Ph 406 Problem Set 7

Due April 2, 1993

1. **Helicity of the Neutrino**. (Phys. Rev. **109** (1958) 1015; see also p. 1423 ff in *Alpha-, Beta-, and Gamma-Ray Spectroscopy*, ed. by K. Siegbahn.)

   Suppose a spin-0 nucleus A decays via electron capture to a spin-1 excited state $B^\star$ of nucleus B along with emission of a neutrino (Gamow-Teller transition). Then suppose the spin-1 state $B^\star$ decays via emission of an electric-dipole photon to the spin-0 ground state of B.

   We wish to infer the helicity of the neutrino by observation of the angular distribution of the photon. To do this, the direction of the neutrino in the laboratory must be singled out by some feature of the apparatus. This is possible by an ingenious argument of Goldhaber *et al.*

   Take the z axis along the direction of the neutrino in the decay $A \to B^\star$. Then in the $V - A$ theory of the weak interaction, the neutrino must have spin component $S_z = -\frac{1}{2}$. The nucleus $B^\star$ could have $S_z = -1,\ 0$ or 1, leading to J_z for the final state of $-\frac{3}{2},\ -\frac{1}{2}$, or $\frac{1}{2}$. (Recall that $L_z = 0$ for any two-body state where the z axis is along the direction of momentum in the rest frame.) But the initial state has only spin $\frac{1}{2}$ from the electron, assuming electron capture from an S-wave orbital, so only $S_z = 0$ or 1 are possible for nucleus $B^\star$.

   Let θ be the angle of emission of the photon with respect to the $-z$ axis in the rest frame of state $B^\star$. (In the lab frame, $B^\star$ moves along the $-z$ axis.) Use the spin-1 rotation matrix, given in the table appended to page 191 of the Ph529 notes, to show that the angular distribution of the $E1$ photons is $\sin^2\theta$ if the $B^\star$ has $S_z = 0$, and $(1+\cos^2\theta)/2$ when the $B^\star$ has $S_z = 1$. Remember that the photon can only have $S_{z'} = \pm 1$ along the z' axis which is along the photon's direction in the $B^\star$ rest frame.

   In particular, show that photons emitted at $\theta = 0$ or 180° can only have $S_z = +1$. (You can do this either via details of the rotation matrix, or directly from conservation of angular momentum. Be sure you can do it both ways!)

   Let E_K be the energy of the neutrino emitted in the decay of A, and E_0 be the excitation energy of $B^\star$ with respect to ground state B. Deduce the energy of the photon in the lab frame as a function of $\theta,\ E_K$ and E_0. You may approximate $M_A \approx M_{B^\star} \approx M_B \equiv M$. The highest photon energy occurs for $\theta = 0$, for which these photons have $S_z = 1$ (and negative helicity as these photons are moving along the $-z$ axis). If the neutrino had positive helicity, these photons would have positive helicity also, by conservation of angular momentum.

   So if we can measure the helicity of the highest-energy photons, we determine the helicity of the neutrino.

   The helicity of photons can be determined by passing them through a filter consisting of magnetized iron, which attenuates photons of +1 and −1 helicity by different amounts. The reaction here is just Compton scattering of polarized electrons and photons. Because the electron has spin $\frac{1}{2}$, an electron can only absorb a photon whose spin is opposite – which flips the spin of the intermediate electron prior to the radiation (scattering) of the final photon.

But we want to determine the helicity of only the highest energy photons, so a final trick is needed. Suppose the photons from the $B^\star$ decay impinge upon other ground-state B nuclei that are at rest in the lab. Calculate the energy of the photons such that the nucleus can be excited to the level $B^\star$. The latter states decay back to the ground state by photon emission, scattering only a certain subset of the photons from the first $B^\star$ decay into the detector.

For the historical experiment, $A = ^{152}$Eu, and $B = ^{152}$Sm, for which $E_K = 840$ keV, while $E_0 = 961$ keV. Due to recoil effects, you should have found that even the highest energy photons from the first $B^\star$ decay have insufficient energy to re-excite B nuclei at rest. However, the lifetime of the spin-1 ^{152}Sm excited state was measured to be 7×10^{-14} s. Convert this to a width in keV. What fraction of the photons from the first $B^\star$ decay overlap the Breit-Wigner resonance curve for $B^\star$ excitation (assuming the spectrum of photon energies is flat between max and min energies?

If the lifetime of the $B^\star$ level had been too long, the overlap would be too small for the experiment to work. Also, if the lifetime were long, the $B^\star$ atom might have collided with another atom and changed its momentum prior to the photon decay. Then to correlation between the decay-photon helicity and the neutrino helicity would have been lost.

So it's a small miracle that any system exists in nature that permits this measurement!

2. **Positronium**

The concept of positronium, an atom made of an electron and a positron, was invented by Princetonian John Wheeler – who called it a polyelectron (Ann. N.Y. Acad. **48** (1946) 219).

What symmetry principles would be violated if the following 1-photon transitions between excited stated of positronium were observed (via microwave pumping)?

(a) $1^3S_1 \to 2^3S_1$

(b) $1^3S_1 \to 2^1S_0$

(c) $1^3S_1 \to 2^1P_1$

(d) $1^3S_1 \to 2^3P_1$

Polarized 1^3S_1 positronium can be formed when positrons from ^{22}Na β-decay combine with atomic electrons. This state decays to 3 photons (do you recall why 2-photon decay is forbidden?). Let $\hat{k}_1$ and $\hat{k}_2$ be the directions of the two higher-energy decay photons, and $\hat{S}$ be the direction of the positronium spin. What symmetries would be violated by angular correlations of the form?

(a) $\hat{S} \cdot \hat{k}_1 \times \hat{k}_2$

(b) $\hat{S} \cdot \hat{k}_1$

(c) $(\hat{S} \cdot \hat{k}_1)(\hat{S} \cdot \hat{k}_1 \times \hat{k}_2)$

(d) $\hat{S} \cdot \hat{\epsilon}_1 \times \hat{k}_2$

where $\hat{\epsilon}_1$ is along the direction of polarization of final-state photon 1.

3. Λ-hyperons are produced by a pion beam in the reaction $\pi^- + p \rightarrow K^0 + \Lambda$, and observed via their decay $\Lambda \rightarrow p + \pi^-$. Let J denote the spin of the Λ, z the beam direction, and θ the angle of a decay product, relative to z, measured in the Λ rest frame. (a) In the case where the Λ is produced exactly along the z-direction, what are the possible values of J_z? (b) Show that, for unpolarized protons, the decay angular distributions for the forward-produced Λs as a function of Λ spin will be as follows:

$$J_\Lambda = \tfrac{1}{2} \qquad \text{(isotropic)},$$
$$J_\Lambda = \tfrac{3}{2} \qquad (3\cos^2\theta + 1),$$
$$J_\Lambda = \tfrac{5}{2} \qquad (5\cos^4\theta - 2\cos^2\theta + 1).$$

[This method to determine the Λ spin was first proposed by Adair (1955). For a discussion see Sakurai (1964) and Tripp (1965).] (c) State how one might determine the spin of the $\Sigma^\pm$ from the (s-state) capture of negative kaons in hydrogen, $K^- + p \rightarrow \Sigma^\pm + \pi^\mp$.

Direct Evidence for Two-Neutrino Double-Beta Decay in ^{82}Se

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The two-neutrino mode of double-beta decay in ^{82}Se has been observed in a time-projection chamber at a half-life of $(1.1^{+0.8}_{-0.3})\times10^{20}$ yr (68% confidence level). This result from direct counting confirms the earlier geochemical measurements and helps provide a standard by which to test the double-beta-decay matrix elements of nuclear theory. It is the rarest natural decay process ever observed directly in the laboratory.

Helicity of Neutrinos*

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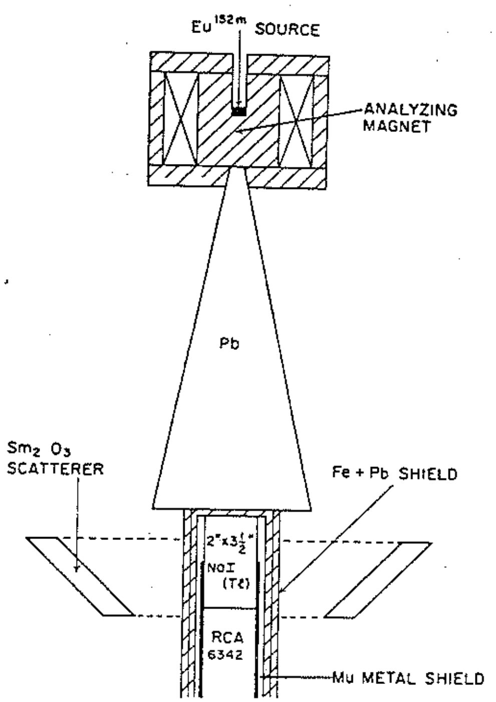

Fig. 6. Schematical arrangement of neutrino helicity experiment. (From Goldhaber *et al.*[1])

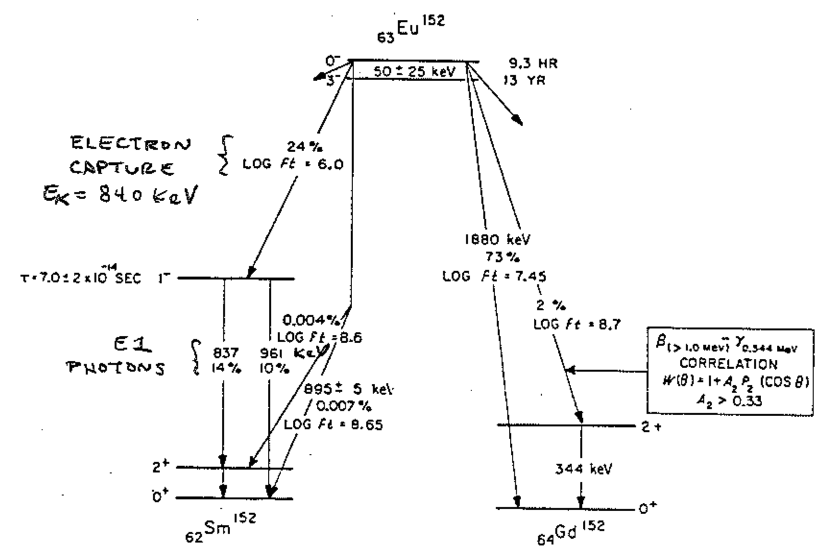

Fig. 4. Partial decay scheme of Eu^{152m}

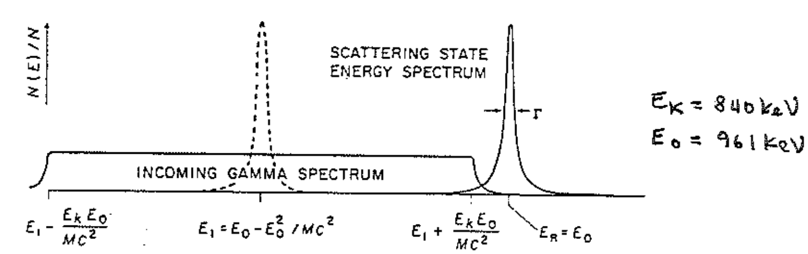

Fig. 5. Schematic diagram of incoming photon spectrum and resonance level width. (Not to scale)

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. .			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1 m_2	J=1, M=+1
+1/2 +1/2	1

m_1 m_2	1, 0	0, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	1, −1
−1/2 −1/2	1

1×1/2

m_1 m_2	3/2, +3/2
+1 +1/2	1

m_1 m_2	3/2, +1/2	1/2, +1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	3/2, −1/2	1/2, −1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	3/2, −3/2
−1 −1/2	1

2×1/2

m_1 m_2	5/2, +5/2
+2 +1/2	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	5/2, +1/2	3/2, +1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	5/2, −1/2	3/2, −1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	5/2, −3/2	3/2, −3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	5/2, −5/2
−2 −1/2	1

3/2×1/2

m_1 m_2	2, +2
+3/2 +1/2	1

m_1 m_2	2, +1	1, +1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	2, 0	1, 0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	2, −1	1, −1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	2, −2
−3/2 −1/2	1

2×1

m_1 m_2	3, +3
+2 +1	1

m_1 m_2	3, +2	2, +2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	3, +1	2, +1	1, +1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	3, 0	2, 0	1, 0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	3, −1	2, −1	1, −1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	3, −2	2, −2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	3, −3
−2 −1	1

1×1

m_1 m_2	2, +2
+1 +1	1

m_1 m_2	2, +1	1, +1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	2, 0	1, 0	0, 0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	2, −1	1, −1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	2, −2
−1 −1	1

3/2×1

m_1 m_2	5/2, +5/2
+3/2 +1	1

m_1 m_2	5/2, +3/2	3/2, +3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	5/2, −3/2	3/2, −3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	5/2, −5/2
−3/2 −1	1

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

3/2×3/2

m_1 m_2	3, +3
+3/2 +3/2	1

m_1 m_2	3, +2	2, +2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	3, +1	2, +1	1, +1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	3, 0	2, 0	1, 0	0, 0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	3, −1	2, −1	1, −1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	3, −2	2, −2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	3, −3
−3/2 −3/2	1

2×3/2

m_1 m_2	7/2, +7/2
+2 +3/2	1

m_1 m_2	7/2, +5/2	5/2, +5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	7/2, −5/2	5/2, −5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	7/2, −7/2
−2 −3/2	1

2×2

m_1 m_2	4, +4
+2 +2	1

m_1 m_2	4, +3	3, +3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	4, +2	3, +2	2, +2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	4, +1	3, +1	2, +1	1, +1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	4, −1	3, −1	2, −1	1, −1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	4, −2	3, −2	2, −2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	4, −3	3, −3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	4, −4
−2 −2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).