton lifetime based on the above model.

Give a similar simple estimate for (3-body) neutron decay, $n \to pe^-\overline{\nu}_e$, whose observed lifetime is ≈ 1000 s.

2. **The M-S-W Effect.** An interesting phenomenon in the K^0-$\overline{K}^0$ system is that the difference in the interaction of a K^0 and a $\overline{K}^0$ with matter permits regeneration of the mass/lifetime eigenstate K_S^0 by a block of matter placed in a beam of Kaons where the K_S^0 component has essentially died out and only K_L^0 remain.[2] Use of two regenerators with variable separation permits measurement of the magnitude of the K_S^0-K_L^0 mass difference,[3,4] and if the two regenerators are placed close to the Kaon source so that the

[1]J.C. Pati and A. Salam, *Lepton number as the fourth "color,"* Phys. Rev. D **10**, 275 (1974), http://kirkmcd.princeton.edu/examples/EP/pati_prd_10_275_74.pdf
H. Georgi and S.L. Glashow, *Unity of All Elementary-Particle Forces*, Phys. Rev. Lett. **32**, 438 (1974), http://kirkmcd.princeton.edu/examples/EP/georgi_prl_32_438_74.pdf.

[2]A. Pais and O. Piccioni, *Note on the Decay and Absorption of the θ^0*, Phys. Rev. **100**, 1487 (1955), http://kirkmcd.princeton.edu/examples/EP/pais_pr_100_1487_55.pdf.

[3]M.L. Good, *Method for Determining the $K_+^0 - K_-^0$ Mass Difference*, Phys. Rev. **110**, 550 (1958), http://kirkmcd.princeton.edu/examples/EP/good_pr_110_550_58.pdf.

[4]R.H. Good *et al.*, *Regeneration of Neutral K Mesons and Their Mass Difference*, Phys. Rev. **124**, 1223 (1961), http://kirkmcd.princeton.edu/examples/EP/good_pr_124_1223_61.pdf;
V.L. Fitch *et al.*, *Mass Difference of Neutral K Mesons*, Nuovo Cim. **22**, 1160 (1961), http://kirkmcd.princeton.edu/examples/EP/fitch_nc_22_1160_61.pdf.

K_S^0 component is still significant at the first regenerator, the sign of the mass difference can be determined.[5]

In 1978, Wolfenstein noted[6] that an electron (anti)neutrino can interact with matter via diagrams with either W or Z^0 exchange, while ν_μ and ν_τ can only interact via Z^0 exchange, which leads to "matter effects" for neutrinos that are equivalent to regeneration in the K^0-$\overline{K}^0$ system.

In this problem you should deduce some consequences of these matter effects in a two-neutrino scenario, supposing that an electron or muon (anti)neutrino is produced at time $t = 0$ with a definite momentum, $m_W \gg P_\nu \gg m_i$, such that the neutrino-mass eigenstates ν_1 and ν_2 have energies $E_i \approx P + m_i^2/2P \approx P + m_i^2/2E$ for $i = 1, 2$.[7] Note that

$$E \equiv \frac{E_1 + E_2}{2}, \qquad E_{1,2} \approx E \pm \frac{\Delta m^2}{4E}, \qquad \text{with} \qquad \Delta\mathrm{m}^2 = \Delta\mathrm{m}_{12}^2 \equiv \mathrm{m}_1^2 - \mathrm{m}_2^2. \tag{1}$$

The goal of this exercise is to display how the oscillations of a beam of initial muon neutrinos that propagates through uniform matter are sensitive to the sign of Δm_{12}^2.

The task is to identify the mass eigenstates of neutrinos propagating inside matter. An approximate analysis of the neutrino wavefunction, $|\psi(t)\rangle = a_e(t)|\nu_e\rangle + a_\mu(t)|\nu_\mu\rangle = b_1(t)|\nu_1\rangle + b_2(t)|\nu_2\rangle$, is that it obeys the Schrödinger equation $i\dot{\psi} = H\psi = (H_0 + H')\psi$, where the Hamiltonian (a 2×2 matrix) H_0 applies for propagation in vacuum, and H' describes the additional effect of propagation through matter. Of course, the amplitudes in the flavor basis and the mass basis are related by

$$\begin{pmatrix} a_e \\ a_\mu \end{pmatrix} = U \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \tag{2}$$

where $\theta = \theta_{12}$ is the neutrino-mixing angle.

Problem: Give the forms of the vacuum Hamiltonian H_0 in the mass basis and in the flavor basis for a neutrino of momentum P. Show that in the flavor basis H_0 is the average energy E times the unit matrix $\mathbf{I}$ plus $\Delta m^2/4E$ times a symmetric matrix that is a function of 2θ. Note a relation between the mixing angle and all four of the

[5] J.V. Jovanovich *et al.*, *Experiment on the Sign and Magnitude of the K_L^0-K_S^0 Mass Difference*, Phys. Rev. Lett. **124**, 1223 (1961), http://kirkmcd.princeton.edu/examples/EP/jovanovich_prl_17_1075_66.pdf;
W.A.W. Melhop *et al.*, *Interference between Neutral Kaons and Their Mass Difference*, Phys. Rev. **172**, 1613 (1968), http://kirkmcd.princeton.edu/examples/EP/melhop_pr_172_1613_68.pdf.

[6] L. Wolfenstein, *Neutrino oscillations in matter*, Phys. Rev. D **17**, 2369 (1978), http://kirkmcd.princeton.edu/examples/neutrinos/wolfenstein_prd_17_2369_78.pdf;
Neutrino oscillations and stellar collapse, Phys. Rev. D **20**, 2634 (1979), http://kirkmcd.princeton.edu/examples/neutrinos/wolfenstein_prd_20_2634_79.pdf.

[7] For aspects of a more detailed description in which the neutrino is produced in an entangled state such that energy is conserved during neutrino oscillations, see K.T. McDonald, *Do Neutrino Oscillations Conserve Energy?* (July 21, 2013), http://kirkmcd.princeton.edu/examples/neutrino_osc.pdf.

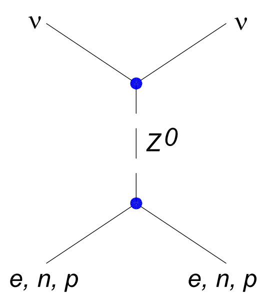
elements of the matrix H_0, which relation can be used to identify the effective mixing angle θ' for the full Hamiltonian H once you have found this.

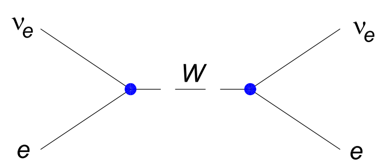
Next, we need to add the effect of neutrino interactions with matter. One way of thinking about this is that the matter interactions scatter neutrinos out of the beam, such that the beam is attenuated, which effect seems equivalent to neutrino decay. However, another view is that the matrix elements of the Hamiltonian (in, say, the flavor basis) represent the transition amplitude that a neutrino of one flavor becomes a neutrino of another (or the same) flavor. Since a matter interaction cannot (directly) change the flavor of a neutrino, we can/should consider the amplitude that after a matter interaction the neutrino of a given flavor remains one of the same flavor.

These two viewpoints are related by the so-called optical theorem[8] that the total scattering cross section is proportional to the imaginary part of the forward elastic scattering cross section.

A technical detail that has been omitted in this course is that the propagators in Feynman diagrams include a factor i, such that the amplitudes that we have discussed for, say, elastic scattering are actually imaginary, as to be consistent with the optical theorem.

Then, the matrix H' in the Hamiltonian $H_0 + H'$ that characterizes the matter effects (in the flavor basis) has $H'_{e\mu} = H'_{\mu e} = 0$, and H'_{ee} and $H'_{\mu\mu}$ as the amplitudes from the Feynman diagrams for forward elastic scattering of the neutrinos off the electrons, neutrons, and protons in matter.





Consider low energies, where the term q^2 in the propagators is negligible compared to $m^2_{W,Z}$, and note that lefthanded protons and neutrons form a weak doublet with weak isospin $I_3 = 1/2$ and weak hypercharge $Y = 1$, while righthanded protons have weak $I_3 = 0$, weak $Y = 2$ and righthanded neutrons have weak $I_3 = 0$, weak $Y = 0$.

The forward scattering amplitudes are then proportional to the sum of amplitudes for scattering by e, p and n (since in forward scattering one can't tell while particle did the scattering; the forward scattering amplitude is coherent), where each subamplitude is the product of the number density N/cm^3 of scatterers, the vertex factors, and the (simplified) propagator. Recall (p. 388, Lecture 22 of the Notes) that introducing the

[8]A historical survey is given in R.G. Newton, *Optical theorem and beyond*, Am. J. Phys. **44**, 639 (1976), http://kirkmcd.princeton.edu/examples/EM/newton_ajp_44_639_76.pdf.
For discussion of the optical theorem in classical electromagnetism, see sec. 10.11 of J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf.

coupling factors C_L and C_R permits summing over left- and righthanded states j in the amplitudes, so the vertex factors for Zjj coupling are proportional to the relevant sum $C_L + C_R$.

Problem: Deduce the form of the (diagonal) transition Hamiltonian H', and give an expression for the effective mixing angle θ' for ν_e and ν_μ that propagate through uniform-density matter. Note that this angle depends on the sign of Δm^2, which could thereby be measured from observation of oscillations of neutrinos that propagate through the Earth from a source to a remote detector, both on the surface. Estimate the characteristic distance scale (oscillation length $L_{\rm matter}$) over which matter effects become prominent, noting that this length $\propto 1/\text{energy} \approx 1/H'$.

Your result should indicate that the effective mixing angle θ' goes to zero in very high-density matter (a kind of "quantum watchdog" effect[9]), and that there exists a "resonance" condition for the matter density at which $\theta' = \pm 45^\circ$. For example, with $E = 3$ MeV as representative of solar or reactor (anti)neutrinos, and the empirical results that $\Delta m^2_{12} \approx 8 \times 10^{-5}$ eV^2, $\sin^2 2\theta_{12} = 0.86$, $\cos 2\theta_{12} \approx 0.4$, the "resonance" occurs (I believe) for density similar to that of gold. These effects complicate the analysis of oscillations of neutrinos produced in stars, as pointed out by Mikheyev and Smirnov.[10]

3. **The See-Saw Mechanism.** An appealing qualitative "explanation" of why the observed neutrinos have very small mass is that each has a partner in some grand-unified theory, and in principle there could be oscillations between these two states governed by a mass matrix (Hamiltonian) of the form

$$\begin{pmatrix} m_1 & m_{12} \\ m_{12} & m_2 \end{pmatrix}, \tag{3}$$

where the "ideal" mass m_1 of the ordinary neutrino could be 0 while that of the grand-unified partner is m_2, and m_{12} describes the coupling between the two "ideal" states. Deduce the "physical" mass eigenvalues m_ν and m'_ν in the approximation that $m_1 \ll m_{12} \ll m_2$, and that the coupling m_{12} has some effect on m_ν. What is the resulting m_ν for $m_2 \approx 10^{15}$ GeV, the grand-unified energy scale, and $m_{12} \approx m_{\rm Higgs}$ as representative of the electroweak energy scale.

The "discovery" of the see-saw mechanism is attributed to various people. The earliest paper I have found wth some form of it is H. Fritzsch, M. Gell-Mann and P. Minkowski, *Vectorlike Weak Currents and New Elementary Fermions*, Phys. Lett. B **59**, 256 (1975), http://kirkmcd.princeton.edu/examples/EP/fritzsch_pl_59b_256_75.pdf. see also, H. Fritzsch and P. Minkowski, *Vectorlike Weak Currents, Massive Neutrinos, and Neutrino Beam*

[9] K. Kraus, *Measuring Processes in Quantum Mechanics I. Continuous Observation and the Watchdog Effect*, Found. Phys. **11**, 549 (1981), http://kirkmcd.princeton.edu/examples/QM/kraus_fp_11_549_81.pdf.

[10] S.P. Mikheyev and A.Y. Smirnov, *Resonant Amplification of ν Oscillations in Matter and Solar-Neutrino Spectroscopy*, Nuovo Cim. **9C**, 17 (1986), http://kirkmcd.princeton.edu/examples/neutrinos/mikheyev_nc_9c_17_86.pdf.

Oscillations, Phys. Lett. B **62**, 72 (1976),
http://kirkmcd.princeton.edu/examples/EP/fritzsch_pl_62b_72_76.pdf. The term "see-saw" seems not to have been used in formal papers prior to 1986.

Solutions

1. **Proton and Neutron Decay.** A simple estimate for the proton-decay rate, based on the diagrams given in prob. 1, is that the decay matrix element is proportional to $g_1 g_2/m_X^2$, and hence

$$\Gamma_{p\to\mathrm{all}} \approx \frac{g_1^2 g_2^2}{m_X^4} m_p^5 \approx \frac{\alpha_{\mathrm{EM}}\alpha_s}{10^{60}}\ \mathrm{GeV} \approx \frac{10^{-63}}{10^{-24}\ \mathrm{s}} = 10^{-39}\ \mathrm{s}^{-1} \approx 10^{-32}\ \mathrm{year}^{-1}, \tag{4}$$

recalling Set 1, Prob. 1, and that a year has $\approx \pi \times 10^7$ s. That is, we estimate the proton lifetime in a grand-unified theory to be about 10^{32} years.

A similar simple estimate for neutron decay via the weak interaction $n \to pe^-\overline{\nu}_e$ whose 3-body final state has $Q \approx m_n - m_p - m_e \approx m_e$ is that

$$\Gamma_{n\to pe^-\overline{\nu}_e} \approx G^2 Q^5 \approx 10^{-10}\frac{Q^5}{m_p^5} m_p \approx 10^{-10}\cdot 10^{-17}\ \mathrm{GeV} \approx \frac{10^{-27}}{10^{-24}\ \mathrm{s}} = 10^{-3}\ \mathrm{s}^{-1}, \tag{5}$$

such that the neutron lifetime is be about 10^3 s, in reasonable agreement with observation.

2. **The M-S-W Effect.** The Hamiltonian H_0 for neutrinos $\nu_{1,2}$ of definite mass and definite momentum P in vacuum is, of course, the diagonal matrix

$$H_{0,\mathrm{mass}} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \approx \begin{pmatrix} E + \frac{\Delta m^2}{4E} & 0 \\ 0 & E - \frac{\Delta m^2}{4E} \end{pmatrix}, \tag{6}$$

where

$$E_i \approx P + \frac{m_i^2}{2E}, \qquad E \equiv \frac{E_1+E_2}{2}, \qquad E_{1,2} \approx E \pm \frac{\Delta m^2}{4E}, \qquad \Delta m^2 \equiv m_1^2 - m_2^2. \tag{7}$$

such that in vacuum, where $\psi_{\mathrm{flavor}} = U\psi_{\mathrm{mass}}$, and

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{8}$$

the time evolution of a neutrino state ψ_{mass} in the mass basis is

$$i\dot{\psi}_{\mathrm{mass}} = i\,U^{-1}\dot{\psi}_{\mathrm{flavor}} = H_{0,\mathrm{mass}}\psi_{\mathrm{mass}} = H_{0,\mathrm{mass}}U^{-1}\dot{\psi}_{\mathrm{flavor}}. \tag{9}$$

That is, in the flavor basis,

$$i\dot{\psi}_{\mathrm{flavor}} = UH_{0,\mathrm{mass}}U^{-1}\dot{\psi}_{\mathrm{flavor}} = H_{0,\mathrm{flavor}}\dot{\psi}_{\mathrm{flavor}}, \tag{10}$$

with

$$\begin{aligned}
H_{0,\text{flavor}} &= U H_{0,\text{mass}} U^{-1} \\
&\approx \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} E_1 \cos^2\theta + E_2 \sin^2\theta & -(E_1 - E_2)\cos\theta\sin\theta \\ -(E_1 - E_2)\cos\theta\sin\theta & E_2\cos^2\theta + E_1\sin^2\theta \end{pmatrix} \\
&= E\,\mathbf{I} + \frac{\Delta m^2}{4E} \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}. \qquad (11)
\end{aligned}$$

We note that the mixing angle θ is related to the matrix elements by

$$\tan 2\theta = \frac{H_{0,12} + H_{0,21}}{H_{0,22} - H_{0,11}}. \qquad (12)$$

Turning to forward elastic scattering, ν_e can scatter off (lefthanded) electrons via W exchange with amplitude

$$A_{\nu_e e \to W \to \nu_e e} \propto n_e \frac{g}{2\sqrt{2}} \frac{1}{M_W^2} \frac{g}{2\sqrt{2}} = \frac{N_e g^2}{8 M_W^2} = \frac{N_e G_F}{\sqrt{2}}. \qquad (13)$$

Both ν_e and ν_μ can scatter off (lefthand and righthanded) electrons, protons and neutrons via Z^0 exchange. In all cases the $Z\nu\nu$ vertex factor is $g/4\cos\theta_W$, and the propagator is (approximately) $1/m_Z^2$. The remaining vertex factors have the form

$$\frac{g(C_L + C_R)}{4\cos\theta_W}, \qquad \text{where} \qquad \mathrm{C_{L,R}} = \cos^2\theta_\mathrm{W} \mathrm{I_{3,L,R}} - \sin^2\theta_\mathrm{W} \frac{\mathrm{Y_{L,R}}}{2}. \qquad (14)$$

That is, the subamplitudes for scattering via Z^0 exchange have the form (p. 388, Lecture 22 of the Notes)

$$\frac{N g^2 (C_L + C_R)}{(4\cos\theta_W)^2 M_Z^2} = \frac{N G_F (C_L + C_R)}{2\sqrt{2}}, \qquad (15)$$

recalling that $\cos^2\theta_W M_Z^2 = M_W^2$. Lefthanded electrons and muons have weak $I_3 = -1/2$ and weak $Y = -1$, while righthanded electrons and muons have weak $I_3 = 0$ and weak $Y = -2$, such that

$$C_{L,e,\mu} + C_{R,e,\mu} = \cos^2\theta_W(-1/2 + 0) - \sin^2\theta_W \frac{(-1-2)}{2} = -\frac{1}{2} + 2\sin^2\theta_W. \qquad (16)$$

Lefthanded protons have weak $I_3 = 1/2$ and weak $Y = 1$, while righthanded protons have weak $I_3 = 0$ and weak $Y = 2$, such that

$$C_{L,p} + C_{R,p} = \cos^2\theta_W(1/2 + 0) - \sin^2\theta_W \frac{(1+2)}{2} = \frac{1}{2} - 2\sin^2\theta_W. \qquad (17)$$

Lefthanded neutrons have weak $I_3 = -1/2$ and weak $Y = 1$, while righthanded neutrons have weak $I_3 = 0$ and weak $Y = 0$, such that

$$C_{L,n} + C_{R,n} = \cos^2\theta_W(-1/2 + 0) - \sin^2\theta_W \frac{(1 + -)}{2} = -\frac{1}{2}\,. \tag{18}$$

Ordinary matter is electrically neutral, $n_e = n_p$, so the amplitudes cancel for neutrinos scattering off electrons and protons via Z^0 exchange, and the only Z^0-exchange amplitude is that due to neutrino-neutron scattering,

$$A_{\nu_{e,\mu}n\to\nu_{e,\mu}n} \propto -\frac{N_n G_F}{4\sqrt{2}}\,. \tag{19}$$

Altogether, the Hamiltonian $H = H_0 + H'$ has the form (in the flavor basis)

$$H_{\rm flavor} = \left(E - \frac{N_n G_F}{4\sqrt{2}}\right)\mathbf{I} + \frac{\Delta m^2}{4E}\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix} + \begin{pmatrix} \frac{N_e G_F}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{20}$$

The mixing angle θ' for neutrinos propagating through uniform matter is

$$\tan 2\theta' = \frac{H_{12} + H_{21}}{H_{22} - H_{11}} = \frac{\Delta m^2 \sin 2\theta}{\Delta m^2 \cos 2\theta - 2\sqrt{2}N_e G_F E} = \frac{\tan 2\theta}{1 - 2\sqrt{2}N_e G_F E/\Delta m^2 \cos 2\theta}. \tag{21}$$

Hence, the effective mixing angle observed in experiments involving neutrino propagation (over sufficient distance) in matter is larger or smaller than the mixing angle θ as Δm^2 is positive or negative.

Since energy $\propto$ 1/length, the length scale over which matter effects have a notable impact on neutrino oscillations is

$$\begin{aligned} L_{\rm matter} &\approx \frac{1}{H'_{11}} \approx \frac{1}{N_e G_F} \approx \frac{\text{cm}^3 \cdot \text{GeV}^2}{6\times 10^{23}\cdot 10^{-5}} \cdot \frac{1}{(200\ \text{MeV}\cdot\text{f})^2} \\ &= \frac{\text{cm}^3\cdot\text{MeV}^2}{6\times 10^{12}} \cdot \frac{1}{4\times 10^{-22}\ \text{MeV}^2\cdot\text{cm}^2} = 4\times 10^8\ \text{cm}^2 = 4000\ \text{km}, \end{aligned} \tag{22}$$

independent of the neutrino energy, assuming that $N_e \approx N_{\rm Avagadro}/\text{cm}^3$ as holds approximately for matter in the Earth (and for the Sun on average, while the density at the center of the Sun is about 100 times larger).

While eq. (21) indicates that the effective mixing angle θ' is essentially zero for very large electron density N_e, there is a "resonance" condition for any given neutrino energy E such that $\theta' = \pm 45^\circ$. For example, with $E = 3$ MeV as representative of solar or reactor (anti)neutrinos, and the empirical results that $\Delta m^2_{12} \approx 8\times 10^{-5}$ eV^2, $\sin^2 2\theta_{12} = 0.86$, $\cos 2\theta_{12} \approx 0.4$, the "resonance" occurs for

$$N_e = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}G_F E} \approx \frac{8\times 10^{-5}\ \text{eV}^2\cdot 0.4}{3\cdot 10^{-5}\ \text{GeV}^{-2}\cdot 3\ \text{MeV}} \approx \frac{4\times 10^{-7}\ \text{MeV}^3}{(200\ \text{MeV-f})^3} \approx 5\times 10^{25}\ \text{cm}^{-3}, \tag{23}$$

which is similar to the electron density in gold.

3. **The See-Saw Mechanism.** The eigenvalues λ of the mass matrix (3) are the roots of the determinant equation,

$$\begin{vmatrix} m_1 - \lambda & m_{12} \\ m_{12} & m_2 - \lambda \end{vmatrix} = \lambda^2 - (m_2 + m_1)\lambda + m_1 m_2 - m_{12}^2 = 0. \tag{24}$$

Then, for $m_1 \ll m_{12} \ll m_2$,

$$\begin{aligned} \lambda &= \frac{m_2 + m_1 \pm \sqrt{(m_2 + m_1)^2 - 4m_1 m_2 + 4m_{12}^2}}{2} \\ &= \frac{m_2 + m_1 \pm \sqrt{(m_2 - m_1)^2 + 4m_{12}^2}}{2} \approx \frac{m_2 + m_1}{2} \pm \frac{m_2 - m_1}{2}\left(1 + \frac{2m_{12}^2}{(m_2 - m_1)^2}\right) \\ &\approx m_2,\ m_1 - \frac{m_{12}^2}{m_2}\,. \end{aligned} \tag{25}$$

The mass of the heavier state is essentially unaffected, $m_{\nu'} \approx m_2$, while the mass of the "ordinary" neutrino, $m_\nu \approx m_1 - m_{12}^2/m_2$, is affected by the coupling m_{12} only if $m_1 \approx m_{12}^2/m_2 \approx m_\nu$.

The "prediction" here is both a bit tentative and qualitative, with
$m_\nu \approx (100\ \text{GeV})^2/10^{15}\ \text{GeV} = 10^{-11}\ \text{GeV} = 0.01\ \text{eV}$,
for $m_2 = 10^{15}$ GeV and $m_{12} = 100$ GeV.

This prescription as to how a very light mass might appear in a theory involving two larger, and disparate mass scales is generically called the "see-saw" mechanism, and many variants appear in the literature.

This elementary analysis does not reveal why it may be essential that the electrically neutral particles 1 and 2 be Majorana states.