ENERGY CONSIDERATIONS

STORED 'MAGNETIC' ENERGY (BECKER SEC 55)

IN ELECTROSTATICS WE FOUND THAT THE WORK REQUIRED TO ASSEMBLE A CHARGE DISTRIBUTION IS

$$U = \frac{1}{2}\int \rho\,\phi\, d\mathrm{vol}.$$

WE WERE ABLE TO TRANSFORM THIS INTO A RELATION INVOLVING ONLY THE FIELDS,

$$U = \frac{1}{8\pi}\int \bar{E}\cdot\bar{D}\, d\mathrm{vol} \qquad \text{(FOR LINEAR DIELECTRICS)}.$$

WE TURN NOW TO THE QUESTION OF ENERGY IN SITUATIONS WHERE FLOWING CURRENTS HAVE CREATED MAGNETIC FIELDS. (WE HAVE ALREADY DISCUSSED THE ENERGY OF PERMANENT MAGNETIC DIPOLES IN EXTERNAL FIELDS, P.87)

ON P.77 WE REMARKED HOW ENERGY MUST BE SUPPLIED TO MAINTAIN STEADY CURRENTS IN CONDUCTORS OF FINITE CONDUCTIVITY — DUE TO THE JOULE HEATING. THIS ENERGY IS SUPPLIED BY SOME SOURCE OF E.M.F. SUCH AS A BATTERY. WE NOW SHOW THAT AS THE BATTERIES SET THE CURRENTS IN MOTION, THEY DO EXTRA WORK BEYOND THAT LOST TO JOULE HEATING. IT IS THIS EXTRA TERM WHICH WE WILL CALL THE MAGNETIC ENERGY.

AS BEFORE, WE CALL $\bar{E}'$ THE NON-ELECTROMAGNETIC "FIELD" CREATED INSIDE THE SOURCE OF E.M.F. WHICH DRIVES THE CURRENT.

THEN, $\bar{J} = \sigma(\bar{E} + \bar{E}')$.

NOW $\bar{E}$ IS THE REST OF THE ELECTRIC FIELD, CONSISTING OF THE ELECTROSTATIC FIELD $-\bar{\nabla}\phi$, AND THE INDUCED FIELD $-\frac{1}{c}\frac{\partial \bar{A}}{\partial t}$ ARISING FROM THE FARADAY EFFECT. [$-\frac{1}{c}\frac{\partial \bar{A}}{\partial t}$ IS NON-ELECTROSTATIC, SO OUR PRESENT NOTATION IS SLIGHTLY DIFFERENT THAN BEFORE.]

THE BATTERY DOES WORK ON THE CHARGES IT MOVES AT RATE

$$\frac{dU_{BAT}}{dt} = \bar{F}'\cdot\bar{v} = \rho\bar{E}'\cdot\bar{v} = \bar{J}\cdot\bar{E}' \qquad \text{PER UNIT VOLUME}$$

$$= \frac{J^2}{\sigma} - \bar{J}\cdot\bar{E}, \qquad \text{NOTING } \bar{E}' = \frac{\bar{J}}{\sigma} - \bar{E}.$$

WE IDENTIFY THE 2ND TERM AS THE RATE OF INCREASE OF MAGNETIC ENERGY,

$$\frac{dU_{MAG}}{dt} = -\bar{J}\cdot\bar{E} = \bar{J}\cdot\left(\bar{\nabla}\phi + \frac{1}{c}\frac{\partial\bar{A}}{\partial t}\right).$$

[THE FIRST TERM IS THE JOULE HEATING.]

THE FIRST TERM, $\bar{J}\cdot\bar{\nabla}\phi$, DOES NOT APPEAR TO HAVE MUCH TO DO WITH MAGNETISM. IT IS FORTUNATE FOR OUR ARGUMENT THAT IT VANISHES ON INTEGRATION OVER VOLUME

$$\int \bar{J}\cdot\bar{\nabla}\phi \, dvol = \int \bar{\nabla}\cdot(\bar{J}\phi)\, dvol - \int \phi(\underbrace{\bar{\nabla}\cdot\bar{J}}_{\sim 0})\, dvol$$

$$= \oint \phi \bar{J}\cdot d\bar{S} = 0 \quad \text{FOR A SURFACE AT } \infty.$$

WE HAVE SET $\bar{\nabla}\cdot\bar{J} = 0$ ABOVE, WHICH STRICTLY HOLDS ONLY IN MAGNETOSTATICS. BUT IN BUILDING UP OUR FINAL CURRENT DISTRIBUTIONS, WE SUPPOSE THAT THE CHANGES OCCUR SO SLOWLY THAT $\bar{\nabla}\cdot\bar{J} \sim 0$ AT ALL TIMES.

WE ARE LEFT WITH $\dfrac{dU_{MAG}}{dt} = \dfrac{1}{c}\displaystyle\int \bar{J}\cdot\dfrac{\partial\bar{A}}{\partial t}\, dvol.$

WE INTEGRATE THIS OVER TIME, NOTING THAT IF $\bar{\nabla}\cdot\bar{J} \sim 0$ ALWAYS THEN $\bar{A} = \dfrac{1}{c}\displaystyle\int \dfrac{\bar{J}\, dvol}{r}$, SO $\displaystyle\int \bar{J}\cdot\frac{\partial\bar{A}}{\partial t} = \int \bar{A}\cdot\frac{\partial\bar{J}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\int \bar{J}\cdot\bar{A}$.

HENCE, $\underline{U_{MAG} = \dfrac{1}{2c}\displaystyle\int \bar{J}\cdot\bar{A}\, dvol}$.

THIS MAY BE CONVERTED TO AN INTEGRAL OF THE FIELDS ONLY BY NOTING $\bar{\nabla}\times\bar{H} = \dfrac{4\pi\bar{J}}{c}$ SUPPOSING $\dfrac{\partial\bar{E}}{\partial t} \approx 0$

$$U_{MAG} = \frac{1}{8\pi}\int (\bar{\nabla}\times\bar{H})\cdot\bar{A}\, dvol = \frac{1}{8\pi}\underbrace{\int \bar{\nabla}\cdot(\bar{H}\times\bar{A})\, dvol}_{\oint_{SURFACE} \bar{H}\times\bar{A}\cdot d\bar{S} \to 0} + \frac{1}{8\pi}\int \bar{H}\cdot\bar{\nabla}\times\bar{A}\, dvol$$

SO $\underline{U_{MAG} = \dfrac{1}{8\pi}\displaystyle\int \bar{B}\cdot\bar{H}\, dvol}$

THESE RELATIONS HAVE BEEN OBTAINED SUPPOSING THE CURRENTS AND FIELDS VARY SLOWLY WITH TIME. WE WILL SHORTLY RELAX THIS CONSTRAINT, FINDING THE SAME FORMS HOLD IN GENERAL.

Ph 206 LECTURE 10

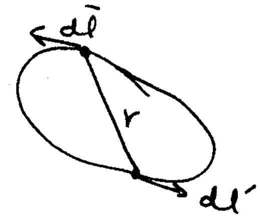
ENERGY IN CIRCUITS - INDUCTANCE (BECKER CHAPTER CIV)

WE CONSIDER OUR ENERGY RELATION, $U_{MAG} = \frac{1}{2c}\int \bar{J}\cdot\bar{A}\, dvol$,

IN SITUATIONS WHERE ALL THE CURRENT FLOWS IN CIRCUITS.

SUPPOSE THERE IS ONLY A SINGLE CIRCUIT CONTAINING CURRENT I.

THEN $\bar{A} = \frac{I}{c}\oint \frac{d\bar{l}}{r}$



SO $U_{MAG} = \frac{I^2}{2c^2}\oint\oint \frac{d\bar{l}\cdot d\bar{l}'}{r}$

↖ BOTH INTEGRALS AROUND THE SAME CIRCUIT

THIS IS USUALLY WRITTEN

$$U_{MAG} = \frac{1}{2} L I^2 \quad \text{OR} \quad \frac{1}{2} L_{11} I^2$$

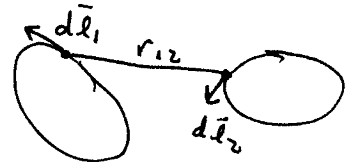
WHERE $L = \frac{1}{c^2}\oint\oint \frac{d\bar{l}\cdot d\bar{l}'}{r}$ ≡ INDUCTANCE (OR SELF-INDUCTANCE).

THE INDUCTANCE IS A PROPERTY OF THE GEOMETRY OF THE CIRCUIT. ITS DIMENSIONS ARE [LENGTH]/c^2 IN GAUSSIAN UNITS.

IF WE HAVE TWO CIRCUITS, CARRYING CURRENTS I_1 AND I_2 WE CONSIDER ONLY THE ENERGY DUE TO THE EFFECT OF ONE CIRCUIT ON THE OTHER:

$$U_{INTERACTION} = \frac{1}{2c}\int \bar{J}_1\cdot\bar{A}_2\, dvol_1 + \frac{1}{2c}\int \bar{J}_2\cdot\bar{A}_1\, dvol_2$$

$$= \frac{I_1 I_2}{c^2}\oint\oint \frac{d\bar{l}_1\cdot d\bar{l}_2}{r_{12}}$$



$$\equiv I_1 I_2 L_{12}$$

[THE SENSE OF INTEGRATION SHOULD FOLLOW THE DIRECTION OF THE CURRENT]

WHERE $L_{12} = \frac{1}{c^2}\oint\oint \frac{d\bar{l}_1\cdot d\bar{l}_2}{r_{12}}$ ≡ MUTUAL INDUCTANCE.

THESE RATHER FORMIDABLE EXPRESSIONS ARE SIMPLY RELATED TO THE MAGNETIC FLUX PASSING THRU THE CIRCUITS.

WE CLAIM $\frac{1}{c}\Phi_{\text{THRU 1 DUE TO 2}} = I_2 L_{12}$

NOW $\Phi_{1 \text{ FROM } 2} = \int \bar{B}_2 \cdot d\bar{S}_1 = \int \bar{\nabla} \times \bar{A}_2 \cdot d\bar{S}_1 = \oint \bar{A}_2 \cdot d\bar{l}_1$

$$= \frac{I_2}{c} \oint \oint \frac{d\bar{l}_2 \cdot d\bar{l}_1}{r_{12}}$$

[THE RELATION $\Phi_{MAG} = \oint \bar{A} \cdot d\bar{l}$ GIVES AN ADDITIONAL MEANING TO THE VECTOR POTENTIAL $\bar{A}$]

$= c I_2 L_{12}$ AS CLAIMED.

OF COURSE, $\frac{1}{c}\Phi = IL$ IS THE FLUX LINKING A CIRCUIT DUE TO ITS OWN MAGNETIC FIELD.

IT IS OFTEN EASIER TO USE THE FLUX RELATIONS TO CALCULATE THE INDUCTANCES THAN DIRECTLY USING THE INTEGRALS GIVEN ON P. 114.

THE INDUCTANCES FIND THEIR GREATEST USE IN CIRCUIT EQUATIONS:

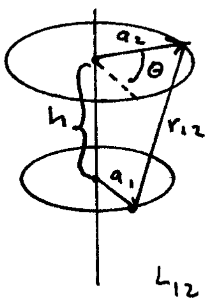
$$\mathcal{E} = IR + \frac{Q}{C}$$

R = RESITANCE
C = CAPACITANCE

THE EMF IS $\mathcal{E} = V_{BAT} - \frac{1}{c}\dot{\Phi}_{MAG} = V_{BAT} - L\dot{I}$

SO $V = L\dot{I} + IR + \frac{Q}{C}$

EXAMPLE: <u>MUTUAL INDUCTANCE OF TWO COAXIAL CIRCULAR LOOPS</u> (BECKER SEC 49a)



FROM P. 114 WE HAVE $L_{12} = \frac{1}{c^2}\oint_1 \oint_2 \frac{d\bar{l}_1 \cdot d\bar{l}_2}{r_{12}}$, WHERE

$r_{12} = \sqrt{h^2 + a_1^2 + a_2^2 - 2a_1a_2\cos\theta}$.

WRITING $dl_1 = a_1 d\phi_1$ AND $dl_2 = a_2 d\phi_2$ IN TERMS OF AZIMUTHS ϕ_1 & ϕ_2,

$$L_{12} = \frac{1}{c^2}\int_0^{2\pi}\int_0^{2\pi} \frac{a_1a_2\cos\theta \, d\phi_1 d\phi_2}{\sqrt{h^2+a_1^2+a_2^2-2a_1a_2\cos\theta}} = \frac{4\pi}{c^2}\int_0^{\pi} \frac{a_1a_2\cos\theta \, d\theta}{\sqrt{h^2+a_1^2+a_2^2-2a_1a_2\cos\theta}},$$

NOTING $\phi_2 = \phi_1 + \theta$. THIS IS AN <u>ELLIPTIC</u> <u>INTEGRAL</u> $\Rightarrow$ USE TABLES IN GENERAL.

CASE a). $h >> \max\{a_1, a_2\}$. IN THIS CASE WE CAN IGNORE THE TERMS $a_1^2 + a_2^2$, AND EXPAND THE DENOMINATOR TO FIND

$$L_{12} \approx \frac{4\pi a_1 a_2}{c^2 h}\int_0^{\pi} \cos\theta \, d\theta \left(1 + \frac{a_1a_2\cos\theta}{h^2}\right) = \frac{2\pi a_1^2 a_2^2}{c^2 h^3}.$$

[AS ALWAYS, AN INDUCTANCE HAS DIMENSIONS $\frac{\text{LENGTH}}{c^2}$ IN C.G.S. UNITS.]

CASE b) $h \ll \min\{a_1, a_2\}$, AND $a_1 \approx a_2 \approx a$, i.e., LOOPS ALMOST IDENTICAL.

$$L_{12} \approx \frac{4\pi a^2}{c^2}\int_0^{\pi} \frac{\cos\theta \, d\theta}{\sqrt{\underbrace{h^2+(a_1-a_2)^2}_{\equiv b^2} + 4a^2 \sin^2 \theta/2}} \overset{(\alpha = \frac{\theta}{2})}{=} \frac{4\pi a}{c^2}\int_0^{\pi/2} \frac{\cos 2\alpha \, d\alpha}{\sqrt{\sin^2\alpha + \frac{b^2}{4a^2}}}$$

TRICK: SPLIT THE INTEGRAL INTO TWO PARTS INTRODUCING ϵ SUCH THAT $\frac{b}{2a} \ll \epsilon \ll 1$

THEN FOR $0<\alpha<\epsilon$, $\cos 2\alpha \approx 1$, $\sin^2\alpha \approx \alpha^2$; WHILE ON $\epsilon<\alpha<\pi/2$, $\sqrt{\sin^2\alpha + \frac{b^2}{4a^2}} \approx \sin\alpha$

$$\text{THUS } L_{12} \approx \frac{4\pi a}{c^2}\left\{\int_0^{\epsilon} \frac{d\alpha}{\sqrt{\alpha^2 + \frac{b^2}{4a^2}}} + \int_{\epsilon}^{\pi/2} \frac{1-2\sin^2\alpha}{\sin\alpha}\, d\alpha\right\}$$

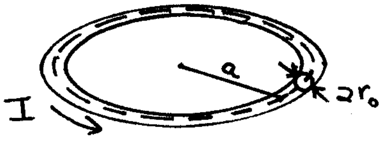
$$= \frac{4\pi a}{c^2}\left\{ \ln\left(\alpha + \sqrt{\alpha^2 + \frac{b^2}{4a^2}}\right)\Big|_0^{\epsilon} + \left(\ln\tan\frac{\alpha}{2} + 2\cos\alpha\right)\Big|_{\epsilon}^{\pi/2}\right\}$$

$$= \frac{4\pi a}{c^2}\left\{\ln 2\epsilon - \ln\frac{b}{2a} - \ln\frac{\epsilon}{2} - 2\right\} = \underline{\frac{4\pi a}{c^2}\left(\ln\frac{8a}{b} - 2\right)} \quad \text{INDEPENDENT OF } \epsilon$$

$b = \sqrt{h^2 + (a_1-a_2)^2}$ = CLOSEST DISTANCE BETWEEN THE TWO LOOPS.

EXAMPLE: SELF INDUCTANCE OF A CIRCULAR RING (BECKER SEC. 49b)

THE RING HAS RADIUS a AND IS MADE FROM A WIRE OF RADIUS $r_0 \ll a$



RECALL THAT THE MEANING OF THE SELF INDUCTANCE L IS THAT THE FLUX THROUGH THE LOOP SHOULD BE RELATED TO THE CURRENT I IN THE LOOP ACCORDING TO $\Phi = cLI$. BUT IN THE LIMIT $r_0 \to 0$ THE FLUX BECOMES INFINITE SINCE FIELD $B(r_0) \to \infty$. TO OBTAIN A MEANINGFUL VALUE OF L WE MUST EXTEND OUR CONCEPT OF INDUCTANCE TO INCLUDE CONDUCTORS WITH FINITE CROSS SECTION.

WE FIRST ARGUE THAT A CONSISTENT DEFINITION OF THE FLUX THROUGH A LOOP OF FINITE CROSS SECTIONAL AREA A ($=\pi r_0^2$ IN OUR CASE) IS

$$\Phi = \frac{1}{A}\sum_i A_i \Phi_i,$$

WHERE WE SUBDIVIDE THE LOOP INTO A LARGE NUMBER OF LOOPS EACH OF CROSS SECTIONAL AREA A_i, AND WHERE THE FLUX THRU LOOP i DUE TO CURRENT I_j IN LOOP j IS $\Phi_{ij} = cL_{ij}I_j$ AND L_{ij} CAN BE EVALUATED AS IN THE PREVIOUS EXAMPLE.

IF THE ENTIRE RING HAS RESISTANCE R THEN THE RESISTANCE OF LOOP i IS $R_i = \frac{RA}{A_i}$, SUPPOSING THE CURRENT IN LOOP i IS $I_i = \frac{IA_i}{A}$.

FARADAY'S LAW + OHM'S LAW FOR LOOP i TELLS US THAT IF FLUX Φ_i IS CHANGING, Then $I_i = -\frac{1}{cR_i}\frac{d\Phi_i}{dt}$, so $I = \sum_i' I_i = -\frac{1}{c}\sum_i \frac{1}{R_i}\frac{d\Phi_i}{dt} = -\frac{1}{cRA}\sum_i' A_i \frac{d\Phi_i}{dt}$

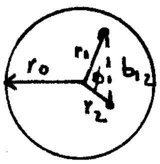
But we also expect $I = -\frac{1}{cR}\frac{d\Phi}{dt}$ considering the ring as a whole.

Hence $\Phi = \frac{1}{A}\sum_i' A_i \Phi_i$ is a consistent statement.

As noted, Φ_i can be related to currents in the other sub-loops by

$$\Phi_i = c\sum_j' L_{ij} I_j = \frac{cI}{A}\sum_j L_{ij} A_j, \text{ so } \Phi = \frac{cI}{A^2}\sum_{i,j}' L_{ij} A_i A_j$$

Hence the self inductance L obeys $L = \frac{1}{A^2}\sum_{i,j}' L_{ij} A_i A_j$.



As the number of sub loops increases this becomes $L = \frac{1}{A^2}\iint L_{12}\, dA_1 dA_2$

where 1 and 2 label positions within a circle of radius r_0

From the previous example, $L_{12} \approx \frac{4\pi a}{c^2}(\ln 8a - \ln b_{12} - 2)$

where $b_{12} = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2\cos\phi}$. Thus

$$L \approx \frac{4\pi a}{c^2}\left\{\ln 8a - 2 - \frac{2\pi}{\pi^2 r_0^4}\int_0^{r_0} r_1\, dr_1 \int_0^{r_0} r_2\, dr_2 \int_0^{2\pi} d\phi\, \ln\sqrt{r_1^2 + r_2^2 - 2r_1 r_2\cos\phi}\right\}$$

The integral can be evaluated by the trick of splitting the r_2 interval:

$$\text{INT} = \int_0^{r_0} r_1\, dr_1 \left\{\int_0^{r_1} r_2\, dr_2 + \int_{r_1}^{r_0} r_2\, dr_2\right\}\int_0^{2\pi} d\phi\left[\ln r_> + \frac{1}{2}\ln\left(1 - \frac{r_<}{r_>}e^{i\phi}\right) + \frac{1}{2}\ln\left(1 - \frac{r_<}{r_>}e^{-i\phi}\right)\right.$$

$r_> = \max\{r_1, r_2\}$, $r_< = \min\{r_1, r_2\}$

Noting that $\ln(1-x) = -\sum \frac{x^n}{n}$, we have

$$\int_0^{2\pi} d\phi\, [\cdots] = 2\pi \ln r_> - \int_0^{2\pi} d\phi \sum' \left(\frac{r_<}{r_>}\right)^n \frac{\cos n\phi}{n} = 2\pi \ln r_>$$

$$\text{so INT} = 2\pi\int_0^{r_1} r_1\, dr_1 \left\{\int_0^{r_1} r_2\, dr_2 \ln r_1 + \int_{r_1}^{r_0} r_2\, dr_2 \ln r_2\right\} = 2\pi\int_0^{r_0} r_1\, dr_1\left\{\frac{r_0^2 \ln r_0}{2} - \frac{r_0^2}{4} + \frac{r_1^2}{4}\right\}$$

$$= \frac{\pi r_0^4}{2}\left(\ln r_0 - \tfrac{1}{4}\right)$$

And finally, $L = \frac{4\pi a}{c^2}\left(\ln\frac{8a}{r_0} - \frac{7}{4}\right)$, little different from the mutual inductance of two coaxial loops of radius a at a distance r_0 apart.

If the current is confined to the surface of the ring, as for a superconducting ring, the self inductance is again

$L = \frac{4\pi a}{c^2}\left(\ln\frac{8a}{r_0} - 2\right)$. See V. Fock, Phys. Z. Sow. U. 1, 215 (1932).

ANOTHER VIEW IS THAT THE SELF INDUCTANCE OF THE RING IS PARTLY DUE TO THE INTERIOR OF THE WIRE AND PARTLY DUE TO ITS EXTERIOR:

$$L = L_{IN} + L_{OUT}.$$

THE MAGNETIC FIELD EXTERIOR TO THE WIRE IS THE SAME AS IF ALL THE CURRENT WERE CONCENTRATED ON THE AXIS. HENCE, WE CAN CALCULATE $L_{OUT} = \Phi_{OUT}/I$ BY CONSIDERING THE FLUX LINKED BY ANY LOOP ON THE SURFACE OF THE WIRE DUE TO CURRENT I ON THE AXIS. FROM PP 115-115a THIS GIVE $L_{OUT} = \frac{4\pi a}{c^2}\left(\ln\frac{8a}{r_0} - 2\right)$.

INSIDE THE WIRE WE CAN USE AN ENERGY METHOD:

$$U_{IN} = \frac{1}{2} L_{IN} I^2 = \int_{IN} \frac{B^2}{8\pi}\, dVol = \frac{2\pi a}{8\pi}\int_0^{r_0}\left(\frac{2Ir}{c r_0^2}\right)^2 2\pi r\, dr = \frac{\pi a I^2}{2c^2},$$

THUS, $L_{IN} = \frac{\pi a}{c^2}$, AND $L = L_{IN} + L_{OUT} = \frac{4\pi a}{c^2}\left(\ln\frac{8a}{r_0} - \frac{7}{4}\right)$ AS BEFORE.

WE CAN ALSO MAKE A ROUGH ESTIMATE OF THE SELF INDUCTANCE BY THE ENERGY METHOD:

$$U = \frac{1}{2} L I^2 \sim \frac{1}{2} L_{OUT} I^2 \sim \int_{OUT} \frac{B^2}{8\pi}\, dVol \approx \frac{2\pi a}{8\pi}\int_{r_0}^{a}\left(\frac{2I}{cr}\right)^2 2\pi r\, dr = \frac{2\pi a I^2}{c^2}\ln\frac{a}{r_0}$$

SO $L \sim \frac{4\pi a}{c^2}\ln\frac{a}{r_0} \quad \left(= \frac{4\pi a}{c^2}\left(\ln\frac{8a}{r_0} - \frac{7}{4} + \frac{1}{3}\right)\right)$.

FORCES ON CIRCUITS (BECKER SEC 52)

WE EXPECT THAT THE MAGNETIC FORCE ON A CIRCUIT CAN BE OBTAINED BY TAKING THE GRADIENT OF THE MAGNETIC ENERGY

$$\bar{F} = -\bar{\nabla} U_{MAG}.$$

BUT WE MUST BE CAREFUL. TYPICALLY THE CURRENTS DO NOT MAINTAIN THEMSELVES (BECAUSE OF JOULE HEATING LOSSES), BUT ARE DRIVEN BY BATTERIES. AS WAS THE CASE IN ELECTROSTATICS, CONSIDERATION OF THE WORK DONE BY THE BATTERIES WILL LEAD

TO $\bar{F} = +\bar{\nabla} U_{MAG}.$ (BATTERIES PRESENT)

TO SEE THIS, SUPPOSE CIRCUIT i IS IN MOTION WITH VELOCITY $\bar{v}_i$, WHILE BATTERIES HOLD THE CURRENTS CONSTANT IN ALL CIRCUITS.

TAKING INTO ACCOUNT ALL FORMS OF ENERGY, ENERGY IS CONSERVED. $\therefore \frac{dU_{TOTAL}}{dt} = 0$

So $\quad 0 = \underbrace{\bar{F}_i \cdot \bar{v}_i}_{\substack{\text{MECHANICAL} \\ \text{POWER SUPPLIED} \\ \text{TO CIRCUIT } i}} + \dfrac{dU_{MAG}}{dt} + \underbrace{\sum_j I_j^2 R_j}_{\text{JOULE HEATING}} - \underbrace{\sum_j I_j V_j}_{\substack{\text{POWER PROVIDED BY} \\ \text{THE BATTERIES}}}$

NOW $\quad U_{MAG} = \dfrac{1}{2c}\int \bar{j}\cdot\bar{A}\, d\mathrm{vol} = \dfrac{1}{2c}\sum_j I_j \oint \bar{A}\cdot d\bar{l}_j = \dfrac{1}{2c}\sum_j I_j \Phi_j$

WHERE $\quad \Phi_j = \int \bar{B}\cdot d\bar{S}_j$ = MAGNETIC FLUX THRU CIRCUIT j

HENCE $\quad \dfrac{dU_{MAG}}{dt} = \dfrac{1}{2c}\sum_j I_j \dot{\Phi}_j \qquad$ (CURRENTS HELD CONSTANT)

ALSO NOTE THAT FOR EACH CIRCUIT $\quad V_j = I_j R_j + \dfrac{\dot{\Phi}_j}{c}$

(WE IGNORE ANY CAPACITORS IN THE CIRCUITS) i.e. IGNORE ELECTROSTATIC ENERGY

THUS $\quad 0 = \bar{F}_i\cdot\bar{v}_i + \dfrac{1}{2c}\sum_j I_j\dot{\Phi}_j + \sum_j I_j^2 R_j - \sum_j I_j^2 R_j - \sum_j \dfrac{I_j \dot{\Phi}_j}{c}$

SO $\quad \bar{F}_i\cdot\bar{v}_i = +\dfrac{1}{2c}\sum_j I_j \dot{\Phi}_j = +\dfrac{dU_{MAG}}{dt} = \bar{\nabla}U_{MAG}\cdot\bar{v}$

SO $\quad \underline{\bar{F} = +\bar{\nabla}U_{MAG}}$ AS CLAIMED, WHEN THE CURRENTS ARE HELD CONSTANT.

<u>EXAMPLE</u> CONSIDER TWO CURRENT LOOPS OF THE SAME SIZE, BUT WITH CURRENTS FLOWING IN OPPOSITE DIRECTIONS

IF THE LOOPS ARE ESSENTIALLY ON TOP OF ONE ANOTHER THE MAGNETIC FIELDS CANCEL TO GOOD APPROXIMATION, AND $U_{MAG} \approx 0$

THE BASIC FORCE LAW FOR CIRCUITS (p80) INDICATES THAT THE TWO LOOPS REPEL ONE ANOTHER DUE TO MAGNETIC EFFECTS

SUPPOSE BATTERIES HOLD THE CURRENTS CONSTANT WHILE THE LOOPS FLY APART. AS THE LOOPS SEPARATE THE CANCELLATION OF THE $\bar{B}$ FIELDS IS REDUCED, SO $U_{MAG} = \dfrac{1}{8\pi}\int B^2\, d\mathrm{vol}$ INCREASES.

THIS IS CONSISTENT WITH $\bar{F} = +\bar{\nabla}U_{MAG}$.

SUPPOSE INSTEAD THERE ARE NO BATTERIES HOLDING THE CURRENTS FIXED. FOR AN ORDINARY CONDUCTOR, THE JOULE HEAT LOSS WOULD CONSUME THE MAGNETIC ENERGY, CAUSING THE CURRENTS TO DIE OUT QUICKLY. FOR THE SAKE OF ARGUMENT, WE CONSIDER SUPERCONDUCTING RINGS, WHICH HAS NO JOULE LOSS.

THE LOOPS STILL REPEL EACH OTHER, AND IF I_1, I_2 REMAINED CONSTANT U_{MAG} WOULD RISE AS THEY SEPARATE. THIS HOWEVER WOULD BE A VIOLATION OF CONSERVATION OF ENERGY! BOTH THE MAGNETIC & KINETIC ENERGIES OF THE SYSTEM HAVE INCREASED!

THE WAY OUT MUST INVOLVE A DECREASE OF I_1 & I_2, AND HENCE U_{MAG} ALSO. WE CANNOT ASCRIBE THIS TO JOULE LOSSES BY ASSUMPTION OF SUPERCONDUCTIVITY. ALSO THE MAGNETIC FORCES OF ONE CURRENT ON THE OTHER CANNOT CHANGE THE CHARGES SPEED.

THERE IS A GENERAL RESULT WORTH NOTING: $\bar{F}_{MAG} = e\frac{\bar{v}}{c}\times\bar{B} = \overset{\text{MASS}}{M}\frac{d\bar{v}}{dt}$

$\therefore$ $d\bar{v}$ IS $\perp$ TO $\bar{v}$ AND $|\bar{v}|$ IS UNCHANGED. ANOTHER WAY OF NOTING THE RESULT IS $\frac{dU_{KINETIC}}{dt} = \bar{F}\cdot\bar{v} = \bar{v}\cdot\left(e\frac{\bar{v}}{c}\times\bar{B}\right) = 0.$

$\Rightarrow$ MAGNETIC FIELDS CAN CHANGE A PARTICLE'S MOMENTUM, BUT NOT ITS ENERGY!

THE PARADOX IS RESOLVED BY FARADAY'S LAW! A NET EMF CAN BE INDUCED AROUND THE LOOP, WHICH CAN INDEED CHANGE THE CHARGES' VELOCITY. LENZ' LAW CONFIRMS THAT THE SIGN OF THE EFFECT IS TO REDUCE THE VELOCITIES & CURRENTS: AS LOOP 1 MOVES AWAY FROM LOOP 2 IT SEES SMALLER $\bar{B}_2$ FROM LOOP 2. THIS INDUCES A CHANGE IN I_1 WHICH SHOULD TRY TO RESTORE $\bar{B}$ TO THAT OF $\bar{B}_2$ ORIGINALLY. i.e., ΔI_1 HAS THE SAME SENSE AS I_2 $\Rightarrow$ I_1 IS REDUCED ....

THE POYNTING VECTOR (BECKER SEC. 54)

WE RETURN TO THE CONSIDERATIONS OF ENERGY AT THE BEGINNING OF THIS LECTURE, BUT ARGUE IN A DIFFERENT MANNER SO AS TO REMOVE THE RESTRICTION THAT $\bar{\nabla}\cdot\bar{j}\sim 0$.

WE SAW THAT $\frac{dU_{BAT}}{dt} = \frac{j^2}{\sigma} - \bar{j}\cdot\bar{E}$ = JOULE HEATING + $\frac{dU_{FIELD}}{dt}$, SO $\frac{dU_{FIELD}}{dt} = -\bar{j}\cdot\bar{E}$

IN ANOTHER VIEW, WE IGNORE BATTERIES AND JOULE HEATING AND WRITE

$\frac{dU_{TOT}}{dt} = 0 = \frac{dU_{FIELD}}{dt} + \frac{dU_{MECH}}{dt}$ WHERE $\frac{dU_{MECH}}{dt} = \bar{f}\cdot\bar{v} = \rho\bar{E}\cdot\bar{v} + \frac{\bar{j}}{c}\times\bar{B}\cdot\bar{v} = \bar{j}\cdot\bar{E}$ SINCE $\bar{j}\parallel\bar{v}$

AGAIN, $\frac{dU_{FIELD}}{dt} = -\bar{j}\cdot\bar{E}$. [TO CONNECT THE TWO VIEWS, NOTE THAT IF THE CONDUCTORS AREN'T MOVING, THE CHANGING MECHANICAL ENERGY IS CONVERTED TO HEAT.]

AGAIN WE CAN REPLACE THE CURRENT $\bar{j}$ BY AN EXPRESSION INVOLVING FIELDS ONLY

$$\bar{\nabla}\times\bar{H} = \frac{4\pi}{c}\bar{j} + \frac{1}{c}\frac{\partial\bar{D}}{\partial t} \Rightarrow \bar{j} = \frac{c}{4\pi}\bar{\nabla}\times\bar{H} - \frac{1}{4\pi}\frac{\partial\bar{D}}{\partial t}$$

$$\frac{dU_{FIELD}}{dt} = -\frac{c}{4\pi}\,\bar{E}\cdot(\bar{\nabla}\times\bar{H}) + \frac{1}{4\pi}\,\bar{E}\cdot\frac{\partial\bar{D}}{\partial t}$$

$$= \frac{c}{4\pi}\,\bar{\nabla}\cdot(\bar{E}\times\bar{H}) - \underbrace{\frac{c}{4\pi}\,\bar{H}\cdot(\nabla\times E)}_{+\frac{1}{4\pi}\bar{H}\cdot\frac{\partial\bar{B}}{\partial t} = \frac{1}{8\pi}\frac{\partial(\bar{B}\cdot\bar{H})}{\partial t}} + \frac{1}{8\pi}\,\frac{\partial(\bar{E}\cdot\bar{D})}{\partial t}$$

[IN MEDIA WHERE $\bar{D}=\epsilon\bar{E}$ & $\bar{B}=\mu\bar{H}$]

IF WE INTEGRATE THIS OVER A VOLUME, WE HAVE

$$\frac{dU_{FIELD}}{dt} = \frac{1}{8\pi}\int\frac{\partial}{\partial t}(\bar{E}\cdot\bar{D}+\bar{B}\cdot\bar{H})\,dvol + \frac{c}{4\pi}\int\bar{\nabla}\cdot(\bar{E}\times\bar{H})\,dvol$$

$$= \frac{d}{dt}\int\frac{(\bar{E}\cdot\bar{D}+\bar{B}\cdot\bar{H})}{8\pi}\,dvol + \frac{c}{4\pi}\oint_{SURFACE}\bar{E}\times\bar{H}\cdot d\,\text{AREA}$$

RECALLING THE ORIGIN OF THIS DERIVATION, $\frac{dU_{FIELD}}{dt}$ IS THE POWER ADDED BY OUTSIDE SOURCES IN ORDER TO CAUSE CHANGES IN THE CURRENTS AND FIELDS. WE RECOGNIZE A SIMILARITY BETWEEN THE ABOVE EXPRESSION AND THAT WHICH WOULD HOLD IF WE COULD CREATE NEW CHARGES WITHIN SOME VOLUME:

$$\text{RATE OF CREATION OF CHARGE} = \frac{d}{dt}\int\rho\,dvol + \underbrace{\int\bar{J}\cdot d\bar{S}}_{\text{RATE OF FLOW OF CHARGE}}$$

RATE OF ACCUMULATION OF CHARGE

WE THEREFORE INTERPRET $u = \frac{1}{8\pi}(\bar{E}\cdot\bar{D}+\bar{B}\cdot\bar{H})$ = FIELD ENERGY DENSITY

THIS IS CONSISTENT WITH OUR PREVIOUS DISCUSSION OF ELECTRIC AND MAGNETIC ENERGY.

WE HAVE A NEW TERM: $\bar{S} = \frac{c}{4\pi}\bar{E}\times\bar{H}$ = POYNTING VECTOR

WE ARE LED TO THE INTERPRETATION THAT THIS REPRESENTS A FLOW OF FIELD ENERGY.

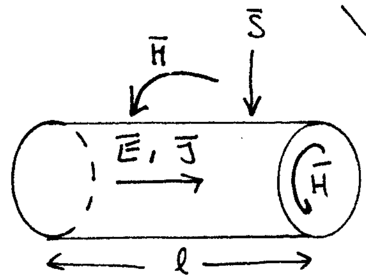
FOR EXAMPLE, IF THE BATTERY SUPPLIES ENERGY SO THAT $\frac{dU_{FIELD}}{dt} > 0$ IN SOME VOLUME, BUT U_{FIELD} IS CONSTANT IN TIME, THEN ENERGY MUST BE FLOWING OUT OF THE VOLUME TO MAINTAIN THE ENERGY BALANCE. $\bar{S} = +\frac{c}{4\pi}\bar{E}\times\bar{H}$ DESCRIBES THIS QUANTITATIVELY.

$\bar{S}$ HAS DIMENSIONS $\frac{\text{ENERGY}}{\text{AREA}\cdot\text{TIME}}$ OR $\frac{\text{ENERGY}}{\text{VOLUME}}\cdot$ VELOCITY

THE POYNTING VECTOR WILL HAVE ITS GREATEST UTILITY AND MEANING IN SITUATIONS INVOLVING ELECTROMAGNETIC WAVES.

THERE ARE A FEW QUASI-STATIC EXAMPLES IN WHICH $\bar{S}\neq 0$, WHICH LEAD TO POSSIBLY AMUSING INTERPRETATIONS.

EXAMPLE ENERGY FLOW IN A RESISTOR.



$\bar{J} = \sigma \bar{E}$

AMPERE: $2\pi r H = \frac{4\pi}{c} j \pi r^2 \Rightarrow H = \frac{2\pi r j}{c} = \frac{2\pi r \sigma E}{c}$

$\bar{E} \times \bar{H} = \frac{2\pi r \sigma E^2}{c}$ POINTS RADIALLY INWARD

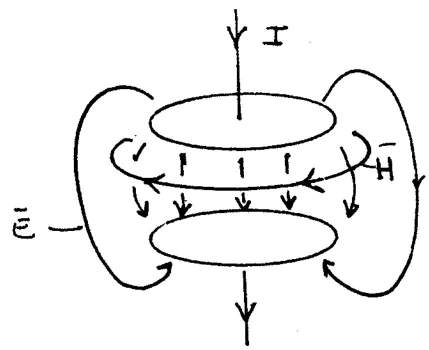
$$\frac{c}{4\pi} \int \bar{E} \times \bar{H} \cdot ds = \frac{r \sigma E^2}{2} \cdot 2\pi r \ell = \pi r^2 \ell \sigma E^2 = \frac{\pi r^2}{\ell} \sigma (E\ell)^2 = \frac{V^2}{R} = I^2 R$$

SO "ENERGY FLOW" IN = JOULE HEATING.

THIS SEEMS FINE. BUT NOTE THAT POYNTING CLAIMS THAT THE ENERGY CAME IN THRU THE SIDES OF THE RESISTOR, NOT ALONG THE WIRE AS WE MIGHT HAVE EXPECTED.

ON THE HOMEWORK SET, YOU ARE ENCOURAGED TO EXAMINE $\bar{E}$ AND $\bar{H}$ OUTSIDE A RESISTIVE WIRE TO SHOW THAT THIS VIEW IS NOT UTTERLY RIDICULOUS.

EXAMPLE CHARGING A CAPACITOR



AS A CAPACITOR CHARGES UP IT ACCUMULATES ENERGY $\frac{1}{8\pi} \int E^2 \, dvol$.

WHILE IT IS CHARGING, A MAGNETIC FIELD EXISTS AS SHOWN. AGAIN $\bar{E} \times \bar{H}$ POINTS INWARD, AND CAN BE SAID TO INDICATE THE FLOW OF ENERGY INTO THE GAP OF THE CAPACITOR.

FIELD MOMENTUM (BECKER SEC 56)

FURTHER RESULTS IN THE SPIRIT OF POYNTING CAN BE OBTAINED BY EXAMINING MOMENTUM BALANCE WHEN FIELDS ARE PRESENT.

WE CONSIDER A SITUATION IN WHICH ONLY FREE CHARGES AND CURRENTS ARE PRESENT, DESCRIBING THESE BY ρ AND $\bar{J}$.

THEN THE FORCE PER UNIT VOLUME ON THESE CHARGES IS

$$\bar{f} = \rho \bar{E} + \frac{\bar{J}}{c} \times \bar{B} = \frac{d \bar{P}_{MECH}}{dt}$$

WHERE $\bar{P}_{MECH}$ IS THE MOMENTUM OF THE MOVING CHARGES AS CONSIDERED IN CLASSICAL MECHANICS.

MAXWELL SEZ: $\rho = \frac{1}{4\pi}\bar{\nabla}\cdot\bar{D}$ AND $\bar{J} = \frac{c}{4\pi}\bar{\nabla}\times\bar{H} - \frac{1}{4\pi}\frac{\partial\bar{D}}{\partial t}$ [LINEAR MEDIUM]

SO $$\frac{d\bar{P}_m}{dt} = \frac{1}{4\pi}\left\{\bar{E}(\bar{\nabla}\cdot\bar{D}) + \frac{1}{c}\bar{B}\times\frac{\partial\bar{D}}{\partial t} - \bar{B}\times(\bar{\nabla}\times\bar{H})\right\}.$$

NOW $$\bar{B}\times\frac{\partial\bar{D}}{\partial t} = \bar{D}\times\frac{\partial\bar{B}}{\partial t} - \frac{\partial}{\partial t}(\bar{D}\times\bar{B}) = -c\,\bar{D}\times(\bar{\nabla}\times\bar{E}) - \frac{\partial}{\partial t}(\bar{D}\times\bar{B}),$$

AND $\bar{H}(\bar{\nabla}\cdot\bar{B}) = 0$, SO WE CAN ADD IT IN TO FIND

$$\bar{f} = \frac{d\bar{P}_m}{dt} = \frac{1}{4\pi}\left\{\bar{E}(\bar{\nabla}\cdot\bar{D}) + \bar{H}(\bar{\nabla}\cdot\bar{B}) - \bar{D}\times(\bar{\nabla}\times\bar{E}) - \bar{B}\times(\bar{\nabla}\times\bar{H})\right\} - \frac{1}{4\pi c}\frac{\partial}{\partial t}(\bar{D}\times\bar{B})$$

IT IS TEMPTING TO INTERPRET THIS AS

$$\frac{d\bar{P}_M}{dt} + \frac{d\bar{P}_{FIELD}}{dt} = \text{FORCE DENSITY}$$

WHERE $\underline{\bar{P}_{FIELD} = \frac{1}{4\pi c}\bar{D}\times\bar{B}} = \epsilon\mu\frac{\bar{S}}{c^2}$ FOR A LINEAR MEDIUM WHERE $\bar{D} = \epsilon\bar{E}$ & $\bar{B} = \mu\bar{H}$.

THAT IS, WE ASCRIBE A MOMENTUM TO THE FIELDS IF THE POYNTING VECTOR IS NON-ZERO.

AGAIN, THIS MAKES THE MOST SENSE WHEN WE ARE DEALING WITH ELECTROMAGNETIC WAVES.

THESE RELATIONS ARE SEEN TO BE CONSISTENT WITH THE THEORY OF RELATIVITY IN THAT $S \sim \frac{\text{ENERGY}}{\text{VOLUME}}\cdot\text{VELOCITY}$

SO THAT WITH ENERGY = "MASS" $\cdot\, c^2$

$$P_{FIELD} \sim \frac{S}{c^2} \sim \frac{\text{"MASS"}\cdot\text{VELOCITY}}{\text{VOLUME}}$$

HOWEVER THEY PREDATE EINSTEIN'S WORK.

THE CONCEPT OF FIELD MOMENTUM WILL ALLOW US TO RESOLVE APPARENT VIOLATIONS OF NEWTON'S 3RD LAW IN THE INTERACTION BETWEEN MOVING CHARGES.....

WE TURN NOW TO THE LONG EXPRESSION FOR THE FORCE DENSITY. AS IN LECTURE 3, WE LOOK FOR A STRESS TENSOR T_{ij} SUCH THAT THE TOTAL FORCE ON A VOLUME IS

$$F_i = \int f_i \, dvol = \int_{\text{SURFACE}} T_{ij} \, dS_j \qquad (f_i = \text{FORCE DENSITY})$$

THIS WILL BE POSSIBLE WITH $f_i = \dfrac{\partial T_{ij}}{\partial x_j}$

NOW $\left[\bar{E}(\bar{\nabla}\cdot\bar{D}) - \bar{D}\times(\bar{\nabla}\times\bar{E})\right]_i = E_i \dfrac{\partial D_j}{\partial x_j} - \epsilon_{ijk} D_j \epsilon_{klm} \dfrac{\partial E_m}{\partial x_l}$

$$= E_i \frac{\partial D_j}{\partial x_j} - D_j \frac{\partial E_j}{\partial x_i} + D_j \frac{\partial E_i}{\partial x_j}$$

IF $\dfrac{\partial \epsilon}{\partial x} = 0 \longrightarrow$

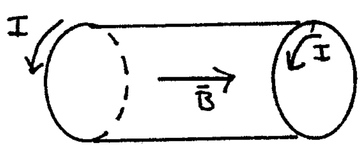
$$= \frac{\partial}{\partial x_j} E_i D_j - \frac{1}{2} \frac{\partial \bar{E}\cdot\bar{D}}{\partial x_i}$$

HENCE, WE SEE AT ONCE WE CAN WRITE (IF $\partial\mu/\partial x = 0$ ALSO)

$$T_{ij} = \frac{1}{4\pi}\left[E_i D_j + B_i H_j - \frac{1}{2}\delta_{ij}(\bar{E}\cdot\bar{D} + \bar{B}\cdot\bar{H})\right]$$

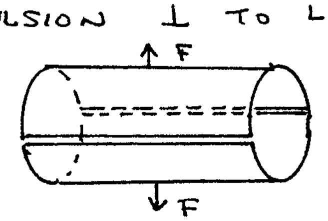
SUPPOSING $\bar{D} = \epsilon\bar{E}$ AND $\bar{B} = \mu\bar{H}$, BUT IGNORING ELECTROSTRICTION AND 'MAGNETOSTRICTION'.

<u>EXAMPLE</u> <u>FORCES ON A SOLENOIDAL MAGNET</u>

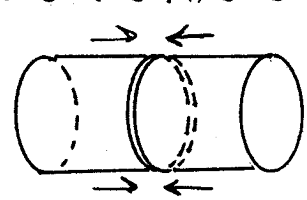


AS IN THE ELECTRIC CASE, WE HAVE
TENSION ALONG LINES OF $\bar{B}$
REPULSION $\perp$ TO LINES OF B

$\Big\}$ MAGNETIC 'PRESSURE' $= \dfrac{B^2}{8\pi}$



THE FORCE TENDS TO BLOW THE MAGNET APART RADIALLY



THE MAGNETIC FORCE PULLS THE TWO HALVES TOGETHER LONGITUDINALLY

DUAL ROLES OF THE POYNTING VECTOR $\bar{S}$ AND STRESS TENSOR $\overline{\overline{T}}$

ENERGY DENSITY: $u_{EM} = \dfrac{\bar{E}\cdot\bar{D} + \bar{B}\cdot\bar{H}}{8\pi}$ OBEYS

$$\frac{dU_{MECH}}{dt} + \frac{du_{EM}}{dt} = -\bar{\nabla}\cdot\bar{S} \qquad \text{WHERE } \bar{S} = \frac{c}{4\pi}\bar{E}\times\bar{H}$$

THIS "CONTINUITY EQUATION" FOR ENERGY DENSITY LEADS TO THE INTERPRETATION OF $\bar{S}$ AS ENERGY FLUX

WE ALSO HAVE MOMENTUM DENSITY $\bar{P}_{EM} = \dfrac{\epsilon\mu}{c^2}\bar{S}$ SUCH THAT

$$\frac{d\bar{P}_{MECH}}{dt} + \frac{d\bar{P}_{EM}}{dt} = \bar{f} = \bar{\nabla}\cdot\overline{\overline{T}}$$

WHICH IS ALSO A "CONTINUITY EQUATION", SO WE CAN INTERPRET

$-\overline{\overline{T}}$ AS MOMENTUM FLUX

IN SUM: $\bar{S}$ IS ENERGY FLUX, $\dfrac{\epsilon\mu}{c^2}\bar{S}$ IS MOMENTUM DENSITY

$-\overline{\overline{T}}$ IS MOMENTUM FLUX, $+\overline{\overline{T}}$ IS STRESS TENSOR

(CAREFUL, IN GENERAL, $\bar{S}$ IS NOT PROPORTIONAL TO MOMENTUM FLUX)

"HIDDEN" MOMENTUM: THERE EXIST APPARENTLY STATIC SITUATIONS IN WHICH

$\bar{P}_{EM} \neq 0$ (AN INSTRUCTIVE EXAMPLE IS A COXIAL CABLE WITH A DC CURRENT.)

BUT "STATIC" $\Rightarrow \bar{P}_{TOTAL} = 0$, SO WE EXPECT THAT $\bar{P}_{MECH} \neq 0$.

THIS (SMALL, BUT NONZERO) MECHANICAL MOMENTUM IS SOMETIMES CALLED "HIDDEN" MOMENTUM.

TYPICALLY, THE $\bar{P}_{EM} \sim \dfrac{1}{c^2} \Rightarrow \bar{P}_{HIDDEN} \sim \dfrac{1}{c^2}$ ALSO

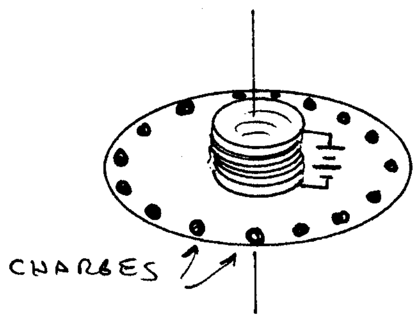
$\Rightarrow$ A RELATIVISTIC CORRECTION

FIELD ANGULAR MOMENTUM

IF WE ARE TO ASCRIBE ENERGY AND MOMENTUM TO THE FIELDS, WE EXPECT ALSO THAT THEY CAN CONTAIN ANGULAR MOMENTUM. IT IS NOT SURPRISING THAT

$$\bar{\mathcal{J}} = \bar{r} \times \frac{(\bar{E} \times \bar{B})}{4\pi c} = \text{ANGULAR MOMENTUM DENSITY.}$$

TO SEE THE NEED FOR SUCH A TERM WE CONSIDER AN EXAMPLE TAKEN FROM FEYNMAN VOL II p17-6.



A SOLENOID COIL IS MOUNTED ON AN INSULATED DISK WHICH HAS A SERIES OF CHARGES DISTRIBUTED ALONG ITS RIM. INITIALLY STEADY CURRENT I FLOWS IN THE COIL. AT SOME MOMENT, THE CIRCUIT IS BROKEN AND THE MAGNETIC FIELD INSIDE THE COIL COLLAPSES.

THEN FOR A LOOP AROUND THE RIM

$$\frac{d\Phi_m}{dt} \neq 0 \Rightarrow \oint \bar{E}\cdot d\bar{l} \neq 0$$

$\Rightarrow$ NET TORQUE ON THE CHARGES

$\Rightarrow$ DISK STARTS TO ROTATE

IN APPARENT VIOLATION OF CONSERVATION OF ANGULAR MOMENTUM!

THE MECHANICAL ANGULAR MOMENTUM OF THE MOVING CHARGES CORRESPONDING TO CURRENT I DOES <u>NOT</u> EXPLAIN THIS EFFECT (UNLIKE THE CASE OF THE EINSTEIN-DE HAAS EXPERIMENT.)

TRY SKETCHING $\bar{E}$, $\bar{B}$, $\bar{S}$ AND $\bar{\mathcal{J}}$ WHILE THE CURRENT IS STILL FLOWING...

WE SKETCH A SOLUTION TO A VARIATION OF THE FEYNMAN DISK PARADOX. ON THE PROBLEM SET WE ENCOURAGE YOU TO TRY TO RESOLVE THE ORIGINAL VERSION.

CONSIDER A CONDUCTING SPHERE OF RADIUS a CARRYING CHARGE Q. THE SPHERE IS MADE OF A PERMANENT MAGNETIC MATERIAL WITH MAGNETIZATION DENSITY $M\hat{z}$ UNIFORM THROUGHOUT THE SPHERE.

THEN
$$\bar{E} = \begin{cases} 0 & r<a \\ \dfrac{Q}{r^2}\hat{r} & r>a \end{cases}$$

AND
$$\bar{B} = \begin{cases} \dfrac{8\pi}{3} M\hat{z} & r<a \\ \dfrac{4\pi a^3 M}{3r^3}\left(3\cos\theta\,\hat{r} - \hat{z}\right) = \dfrac{4\pi a^3 M}{3r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) & r>a \end{cases} \qquad \text{(P. 98)}$$

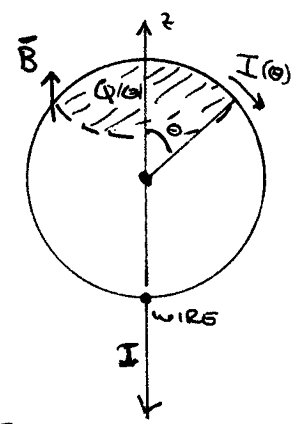
$$\bar{S} = \frac{c}{4\pi}\bar{E}\times\bar{B} = \begin{cases} 0 & r<a \\ \dfrac{c a^3 M Q}{3r^5}\sin\theta\,\hat{\phi} & r>a \end{cases}$$

ONLY THE z-COMPONENT OF ANGULAR MOMENTUM SURVIVES:

$$L_{EM,z} = \int \bar{r}\times\frac{\bar{S}}{c^2}\Big|_z d\text{vol} = \int_a^\infty 2\pi r^2 dr \int_{-1}^{1} d\cos\theta \cdot \frac{a^3 MQ}{3cr^4}\sin^2\theta = \frac{8\pi}{9c}\cdot a^2 MQ$$

NOW SUPPOSE WE DISCHARGE THE SPHERE BY CONNECTING A WIRE TO THE POINT $\theta = \pi$. THE SPHERE IS SUSPENDED FROM A THREAD AT $\theta = 0$, AND IS FREE TO ROTATE.

WE CALCULATE THE TORQUE DUE TO THE INTERACTION OF THE SURFACE CURRENT, $I(\theta)$ WITH THE MAGNETIC FIELD OF THE SPHERE.



$$I(\theta) = \frac{d}{dt}Q(\theta) \quad \text{WHERE} \quad Q(\theta) = \int_\theta^1 Q(t)\frac{d\cos\theta}{2} = \frac{Q(t)}{2}(1-\cos\theta)$$

THE FORCE ON A SURFACE ELEMENT $d\theta\, d\phi$ IS

$$d\bar{F} = -\frac{I}{c}a\,d\theta \cdot \frac{d\phi}{2\pi}\,\frac{8\pi M}{3}\cos\theta\,\hat{\phi}$$
, AND THE z COMPONENT OF THE TORQUE IS
$$d\tau_z = \frac{\dot{Q}(t)}{2c}(1-\cos\theta)\,a^2 d\theta\, d\phi \cdot \frac{4M}{3}\cos\theta\sin\theta.$$

THE RESULTING MECHANICAL ANGULAR MOMENTUM IS

$$L_{MECH}\big|_z = \int \frac{dL_z}{dt}dt = \int dt\,\dot{Q}\,\frac{2Ma^2}{3c}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin\theta\cos\theta\,(1-\cos\theta) = \frac{8\pi a^2 MQ}{9c}$$

MEANWHILE, L_{EM} HAS VANISHED, SO $L_{TOT} = L_{EM} + L_{MECH}$ IS CONSERVED.

(VARIATION: LEAVE Q FIXED, BUT LET $\bar{B}(t)$ SUBSIDE TO ZERO - BY HEATING...)

[N.L. SHARMA, AM. J. PHYS. 56, 420 (1988)]

Ph 206 Lecture 10

Electromagnetic Momentum of a Single Charge

Lorentz force on charge e is $\bar{F} = e\left(\bar{E} + \frac{\bar{v}}{c}\times\bar{B}\right)$

If we replace the fields by potentials, we have

$$\bar{F} = \frac{d\bar{p}_{MECH}}{dt} = -e\bar{\nabla}\phi - \frac{e}{c}\frac{\partial\bar{A}}{\partial t} + \frac{e}{c}\bar{v}\times(\bar{\nabla}\times\bar{A}) = -e\bar{\nabla}\phi - \frac{e}{c}\frac{\partial\bar{A}}{\partial t} - \frac{e}{c}(\bar{v}\cdot\bar{\nabla})\bar{A} + \frac{e}{c}\bar{\nabla}(\bar{v}\cdot\bar{A})$$

We recognize $\frac{\partial\bar{A}}{\partial t} + (\bar{v}\cdot\bar{\nabla})\bar{A}$ as the total derivative $\frac{d\bar{A}}{dt}$ from the point of view of an observer with velocity $\bar{v}$. Hence, we are led to write

$$\frac{d\bar{P}_{TOT}}{dt} = \frac{d}{dt}\left(\bar{p}_{MECH} + \frac{e}{c}\bar{A}\right) = -\bar{\nabla}\left(e\phi - e\frac{\bar{v}}{c}\cdot\bar{A}\right) = -\bar{\nabla}U$$

where $\bar{P} = \bar{p}_{MECH} + \frac{e\bar{A}}{c}$ = total (or canonical) momentum

$U = e\phi - e\frac{\bar{v}}{c}\cdot\bar{A}$ = interaction energy of the charge with the fields

One use of this is in Lagrangian & Hamiltonian formalisms:

$$L = T - U \Rightarrow P_i = \frac{\partial L}{\partial \dot{q}_i} = p_i + \frac{eA_i}{c} \quad \text{as found above} \quad \left(T = \tfrac{1}{2}mv^2 = \frac{p^2}{2m}\right)$$

$$H = \sum p_i v_i - L = \frac{p^2}{2m} + e\phi = \frac{\left(\bar{P} - \frac{e\bar{A}}{c}\right)^2}{2m} + e\phi$$

[From which the Lorentz force could be deduced]

Example: Charge e constrained to move on a ring of radius R outside a solenoid of radius $a < R$: $\bar{B} = \begin{cases} B_0\hat{z} & 0 < r < a \\ \text{no} & r > a \end{cases}$

$$\int \bar{B}\cdot d\bar{S} = \int \bar{\nabla}\times\bar{A}\cdot d\bar{S} = \oint \bar{A}\cdot d\bar{l} \Rightarrow 2\pi R A_\phi = \pi a^2 B_0 \quad \text{or} \quad A_\phi = \frac{a^2}{2R}B_0$$

Initial canonical momentum $= P_\phi = p_\phi + \frac{e}{c}A_\phi = \frac{e}{c}A_\phi = \frac{ea^2B_0}{2cR}$ if charge at rest

Initial canonical angular momentum: $L_z = RP_\phi = \frac{ea^2B_0}{2c}$

Turn off $B_0 \Rightarrow L_z$ conserved $\Rightarrow$ create $p_{\phi,MECH}$ so that $Rp_\phi = \frac{ea^2B_0}{2c} \Rightarrow mv = \frac{ea^2B_0}{2Rc}$

$\Rightarrow$ motion around ring!

Faraday: $2\pi R E_\phi = \frac{\pi a^2\dot{B}_z}{c} \Rightarrow E_\phi = \frac{a^2\dot{B}_z}{2cR}$

$$mv = p_\phi = \int F_\phi dt = \int \frac{ea^2\dot{B}_z}{2cR}dt = \frac{ea^2B_0}{2cR} \checkmark$$

[Faraday called the field $\bar{A}$ the electromagnetic momentum, not the vector potential]

Example: Particle with charge e in a plane wave, $\bar{E} = E_0\hat{x}e^{i(kz-\omega t)} \Rightarrow \bar{A} = \frac{c\bar{E}}{i\omega}$

$$m\ddot{x} = eE_x \Rightarrow p_x = m\dot{x} = -\frac{eE_x}{i\omega} = -\frac{eA_x}{c} \Rightarrow P_x = p_x + \frac{e}{c}A_x = 0$$

$\Rightarrow$ mechanical transverse momentum is equal & opposite to electromagnetic trans. mom