# Ph 406: Elementary Particle Physics
# Problem Set 5

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1. **Positronium**

   The concept of positronium, an atom made of an electron and a positron, was invented by Princetonian John Wheeler – who called it a polyelectron.[1]

   What symmetry principles would be violated if the following 1-photon transitions between excited states $n^{2s+1}L_j$ of positronium were observed (via microwave pumping)?

   (a) $1^3S_1 \to 2^3S_1$
   (b) $1^3S_1 \to 2^1S_0$
   (c) $1^3S_1 \to 2^1P_1$
   (d) $1^3S_1 \to 2^3P_1$

   Polarized $1^3S_1$ positronium can be formed when positrons from $^{22}$Na $\beta$-decay combine with atomic electrons. This state decays to 3 photons (do you recall why 2-photon decay is forbidden?). Let $\hat{\mathbf{k}}_1$ and $\hat{\mathbf{k}}_2$ be the directions of the two higher-energy decay photons, and $\hat{\mathbf{S}}$ be the direction of the positronium spin. What symmetries would be violated by angular correlations of the following forms?

   (e) $\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1 \times \hat{\mathbf{k}}_2$
   (f) $\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1$
   (g) $(\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1)(\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1 \times \hat{\mathbf{k}}_2)$
   (h) $\hat{\mathbf{S}} \cdot \hat{\boldsymbol{\epsilon}}_1 \times \hat{\mathbf{k}}_2$

   where $\hat{\boldsymbol{\epsilon}}_1$ is along the direction of polarization of final-state photon 1.

2. $\Lambda^0$ hyperons are produced by a pion beam in the reaction $\pi^- p \to K^0\Lambda^0$, and observed by the decay $\Lambda^0 \to p\pi^-$ (which is a weak interaction that does not conserve parity). Let $J$ denote the spin of the $\Lambda$ (considered to be unknown in this problem, while the spins of the $\pi^-$, $p$ and $K^0$ are known), and $\theta$ be the angle of a decay product in the $\Lambda$ rest frame, relative to the direction of the $\Lambda$ in the lab frame. In the case where the $\Lambda$ is produced exactly along the beam direction, what are the possible values of $J_z$?

[1] J.A. Wheeler, *Polyelectrons*, Ann. N.Y. Acad. **48**, 219 (1946), http://kirkmcd.princeton.edu/examples/QM/wheeler_anyas_48_219_46.pdf.

Show that for unpolarized beam protons, and for Λ's produced along the beam direction, the Λ-decay angular distribution depends on $J$ according to

$$\begin{aligned} J &= 1/2, \quad \text{isotropic}, \\ J &= 3/2, \quad 3\cos^2\theta + 1, \\ J &= 5/2, \quad 5\cos^4\theta - 2\cos^2\theta + 1. \end{aligned} \tag{1}$$

*Hints in Sakurai,* Invariance Principles and Elementary Particles *(1964), p. 17.*

3. The lowest-mass strange baryons Λ, Σ, Ξ decay weakly, and isospin is not conserved in these decays. For example, the isospin-0 particle $\Lambda^0$ decays to $p\pi^-$ which can only be in isospin 1/2 and 3/2 states. However, strange-baryon "resonances" such as the $\Lambda^0(1520)$ decay (quickly) via the strong interaction. Use isospin conservation to predict the relative decay rates of this state to $pK^-$ and $n\bar{K}^0$, and the relative rates to $\Sigma^+\pi^-$, $\Sigma^0\pi^0$ and $\Sigma^-\pi^+$. Predict also the relative decay rates for $\Sigma(1660) \to p\bar{K}^0,\ n\bar{K}^0,\ pK^-,\ n\bar{K}^-$, and for $\Sigma(1660) \to \Sigma^+\pi^0,\ \Sigma^0\pi^+,\ \Sigma^+\pi^-,\ \Sigma^0\pi^0,\ \Sigma^-\pi^+,\ \Sigma^-\pi^0,\ \Sigma^0\pi^+$.

   Ignore the effect of small mass differences among particles in an isospin multiplet.

   Table of Clebsch-Gordan Coefficients: http://pdg.lbl.gov/2002/clebrpp.pdf

4. **Meson Theory of Hyperdeuterons**

   Estimate the relative binding energies of the 64 possible pairs of baryons in the spin-1/2 octet: $n,\ p,\ \Lambda,\ \Sigma^-,\ \Sigma^0,\ \Sigma^+,\ \Xi^-,\ \Xi^0$.

   For this, use a simplified one-pion-exchange model that the nuclear force is entirely due to exchanges of a single $\pi$ meson, and that the operator $g^2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ characterizes the charge independence of this interaction.[2] Here $g$ is a coupling constant, and $\boldsymbol{\tau}$ is the isospin-1 operator (because pions form an $I = 1$ multiplet). That is, ignore electromagnetic effects and spin-dependent effects.

   (A harder version of the problem would be to deduce that $g^2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ is the appropriate operator.)

   A hint is that the Hamiltonian relevant to binding of the dibaryons is $H \propto g^2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$. Hence, we should consider the matrix elements $\langle B_1B_2|\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2|B_1B_2\rangle$, where $B$ is any member of the baryon octet. As for electricity, we infer that a negative matrix element implies an attractive force, and bound states, while a positive matrix element implies repulsion.

   Note that
$$\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left(\boldsymbol{\tau}^2 - \boldsymbol{\tau}_1^2 - \boldsymbol{\tau}_2^2\right).$$

   Also, charge independence means you don't have to look at each of the 64 pairs separately, but you can more simply consider pairs of isospin multiplets, each of which leads to one or more multiplets of total isospin exactly as for combinations of ordinary

[2] If this interaction is represented by a Feynman diagram with single pion exchange, then each of the $BB\pi$ vertices has strength $g\boldsymbol{\tau}$.

spin. For this, note that the nucleons, $N$, and the cascade particles, $\Xi$, each form an isodoublet, the $\Lambda$ is an isosinglet, and the $\Sigma$'s form an isotriplet.

Give the isospin wavefunctions of the candidate bound states.

*I found that 11 of the 64 pairs should have bounds states, and that none of these would be more weakly bound than the deuteron.*

*Considerations somewhat similar to those of this problem are given in D.B. Lichtenberg and M.H. Ross, Pion Contribution to Hyperon-Nucleon Forces, Phys. Rev.* **107**, *1714 (1957),* http://kirkmcd.princeton.edu/examples/EP/lichtenberg_pr_107_1714_57.pdf.
*No dibaryon bound state other than the deuteron has ever been observed, although searches continue.*[3] *The lightest known hypernucleus is* ${}^{3}_{\Lambda}H$,[4] *and even its antiparticle has been observed.*[5] *A* $\Sigma$*-hypernucleus is* ${}^{4}_{\Sigma}He$.[6] *A handful of examples of hyper-He nuclei containing two* $\Lambda$*'s have been reported.*[7]

5. The $J/\psi$ meson is an (electrically neutral) $c\bar{c}$ state with mass = 3.1 GeV, $J^{PC} = 1^{--}$ and $I^G = 0^-$. Which of the following possible decay modes are allowed, and if forbidden, what symmetry would be violated if such decay were observed: $N\bar{N}$, $\pi^+\pi^-$, $\pi^0\pi^0$, $\gamma\gamma$, $\pi^0\gamma$, $\pi^0\gamma\gamma$, $\pi^+\pi^-\gamma$, $\pi^+\pi^-\pi^0$, $3\pi^0$, $4\pi^0$, $\pi^+\pi^-\pi^+\pi^-$?

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[3] See, for example, B.H. Kim *et al.*, *Search for an H-Dibaryon with a Mass near* $2m_\Lambda$ *in* $\Upsilon(1S)$ *and* $\Upsilon(2S)$ *Decays*, Phys. Rev. Lett. **110**, 222002 (2013),
http://kirkmcd.princeton.edu/examples/EP/kim_prl_110_222002_13.pdf.
For a review of ongoing modeling of such possible states, see S.R. Beane *et al.*, *Light nuclei and hypernuclei from quantum chromodynamics in the limit of SU(3) flavor symmetry*, Phys. Rev. D **87**, 034506 (2013),
http://kirkmcd.princeton.edu/examples/EP/beane_prd_87_034506_13.pdf.

[4] R.J. Prem and P.H. Steinberg, *Lifetimes of Hypernuclei,* ${}_{\Lambda}H^3$, ${}_{\Lambda}H^4$, ${}_{\Lambda}He^4$, Phys. Rev. **136**, B1803 (1964), http://kirkmcd.princeton.edu/examples/EP/prem_pr_136_B1803_64.pdf.

[5] STAR Collaboration, *Observation of an Antimatter Hypernucleus*, Science **328**, 58 (2010),
http://kirkmcd.princeton.edu/examples/EP/star_science_328_58_10.pdf.

[6] T. Nagae *et al.*, *Observation of a* ${}^{4}_{\Sigma}He$ *Bound State in the* ${}^4He(K^-,\pi^-)$ *Reaction at 600 MeV/c*, Phys. Rev. Lett. **80**, 1605 (1998),
http://kirkmcd.princeton.edu/examples/EP/nagae_prl_80_1605_98.pdf.

[7] See, for example, K. Nakazawa *et al.*, *Double-*$\Lambda$ *Hypernuclei via the* $\Xi^-$ *Hyperon Capture at Rest Reaction in a Hybrid Emulsion*, Nucl. Phys. A **835**, 207 (2010),
http://kirkmcd.princeton.edu/examples/EP/nakazawa_npa_835_207_10.pdf.

# Solutions

1. **Positronium**

   (a) $1^3S_1 \to 2^3S_1$

   These two states both have odd parity, $P_{e^+e^-} = (-1)^{L+1}$ with $L = 0$, and total angular momentum $J = 1$. Hence, the photon must have even parity (assuming that the electromagnetic interaction conserves parity, and noting that $P_\gamma = (-1)^{L_\gamma+1}$), which implies that its orbital angular momentum $L_\gamma$ must be odd. In principle, a $J = 1 \to J = 1$ atomic transition with single-photon absorption(emission) could occur for $J_\gamma = 0, 1, 2$, but $J_\gamma = 0$ (monopole radiation) does not occur.

   $L_\gamma = 1$, $J_\gamma = 1$ corresponds to a magnetic dipole transition, for which the parity $P_{\gamma,M} = (-1)^{J_\gamma+1}$ is even. Hence, this is an allowed transition, considering only conservation of angular momentum and parity.

   $L_\gamma = 1$, $J_\gamma = 2$ corresponds to an electric quadrupole transition, with even parity $P_{\gamma,E} = (-1)^{J_\gamma}$, so this also is allowed, considering only conservation of angular momentum and parity.

   However, we should also consider charge conjugation.

   A single photon has $C_\gamma = -1$.

   An $e^+e^-$ state has $C = (-1)^{L+S}$, so the $1^3S_1$ state has $C = (-1)^{0+1} = -1$, as does also the state $2^3S_1$. To conserve charge conjugation, the photon would have to have even charge conjugation.

   Hence, observation of this transition would be a violation of charge conjugation symmetry in the electromagnetic interaction.

   (b) $1^3S_1 \to 2^1S_0$

   The $2^1S_0$ state has charge conjugation $C = (-1)^{0+0} = 1$, so this transition is allowed by charge conjugation symmetry.

   These two states also both have odd parity, so again $L_\gamma$ must be odd.

   The initial atom has $J = 1$ and the final atom has $J = 0$, so the photon must have $J_\gamma = 1$ (assuming that angular momentum is conserved).

   $L_\gamma = 1$, $J_\gamma = 1$ corresponds to a magnetic dipole transition, for which the parity $P_\gamma$ is even. Hence, this is also an allowed transition.

   Altogether, this transition is allowed by conservation angular momentum, parity, and charge conjugation.

   (c) $1^3S_1 \to 2^1P_1$

   The $2^1P_1$ state has charge conjugation $C = (-1)^{1+0} = -1$, so this transition is forbidden via single-photon absorption by charge conjugation symmetry.

   For what it's worth, it is allowed by conservation of angular momentum and parity.

   The initial state has odd parity, while the final state has even parity, so the photon should have odd parity and hence even $L_\gamma$.

   The $J = 1 \to J = 1$ atomic transition can take place with $J_\gamma = 1, 2$.

   $L_\gamma = 0, 2$, $J_\gamma = 1$ is an E1 transition, with odd parity, so this is allowed.

$L_\gamma = 2$, $J_\gamma = 2$ is an M2 transition, with odd parity, so this also is allowed (but would be very weak compared to the E1 transition).

(d) $1^3S_1 \to 2^3P_1$

The $2^3P_1$ state has charge conjugation $C = (-1)^{1+1} = 1$, so this transition is allowed via single-photon absorption by charge conjugation symmetry.

It is also allowed by conservation of angular momentum and parity, with the argument being the same as for the case $1^3S_1 \to 2^1P_1$ (since these arguments depend on $J$ and $L$ but not on $S$).

(e) $\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1 \times \hat{\mathbf{k}}_2$

As discussed in part (a), the $1^3S_1$ state has odd parity. The 3-photon final state also has odd intrinsic parity. Hence, only even-parity correlations are permitted in the final state, if parity is conserved.

Recall that $P(\hat{\mathbf{S}}) = \hat{\mathbf{S}}$ for a spin vector $\hat{\mathbf{S}}$, while $P(\hat{\mathbf{k}}) = -\hat{\mathbf{k}}$ for a momentum vector $\hat{\mathbf{k}}$.

thus, $P(\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1 \times \hat{\mathbf{k}}_2) = \hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1 \times \hat{\mathbf{k}}_2$, and this correlation is consistent with parity conservation.

(f) $\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1$

$P(\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1) = -\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1$, so observation of this correlation would imply parity violation.

(g) $(\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1)(\hat{\mathbf{S}} \cdot \hat{\mathbf{k}}_1 \times \hat{\mathbf{k}}_2)$

This case combines the correlations of (e) and (f), and would violate parity since (f) violates parity.

(h) $\hat{\mathbf{S}} \cdot \hat{\boldsymbol{\epsilon}}_1 \times \hat{\mathbf{k}}_2$

The photon polarization vector $\hat{\boldsymbol{\epsilon}}$ is along its electric field, so $P(\hat{\boldsymbol{\epsilon}}) = -\hat{\boldsymbol{\epsilon}}$. Thus, $P(\hat{\mathbf{S}} \cdot \hat{\boldsymbol{\epsilon}}_1 \times \hat{\mathbf{k}}_2) = \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\epsilon}}_1 \times \hat{\mathbf{k}}_2$, so this correlation is permitted by parity conservation.

2. This problem is based on R.K. Adair, *Angular Distribution of $\Lambda^0$ and $\theta^0$ Decays*, Phys. Rev. **100**, 1540 (1955), http://kirkmcd.princeton.edu/examples/EP/adair_pr_100_1540_55.pdf.

A two-particle state can only have orbital-angular-momentum component $L_z = 0$ along a $z$-axis.

If the $\Lambda^0$ moves along the beam axis, taken to be the $z$-axis, then so also does the $K^0$, and no matter what is their orbital angular momentum $L$, $L_z = 0$. Of course, the initial $\pi^- p$ state has $L_z = 0$, and $J_z = \pm 1/2$, since the pion is spinless and the proton has spin-1/2. Conservation of angular momentum then implies that $J_z = \pm 1/2$ for the final state; these two states are distinguishable, so it suffices to consider only one, say $J_z = 1/2$.

Similarly, since the initial state can only have $J = n/2$ for odd $n$ this also holds for the final state, which in turn implies that the spin of the $\Lambda^0$ is $m/2$ for odd $m$, since the $K^0$ is spinless.

(a) $J_\Lambda = 1/2$.
In general, the decay final state $\pi^- p$ could have $L = 0$ or 1 such that $J = 1/2$. If the $\Lambda$ has $J_z = \pm 1/2$ in its rest frame, then this couples to the $L = 0$ $\pi^- p$ state according to

$$|1/2, 1/2\rangle = |0, 0\rangle |1/2, \pm 1/2\rangle, \tag{2}$$

and couples to the $\pi^- p$ states with orbital angular momentum $L = 1$ and (proton) spin $S = \pm 1/2$ according to

$$|1/2, 1/2\rangle = \sqrt{\frac{2}{3}}|1, 1\rangle|1/2, -1/2\rangle - \sqrt{\frac{1}{3}}|1, 0\rangle|1/2, 1/2\rangle, \tag{3}$$

$$|1/2, -1/2\rangle = -\sqrt{\frac{2}{3}}|1, -1\rangle|1/2, 1/2\rangle + \sqrt{\frac{1}{3}}|1, 0\rangle|1/2, -1/2\rangle, \tag{4}$$

using the Clebsch-Gordon coefficients from
http://pdg.lbl.gov/2013/reviews/rpp2012-rev-clebsch-gordan-coefs.pdf.
The initial $J_z = \pm 1/2$ states, and the decay final states are all distinguishable by the proton spin component, so we have four amplitudes to consider,

$$\alpha|0, 0\rangle|1/2, 1/2\rangle - \beta\sqrt{\frac{1}{3}}|1, 0\rangle|1/2, 1/2\rangle, \tag{5}$$

$$\beta\sqrt{\frac{2}{3}}|1, 1\rangle|1/2, -1/2\rangle, \tag{6}$$

$$\alpha|0, 0\rangle|1/2, -1/2\rangle + \beta\sqrt{\frac{1}{3}}|1, 0\rangle|1/2, -1/2\rangle, \tag{7}$$

$$-\beta\sqrt{\frac{2}{3}}|1, -1\rangle|1/2, 1/2\rangle, \tag{8}$$

where $\alpha$ is the strength of the interaction with the $L = 0$ state, and $\beta$ is the strength of the interaction with the $L = 1$ state. We square amplitudes (5)-(8) and add to find the angular distribution, noting that the orbital angular momentum states $|L, L_z\rangle$ correspond to spherical harmonics $Y_L^{L_z}(\theta, \phi)$, where $\theta$ is the angle of, say, the decay pion with respect to the $z$-axis in the $\Lambda$ rest frame.

$$Y_0^0 = \sqrt{\frac{1}{4\pi}}, \quad Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}, \quad Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta. \tag{9}$$

The four amplitudes (5)-(8) are then (after multiplying by $\sqrt{4\pi}$),

$$\alpha - \beta\sqrt{\frac{1}{3}}\cos\theta, \quad -\beta\sqrt{\frac{1}{3}}\sin\theta\, e^{i\phi}, \quad \alpha + \beta\sqrt{\frac{1}{3}}\cos\theta, \quad -\beta\sqrt{\frac{1}{3}}\sin\theta\, e^{-i\phi}. \tag{10}$$

Squaring, and adding, leads to the angular distribution

$$2|\alpha|^2 + \frac{2|\beta|^2}{3}(\sin^2\theta + \cos^2\theta) = 2|\alpha|^2 + \frac{2|\beta|^2}{3} = \text{isotropic}. \tag{11}$$

We note that the target protons needed to be unpolarized so that the cases of $J_z = \pm 1/2$ for the initial state are equally likely, and the cross terms between different $L$ in the final $\pi^- p$ state cancel out. We assume this holds for the cases of higher possible $\Lambda$ spin, and consider than contributions to the angular distribution from different $L$ separately.

(b) $J_\Lambda = 3/2$.

In this case the orbital angular momentum of the $\pi^- p$ final state can be $L = 1$ or 2 such that $J = 3/2$. If the $\Lambda$ has $J_z = 1/2$ in its rest frame, then this couples to the $\pi^- p$ final states with orbital angular momentum $L = 1$ and (proton) spin $S = 1/2$ according to

$$|3/2, 1/2\rangle = \sqrt{\frac{1}{3}}|1,1\rangle|1/2,-1/2\rangle + \sqrt{\frac{2}{3}}|1,0\rangle|1/2,1/2\rangle, \qquad (12)$$

which implies an angular distribution proportional to

$$\left|Y_1^1\right|^2 + 2\left|Y_1^0\right|^2 \propto \frac{\sin^2\theta}{2} + 2\cos^2\theta \propto 3\cos^2\theta + 1. \qquad (13)$$

Similarly, the coupling to the $\pi^- p$ final states with orbital angular momentum $L = 2$ is

$$|3/2, 1/2\rangle = \sqrt{\frac{3}{5}}|2,1\rangle|1/2,-1/2\rangle - \sqrt{\frac{2}{5}}|2,0\rangle|1/2,1/2\rangle, \qquad (14)$$

which implies an angular distribution of

$$3\left|Y_2^1\right|^2 + 2\left|Y_2^0\right|^2 \propto 3\frac{15}{2}\sin^2\theta\cos^2\theta + 2\frac{5}{4}(3\cos^2\theta - 1)^2 \propto 3\cos^2\theta + 1, \qquad (15)$$

noting that

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}, \qquad Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1). \qquad (16)$$

Thus, either value of $L$ for the $\pi^- p$ final states leads to the same angular distribution, namely $3\cos^2\theta + 1$.

(c) $J_\Lambda = 5/2$.

In this case the possible orbital angular momenta of the final $\pi^- p$ states are $L = 2$ and 3.

We content ourselves with calculating only $L = 2$.

$$|5/2, 1/2\rangle = \sqrt{\frac{2}{5}}|2,1\rangle|1/2,-1/2\rangle + \sqrt{\frac{3}{5}}|2,0\rangle|1/2,1/2\rangle, \qquad (17)$$

which implies an angular distribution of

$$2\left|Y_2^1\right|^2 + 3\left|Y_2^0\right|^2 \propto 2\frac{15}{2}\sin^2\theta\cos^2\theta + 3\frac{5}{4}(3\cos^2\theta - 1)^2 \propto 5\cos^4\theta - 2\cos^2\theta + 1. \qquad (18)$$

The principle of this problem was used to determine that the $\Lambda^0$ has spin-1/2 by F. Eisler *et al.*, *Experimental Determinations of the $\Lambda^0$ and $\Sigma^-$ Spins*, Nuovo Cim. **7** 222 (1958), http://kirkmcd.princeton.edu/examples/EP/eisler_nc_7_222_58.pdf.

3. (a) $\Lambda^0(1520) \to pK^-,\ n\bar{K}^0$.

In terms of isospin states, this is

$$|0,0\rangle = \sqrt{\frac{1}{2}}|1/2,1/2\rangle|1/2,-1/2\rangle - \sqrt{\frac{1}{2}}|1/2,-1/2\rangle|1/2,1/2\rangle, \tag{19}$$

so isospin conservation implies that the two decay amplitudes have equal but opposite strengths, and hence the two decay rates are equal,

$$\Gamma[\Lambda(1520) \to pK^-] = \Gamma[\Lambda(1520) \to p\bar{K}^0]. \tag{20}$$

(b) $\Lambda^0(1520) \to \Sigma^+\pi^-$, $\Sigma^0\pi^0$ and $\Sigma^-\pi^+$.

In terms of isospin states, this is

$$|0,0\rangle = \sqrt{\frac{1}{3}}|1,1\rangle|1,-1\rangle - \sqrt{\frac{1}{3}}|1,0\rangle|1,0\rangle + \sqrt{\frac{1}{3}}|1,-1\rangle|1,1\rangle, \tag{21}$$

so isospin conservation implies that the three decay amplitudes have equal magnitudes, and hence these three decay rates are equal,

$$\Gamma[\Lambda(1520) \to \Sigma^+\pi^-] = \Gamma[\Lambda(1520) \to \Sigma^0\pi^0] = \Gamma[\Lambda(1520) \to \Sigma^-\pi^+] \tag{22}$$

Isospin invariance does not relate the $N\bar{K}$ decay rates to the $\Sigma\pi$ ones, but SU(3) invariance does, with the prediction that the $N\bar{K}$ decay rates are 3/2 times the $\Sigma\pi$ ones, such that the total decay rates $\Gamma[\Lambda(1520) \to N\bar{K}]$ and $\Gamma[\Lambda(1520) \to \sigma\pi]$ are equal (as holds reasonably well in the data).

(c) $\Sigma(1660) \to p\bar{K}^0,\ n\bar{K}^0,\ pK^-,\ n\bar{K}^-$.

These decays of the $\Sigma(1660)$ have the isospin relations

$$|1,1\rangle = |1/2,1/2\rangle|1/2,1/2\rangle, \tag{23}$$

$$|1,0\rangle = \sqrt{\frac{1}{2}}|1/2,1/2\rangle|1/2,-1/2\rangle - \sqrt{\frac{1}{2}}|1/2,-1/2\rangle|1/2,1/2\rangle, \tag{24}$$

$$|1,-1\rangle = |1/2,-1/2\rangle|1/2,-1/2\rangle, \tag{25}$$

so isospin conservation implies that

$$\Gamma[\Sigma^+(1660) \to p\bar{K}^0] = 2\Gamma[\Sigma^0(1660) \to pK^-] \tag{26}$$

$$= 2\Gamma[\Sigma^0(1660) \to n\bar{K}^0] = \Gamma[\Sigma^-(1660) \to nK^-]. \tag{27}$$

(d) $\Sigma(1660) \to \Sigma^+\pi^0,\ \Sigma^0\pi^+,\ \Sigma^+\pi^-,\ \Sigma^0\pi^0,\ \Sigma^-\pi^+,\ \Sigma^-\pi^0,\ \Sigma^0\pi^+$.
These decays of the $\Sigma(1660)$ have the isospin relations

$$|1,1\rangle = \sqrt{\frac{1}{2}}|1,1\rangle|1,0\rangle - \sqrt{\frac{1}{2}}|1,0\rangle|1,1\rangle, \tag{28}$$

$$|1,0\rangle = \sqrt{\frac{1}{2}}|1,1\rangle|1,-1\rangle - \sqrt{\frac{1}{2}}|1,-1\rangle|1,1\rangle, \tag{29}$$

$$|1,-1\rangle = \sqrt{\frac{1}{2}}|1,-1\rangle|1,0\rangle - \sqrt{\frac{1}{2}}|1,0\rangle|1,-1\rangle,, \tag{30}$$

so isospin conservation implies that

$$\Gamma[\Sigma^+(1660) \to \Sigma^+\pi^0] = \Gamma[\Sigma^+(1660) \to \Sigma^0\pi^+] \tag{31}$$

$$= \Gamma[\Sigma^0(1660) \to \Sigma^+\pi^-] = \Gamma[\Sigma^0(1660) \to \Sigma^-\pi^+] \tag{32}$$

$$= \Gamma[\Sigma^-(1660) \to \Sigma^0\pi^-] = \Gamma[\Sigma^-(1660) \to \Sigma^-\pi^0], \tag{33}$$

while $\Gamma[\Sigma^0(1660) \to \Sigma^0\pi^0] = 0$.
SU(3) invariance implies that $\Gamma[\Sigma^+(1660) \to p\bar{K}^0] = \Gamma[\Sigma^+(1660) \to \Sigma^+\pi^0]$, and hence that the total decay rates obey $\Gamma[\Sigma(1660) \to N\bar{K}] = \Gamma[\Sigma(1660) \to \Sigma\pi]$.

4. To deal with all 64 pairs of dibaryons from the basic spin-1/2 octet, $n$, $p$, $\Lambda$, $\Sigma^-$, $\Sigma^0$, $\Sigma^+$, $\Xi^-$, $\Xi^0$, we need a compact analysis. For this, we note that for $\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_1$,

$$(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2)^2 = \boldsymbol{\tau}^2 = \boldsymbol{\tau}_1^2 + \boldsymbol{\tau}_2^2 + 2\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2, \tag{34}$$

$$\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left(\boldsymbol{\tau}^2 - \boldsymbol{\tau}_1^2 - \boldsymbol{\tau}_2^2\right) = \frac{1}{2}[(\tau(\tau+1) - (\tau_1(\tau_1+1) - (\tau_2(\tau_2+1)], \tag{35}$$

noting that the expectation value of the (iso)spin operator $\boldsymbol{\tau}^2$ is $\tau(\tau+1)$. Thus, the strength of the $\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ interaction is the same for all members of a multiplet of total isospin $\tau$, and in the simplest model is the same for any dibaryon isospin multiplet of the same $\tau$.

So, we consider the possible dibaryon isospin multiplets.

(a) The $1/2 \times 1/2$ multiplets are $NN$, $\Xi\Xi$ and $N\Xi$, which lead to $\tau = 0$ and $\tau = 1$ multiplets with $\tau_1 = \tau_2 = 1/2$.

$$\tau = 0: \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[0(0+1) - \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\right] = -\frac{3}{4}, \tag{36}$$

$$\tau = 1: \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[1(1+1) - \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\right] = \frac{1}{4}. \tag{37}$$

This model suggests that there would be bound isosinglet states $(pn - np)/\sqrt{2}$ (deuteron), $(\Xi^0\Xi^- - \Xi^-\Xi^0)/\sqrt{2}$ and $(p\Xi^- - n\Xi^0)/\sqrt{2}$.

(b) The $1/2 \times 0$ multiplets are the $N\Lambda$ and $\Xi\Lambda$ states, with $\tau = 1/2$, $\tau_1 = 1/2$ and $\tau_2 = 0$.

$$\tau = \frac{1}{2}: \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) - (0)\,(0+1)\right] = 0. \tag{38}$$

These states are not bound in this model.

(c) The $0 \times 0$ multiplet is the state $\Lambda\Lambda$, with $\tau = 0 = \tau_1 = \tau_2 = 0$.

$$\tau = 0 : \qquad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[(0)(0+1) - (0)(0+1) - (0)(0+1)\right] = 0. \tag{39}$$

This state is not bound in this model.

(d) The $1 \times 0$ multiplet is the states $\Sigma\Lambda$, with $\tau = 1 = \tau_1$ and $\tau_2 = 0$.

$$\tau = 1 : \qquad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[(1)(0+1) - (1)(1+1) - (0)(0+1)\right] = 0. \tag{40}$$

These states are not bound in this model.

(e) The $1/2 \times 1$ multiplets are the $N\Sigma$ and $\Xi\Sigma$ states, with $\tau = 1/2$ or $3/2$, $\tau_1 = 1/2$ and $\tau_2 = 1$.

$$\tau = \frac{1}{2}: \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) - 1(1+1)\right] = -1, \tag{41}$$

$$\tau = \frac{3}{2}: \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[\left(\frac{3}{2}\right)\left(\frac{3}{2}+1\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right) - 1(1+1)\right] = \frac{1}{2}. \tag{42}$$

This model suggests that there would be bound isodoublet states $(\sqrt{2}n\Sigma^+ - p\Sigma^0)/\sqrt{3}$, $(n\Sigma^0 - \sqrt{2}p\Sigma^-)/\sqrt{3}$ and $(\sqrt{2}\Xi^-\Sigma^+ - \Xi^0\Sigma^0)/\sqrt{3}$ and $(\Xi^-\Sigma^0 - \sqrt{2}\Xi^0\Sigma^-)/\sqrt{3}$.

(f) The $1 \times 1$ multiplets are the $\Sigma\Sigma$ states, with $\tau = 0$, 1 and $\tau_1 = \tau_2 = 1$.

$$\tau = 0: \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[0(0+1) - 1(1+1) - 1(1+1)\right] = -2, \tag{43}$$

$$\tau = 1: \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[1(1+1) - 1(1+1) - 1(1+1)\right] = -1, \tag{44}$$

$$\tau = 2: \quad \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 = \frac{1}{2}\left[2(2+1) - 1(1+1) - 1(1+1)\right] = 1. \tag{45}$$

This model suggests that the most tightly bound state would be isosinglet $(\Sigma^+\Sigma^- - \Sigma^0\Sigma^0 + \Sigma^-\Sigma^+)/\sqrt{3}$, and the isotriplet $\Sigma^+\Sigma^0$, $\Sigma^+\Sigma^-$, $\Sigma^0\Sigma^-$ is also bound.[8]

Taking Coulomb effects into account (which don't conserve isospin), the $\Sigma^+\Sigma^-$ part of the isosinglet would be the most tightly bound dibaryon in this model.

Unfortunately, the data do not support this model.

5. This problem is a variant on considerations in D.B. Lichtenberg and G.C. Summerfield, *G Parity and the Interactions of Heavy Mesons*, Phys. Rev. **127**, 1806 (1962), http://kirkmcd.princeton.edu/examples/EP/lichtenberg_pr_127_1806_62.pdf.

The $J/\psi(3100)$ meson has $J^{PC} = 1^{--}$ and $I^G = 0^-$.

(a) $N\bar{N}$ is an allowed decay mode. A ${}^3S_1$ state has $J = 1$, $L = 0$, $S = 1$, $P = -(-1)^L = -$, $C = (-1)^{L+S} = -$, and can be in an $I = 0$ state for which $G = (-1)^{L+S+I} = -$.

[8]Note that the state $\Sigma^0\Sigma^0$ does not contribute to the $\Sigma\Sigma$ isotriplet. (Similarly, the isovector meson $\rho^0$ does not decay to $\pi^0\pi^0$.)

(b) $\pi^+\pi^-$ has even $G$-parity, so is forbidden. *A delicacy is that most decays of the $J/\psi$ are electromagnetic, which interaction does not respect isospin, in general. However, it appears that in decays via a single intermediate photon (by far the most probable), that photon behaves as if it has isospin 0 and negative $G$-parity as does the $J/\psi$. This is an aspect of the so-called* vector dominance model *in which virtual photons have some properties of hadrons. However, if the final state includes a (real) photon, that photon does not have a definite isospin or $G$-parity, contrary to a tacit assumption in the paper of Lichtenberg.*

(c) $\pi^0\pi^0$ is forbidden by $G$-parity, and also by charge conservation since $C(\pi^0\pi^0) = +$.

(d) $\gamma\gamma$ is forbidden by rules of angular momentum, which tells us that a spin-1 particle cannot decay to $\gamma\gamma$.

(e) $\pi^0\gamma$ is allowed; the branching fraction is $3.5 \pm 0.3 \times 10^{-5}$. See, http://pdg.lbl.gov/2013/listings/rpp2013-list-J-psi-1S.pdf.
*The paper by Lichtenberg incorrectly says this is forbidden by $G$-parity, but the (real, final-state) $\gamma$ does not have definite isospin or $G$-parity.*

(f) $\pi^0\gamma\gamma$ is forbidden by charge conjugation since $C(\pi^0) = +$.

(g) $\pi^+\pi^-\gamma$ is allowed, I think (although Lichtenberg says not). A few decays of this type have been reported.

(h) $\pi^+\pi^-\pi^0$ is allowed (since $\pi^+\pi^-$ with $L = 1$ have $C = -$).

(i) $3\pi^0$ is forbidden by charge conjugation.

(j) $4\pi^0$ is allowed I think (although Lichtenberg says not). This decay has not been observed, but it is not easy to spot, since the actual signature is 8 photons from the 4 $\pi^0 \to \gamma\gamma$ decays.

(k) $\pi^+\pi^-\pi^+\pi^-$ is allowed (although Lichtenberg says not); branching fraction is $3.6 \pm 0.3 \times 10^{-3}$.