

SOURCES OF THE WAVES - THE RETARDED POTENTIALS (BECKER SEC 66-68)

THUS FAR WE HAVE CONSIDERED THE PROPAGATION OF ELECTROMAGNETIC WAVES WITH ONLY OCCASIONAL REFERENCE TO THE MEANS OF THEIR GENERATION. AS WAS THE CASE IN STATIC SITUATIONS IT PROVES USEFUL TO RELATE THE POTENTIALS TO THE CHARGES AND CURRENTS WHICH CAUSE THEM. THEN WE TAKE DERIVATIVES TO FIND THE FIELDS.

$$\text{THE STATIC SOLUTIONS ARE: } \vec{E} = -\vec{\nabla}\phi, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\text{WITH } \phi = \int \frac{\rho dvol}{r} \quad \vec{A} = \frac{1}{c} \int \frac{\vec{j} dvol}{r}$$

WHEN THE SITUATION IS NOT STATIC, WE FIND THAT CHANGES IN THE FIELDS PROPAGATE WITH THE VELOCITY OF LIGHT. WE EXPECT THIS TO HOLD FOR CHANGES IN THE POTENTIALS ALSO. ON THE OTHER HAND, THE GENERAL FORM OF THE TIME-DEPENDENT POTENTIALS MUST REDUCE TO THE STATIC SOLUTIONS WHEN APPROPRIATE.

WE MAKE AN ENLIGHTENED 'GUESS' THAT IN GENERAL

$$\phi(\vec{x}, t) = \int \frac{\rho(\vec{x}', t' = t - \frac{r}{c})}{r} dvol'; \quad \vec{A} = \frac{1}{c} \int \frac{\vec{j}(\vec{x}', t' = t - \frac{r}{c})}{r} dvol'$$

THE RETARDED POTENTIALS. (LORENZ, 1867)

THE POTENTIALS ARE JUST THE STATIC SOLUTIONS AS "SEEN" BY THE OBSERVER AT \vec{x} AT TIME t . THE OBSERVER 'LOOKS' AT THE CHARGES BY MEANS OF 'LIGHT RAYS' WHICH PROPAGATE WITH VELOCITY c . HENCE THE CHARGE DISTRIBUTION (S)HE SEES IS THAT WHICH ACTUALLY OCCURRED AT AN EARLIER TIME. $t' = t - \frac{r}{c}$

WE NOW SHOW THAT THIS 'GUESS' IS REALLY CORRECT.

FIRST WE RECALL THE RELATION OF THE FIELDS TO THE TIME-DEPENDENT POTENTIALS:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{ALWAYS} \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A} \quad \text{CAN ALWAYS BE TRUE}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} = -\frac{1}{c} \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} \quad \Rightarrow \quad \vec{\nabla} \times \left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0$$

$$\Rightarrow \quad \vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}\phi \quad \text{CAN ALWAYS HOLD} \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}\phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

WE NOW LOOK FOR WAVE EQUATIONS FOR THE POTENTIALS.

$$\nabla \times \bar{B} = \frac{4\pi}{c} \bar{j} + \frac{1}{c} \frac{\partial \bar{E}}{\partial t} \Rightarrow \nabla \times (\nabla \times \bar{A}) = \frac{4\pi}{c} \bar{j} + \frac{1}{c} \left(-\nabla \frac{\partial \phi}{\partial t} - \frac{1}{c} \frac{\partial^2 \bar{A}}{\partial t^2} \right)$$

OR
$$\nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\frac{4\pi}{c} \bar{j} + \nabla \left(\nabla \cdot \bar{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right)$$

ALSO
$$\nabla \cdot \bar{E} = 4\pi \rho \Rightarrow \nabla^2 \phi + \frac{1}{c} \frac{\partial}{\partial t} (\nabla \cdot \bar{A}) = -4\pi \rho$$

WHICH ARE NOT EXACTLY THE WAVE EQUATIONS WE'VE SEEN BEFORE!

WE CAN TIDY THINGS UP BY NOTING A VERY DEEP FEATURE OF THE ELECTROMAGNETIC FIELDS - THE SO-CALLED GAUGE INVARIANCE.

IN STATIC SITUATIONS THE POTENTIALS ARE DEFINED ONLY UP TO A CONSTANT. BUT IN FACT, WE HAVE EVEN MORE FREEDOM:

IF Ω IS ANY SCALAR FUNCTION, THEN THE TRANSFORMATION

$$A' = \bar{A} + \nabla \Omega \quad \phi' = \phi - \frac{1}{c} \frac{\partial \Omega}{\partial t} \quad \text{LEAVES } \bar{E} \text{ AND } \bar{B} \text{ UNCHANGED!}$$

[CLEARLY $\bar{B} = \nabla \times \bar{A} = \nabla \times \bar{A}'$; ALSO $-\nabla \phi' - \frac{1}{c} \frac{\partial \bar{A}'}{\partial t} = -\nabla \phi + \frac{1}{c} \nabla \frac{\partial \Omega}{\partial t} - \frac{1}{c} \frac{\partial \bar{A}}{\partial t} - \frac{1}{c} \frac{\partial \nabla \Omega}{\partial t}$]

$$= -\nabla \phi - \frac{1}{c} \frac{\partial \bar{A}}{\partial t} = \bar{E}$$

WE CAN USE THIS FREEDOM OF DEFINITION IN VARIOUS WAYS

1. WE COULD SET $\nabla \cdot \bar{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$ THE LORENTZ GAUGE CONDITION

SUPPOSE $\nabla \cdot \bar{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \neq 0$. THEN TRANSFORMING TO \bar{A}' AND ϕ'

$$\nabla \cdot \bar{A}' - \nabla^2 \Omega + \frac{1}{c} \frac{\partial \phi'}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \Omega}{\partial t^2} \neq 0$$

BUT WE CAN HAVE $\nabla \cdot \bar{A}' + \frac{1}{c} \frac{\partial \phi'}{\partial t} = 0$ IF $\nabla^2 \Omega - \frac{1}{c^2} \frac{\partial^2 \Omega}{\partial t^2} \neq 0$

THE LATTER IS A WAVE EQUATION WE CAN SURELY SOLVE $\Rightarrow \Omega$ EXISTS.

HENCE WE CAN ALWAYS CHOOSE POTENTIALS \bar{A} AND ϕ TO SATISFY $\nabla \cdot \bar{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$

$$\Rightarrow \nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\frac{4\pi}{c} \bar{j}$$

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho$$

THE WAVE EQUATIONS WHICH RELATE THE POTENTIALS TO THE SOURCES ρ AND \bar{j} .

2. MANY OTHER CHOICES ARE POSSIBLE. FOR EXAMPLE WE CAN ALWAYS ARRANGE THAT $\nabla \cdot \bar{A} = 0$ [COULOMB GAUGE CONDITION]

i.e., IF $\nabla \cdot \bar{A} \neq 0$, CHOOSE $\nabla \cdot \bar{A}' = \nabla \cdot \bar{A} + \nabla^2 \Omega = 0 \Leftrightarrow \nabla^2 \Omega = -\nabla \cdot \bar{A}$, WHICH IS A POISSON-EQ. PROBLEM WHICH SURELY HAS A SOLUTION.

THEN $\nabla^2 \phi = -4\pi \rho \Rightarrow \phi(\bar{x}, t) = \int \frac{\rho(\bar{x}', t)}{r} d^3x'$.

IS THE INSTANTANEOUS STATIC SOLUTION! A PECULIAR RESULT - BUT O.K. MATHEMATICALLY, AFTER WE CALCULATE \bar{E} AND \bar{B} .

CERTAINLY IF $\rho = \text{CONSTANT}$ IN TIME THERE IS NO PROBLEM: THEN FOR \bar{A} WE HAVE $\nabla^2 \bar{A} - \frac{1}{c^2} \frac{\partial^2 \bar{A}}{\partial t^2} = -\frac{4\pi \bar{j}}{c}$ AND ALL TIME DEPENDENT EFFECTS ARE DERIVED FROM \bar{A} .

WE WILL NOT CONSIDER THE COULOMB GAUGE FURTHER.

WE NOW MUST SHOW THAT THE RETARDED POTENTIALS ARE INDEED THE SOLUTIONS OF THE WAVE EQUATIONS FOR THE POTENTIALS.

EACH OF OUR 4 WAVE EQUATIONS WAS THE FORM

$$\nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = -4\pi g(\bar{x}, t).$$

A SUBTLE POINT: ONLY FOR A RECTANGULAR COORDINATE g DOES $(\nabla^2 \bar{A})_q = \nabla^2 (A_q)$ C.F. MORSE & FESHBACH, VOL I, P. 116

A VERY INSTRUCTIVE SOLUTION IS OBTAINED BY FOURIER ANALYSIS

WE DECOMPOSE $g(\bar{x}, t) = \int_{-\infty}^{\infty} g_{\omega}(\bar{x}) e^{-i\omega t} d\omega,$

WHERE $g_{\omega}(\bar{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\bar{x}, t) e^{i\omega t} dt.$

LIKEWISE WE CAN DECOMPOSE OUR DESIRED SOLUTION:

$$\Psi(\bar{x}, t) = \int_{-\infty}^{\infty} \Psi_{\omega}(\bar{x}) e^{-i\omega t} d\omega, \text{ WHERE } \Psi_{\omega}(\bar{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\bar{x}, t) e^{i\omega t} dt.$$

SUBSTITUTING THESE RELATIONS INTO THE WAVE EQUATION:

$$\nabla^2 \Psi_{\omega}(\bar{x}) + \frac{\omega^2}{c^2} \Psi_{\omega}(\bar{x}) = -4\pi g_{\omega}(\bar{x}).$$

THIS IS SOMEWHAT SIMILAR TO POISSON'S EQUATION. SOMETIMES IT IS CALLED HELMHOLTZ' EQUATION.

WE CAN SOLVE THIS BY THE METHOD OF GREEN.

FIRST SUPPOSE $g_w(\vec{x}) = A \delta(\vec{x})$ - A POINT SOURCE AT THE ORIGIN.

$$\text{THEN FOR } \vec{x} \neq 0 \quad \nabla^2 \psi_w + \frac{\omega^2}{c^2} \psi_w = 0,$$

AND ψ MUST BE SPHERICALLY SYMMETRIC: $\psi_w = \psi_w(r)$.

$$\text{THEN } \nabla^2 \psi_w(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi_w}{\partial r} \right) = \frac{1}{r} \frac{\partial^2}{\partial r^2} r \psi_w \quad \text{IN SPHERICAL COORDS,}$$

$$\text{SO } \frac{\partial^2}{\partial r^2} (r \psi_w) + \frac{\omega^2}{c^2} (r \psi_w) = 0 \Rightarrow r \psi_w = B e^{\pm ikr}, \quad \left[k = \frac{\omega}{c} \right],$$

$$\text{OR } \psi_w = \frac{B}{r} e^{\pm ikr}.$$

WE EVALUATE B BY INTEGRATING THE SOLUTION OVER A SMALL VOLUME ABOUT THE ORIGIN:

$$\int_{\text{SPHERE}} (\nabla^2 \psi_w + k^2 \psi_w) d\text{vol} = -4\pi \int A \delta(\vec{x}) d\text{vol} = -4\pi A.$$

FOR $r \gg 0$ $\psi_w \rightarrow B/r$ so $\int \nabla^2 \psi_w d\text{vol} \rightarrow -4\pi B$, while $\int \psi_w d\text{vol} \sim r \rightarrow 0$,

SO $B = A$, AS IS NO GREAT SURPRISE.

WE CAN CLEARLY BUILD UP THE GENERAL SOLUTION TO BE

$$\psi_w(\vec{x}) = \int \frac{g_w(\vec{x}') e^{\pm ikr}}{r(\vec{x}, \vec{x}')} d\text{vol}'$$

$$\text{SO } \psi(\vec{x}, t) = \int_{-\infty}^{\infty} d\omega \int d\text{vol}' \underbrace{\frac{g_w(\vec{x}') e^{-i(\omega t \pm kr)}}{r}}_{\text{SPHERICAL WAVE}}$$

$$\text{DEFINE } t' = t \pm \frac{k}{\omega} r = t \pm \frac{r}{c}$$

$$\text{THEN } \psi(\vec{x}, t) = \int \frac{d\text{vol}'}{r} \underbrace{\int_{-\infty}^{\infty} g_w(\vec{x}') e^{-i\omega t'} d\omega}_{g(\vec{x}', t')} = \int \frac{g(\vec{x}', t') d\text{vol}'}{r}$$

THE CHOICE $t' = t + r/c$ WOULD GIVE THE POTENTIALS NOW IN TERMS OF THE CHARGES IN THE FUTURE. ON THE GROUNDS OF 'COMMON SENSE' WE THROW OUT THIS POSSIBILITY.

(BUT SHOULD WE?? WHAT ABOUT TIME-REVERSAL INVARIANCE.....)

WE KEEP ONLY THE CASE WITH $t' = t - r/c$ - THE RETARDED TIME. OF COURSE, $r = |\vec{r}|$ WHERE $\vec{r} = \vec{x} - \vec{x}'$.

WE INTRODUCE A NOTATIONAL DEVICE

$$[P] \equiv P_{\text{RETARDED}} = \rho(\vec{x}', t - r/c), \quad \left(\begin{array}{l} \text{WHICH DEPENDS ON} \\ \text{THE OBSERVER AS WELL} \\ \text{AS } \rho \text{ ITSELF} \end{array} \right)$$

$$[\vec{j}] = \vec{j}_{\text{RETARDED}} = \vec{j}(\vec{x}', t - r/c).$$

Thus
$$\phi = \int \frac{[P]}{r} dvol', \quad \vec{A} = \frac{1}{c} \int \frac{[\vec{j}]}{r} dvol'.$$

ARE THE GENERAL SOLUTIONS TO THE WAVE EQUATIONS FOR THE POTENTIALS.

THE RADIATION FIELDS

WE PROCEED TO CALCULATE THE FIELDS FROM THE POTENTIALS

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{c} \int \vec{\nabla} \times \left(\frac{[\vec{j}]}{r} \right) dvol'.$$

Now
$$\vec{\nabla} \times \frac{[\vec{j}]}{r} = \vec{\nabla} \left(\frac{1}{r} \right) \times [\vec{j}] + \frac{1}{r} \vec{\nabla} \times [\vec{j}]$$

WE HAVE SEEN BEFORE THAT
$$\vec{\nabla} \frac{1}{r} = -\frac{\vec{r}}{r^3} \equiv -\frac{\hat{n}}{r^2} \quad \left[\hat{n} \equiv \frac{\vec{r}}{r} \right]$$

$$\begin{aligned} \left\{ \vec{\nabla} \times \frac{[\vec{j}]}{r} \right\}_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} j_k(\vec{x}', t' = t - r/c) \\ &= \epsilon_{ijk} \frac{\partial j_k}{\partial t'} \frac{\partial t'}{\partial x_j} = \epsilon_{ijk} [\dot{j}]_k \left(\frac{-x_j}{rc} \right) \\ &= -\frac{1}{rc} (\vec{r} \times [\dot{j}])_i = -\frac{1}{c} (\hat{n} \times [\dot{j}])_i. \end{aligned}$$

$x_j = x'_j$

Thus
$$\vec{B} = \frac{1}{c} \int -\frac{\hat{n} \times [\dot{j}]}{r^2} dvol' + \frac{1}{c} \int -\frac{\hat{n} \times [\dot{j}]}{rc} dvol'.$$

$$\underline{\bar{B}} = \frac{1}{c} \int \frac{[\dot{\mathbf{j}}] \times \hat{n}}{r^2} dvol' + \frac{1}{c^2} \int \frac{[\dot{\mathbf{j}}] \times \hat{n}}{r} dvol'$$

COMPARING TO P. 80, THIS IS LIKE THE BIOT-SAVART LAW USING THE RETARDED CURRENT $[\mathbf{j}]$ RATHER THAN \mathbf{j} , PLUS A NEW TERM IN $[\dot{\mathbf{j}}]/r$ - THE RADIATION FIELD,

WE GIVE TWO FORMS FOR $\bar{E} = -\bar{\nabla}\phi - \frac{1}{c} \frac{\partial \bar{A}}{\partial t}$ BY SIMILAR ARGUMENTS.

FIRST, $\frac{\partial}{\partial t}$ ACTS ONLY ON $[\dot{\mathbf{j}}] = \dot{\mathbf{j}}(\bar{\mathbf{x}}, t - r/c)$, SO $-\frac{1}{c} \frac{\partial \bar{A}}{\partial t} = -\frac{1}{c^2} \int \frac{[\dot{\mathbf{j}}]}{r} dvol'$

$$\text{ALSO, } -\bar{\nabla}\phi = -\int \bar{\nabla} \left(\frac{[\rho]}{r} \right) dvol' = -\int [\rho] \bar{\nabla} \left(\frac{1}{r} \right) dvol' - \int \frac{1}{r} \bar{\nabla} [\rho] dvol'$$

AS BEFORE, $\bar{\nabla} \left(\frac{1}{r} \right) = -\frac{\hat{n}}{r^2}$, WHILE $\bar{\nabla} [\rho] = \bar{\nabla} \rho(\bar{\mathbf{x}}, t - r/c) = [\dot{\rho}] \left(-\frac{\bar{\nabla} r}{c} \right) = -\frac{[\dot{\rho}] \hat{n}}{c}$

$$\text{THUS } \underline{\bar{E}} = \int \frac{[\rho] \hat{n}}{r^2} dvol' + \frac{1}{c} \int \frac{[\dot{\rho}] \hat{n}}{r} dvol' - \frac{1}{c^2} \int \frac{[\dot{\mathbf{j}}]}{r} dvol'$$

AGAIN WE HAVE A LEADING TERM THAT IS THE RETARDED STATIC FIELD, PLUS CORRECTION TERMS.

WE CAN GO FURTHER.

WE WISH TO USE THE CONTINUITY EQUATION $\bar{\nabla} \cdot \bar{\mathbf{j}} = -\dot{\rho}$ TO SET $[\dot{\rho}] = -\bar{\nabla}' \cdot [\bar{\mathbf{j}}]$ BUT THIS IS NOT QUITE RIGHT!

THE RETARDED CURRENT DENSITY $[\bar{\mathbf{j}}]$ DEPENDS ON $\bar{\mathbf{x}}'$ BOTH DIRECTLY, AND IMPLICITLY THROUGH r . THE CONTINUITY EQUATION RELATES ONLY THE DERIVATIVE OF THE DIRECT DEPENDENCE OF $\bar{\mathbf{j}}$ ON $\bar{\mathbf{x}}'$.

$$\text{THAT IS, } \bar{\nabla}' \cdot [\bar{\mathbf{j}}] = [\bar{\nabla}' \cdot \bar{\mathbf{j}}] + [\dot{\mathbf{j}}] \cdot \left(-\frac{\bar{\nabla}' r}{c} \right) = -[\dot{\rho}] + \frac{[\dot{\mathbf{j}}] \cdot \hat{n}}{c}$$

↑ EXTRA PIECE FROM IMPLICIT DEPENDENCE VIA r .

$$\text{THUS } \frac{1}{c} \int \frac{[\dot{\rho}] \hat{n}}{r} dvol' = -\frac{1}{c} \int \frac{(\bar{\nabla}' \cdot [\bar{\mathbf{j}}]) \hat{n}}{r} dvol' + \frac{1}{c^2} \int \frac{([\dot{\mathbf{j}}] \cdot \hat{n}) \hat{n}}{r} dvol'$$

THE FIRST INTEGRAL CAN BE TRANSFORMED BY LOOKING AT A COMPONENT:

$$\begin{aligned} (\bar{\nabla}' \cdot [\bar{\mathbf{j}}]) \frac{\hat{n}}{r} \Big|_i &= \frac{\partial [\bar{\mathbf{j}}]_k}{\partial x'_k} \frac{r_i}{r^2} = \frac{\partial}{\partial x'_k} \left(\frac{[\bar{\mathbf{j}}]_k r_i}{r^2} \right) - [\bar{\mathbf{j}}]_k \frac{\partial}{\partial x'_k} \left(\frac{r_i}{r^2} \right) \\ &= \frac{\partial}{\partial x'_k} \left(\frac{[\bar{\mathbf{j}}]_k r_i}{r^2} \right) + \frac{[\bar{\mathbf{j}}]_i - 2([\bar{\mathbf{j}}] \cdot \hat{n}) \hat{n}_i}{r^2} \quad \text{USING } \frac{\partial}{\partial x'_k} \left(\frac{r_i}{r^2} \right) = -\frac{\delta_{ik}}{r^2} + \frac{2r_i r_k}{r^4} \end{aligned}$$

NOTE ALSO THAT $[\bar{\mathbf{j}}] - 2([\bar{\mathbf{j}} \cdot \hat{n}] \cdot \hat{n}) \hat{n} = -([\bar{\mathbf{j}} \cdot \hat{n}] \hat{n} + ([\bar{\mathbf{j}}] \times \hat{n}) \times \hat{n})$

WE USE GAUSS' THEOREM TO KILL OFF THE INTEGRAL OF THE FIRST TERM:

$$\int \frac{2}{\partial x'_k} \left(\frac{[\dot{j}]_k r_i}{r^2} \right) dvol' = \int_{\text{SURFACE}} d\vec{S}' \cdot \frac{[\dot{j}]}{r^2} r_i \rightarrow 0 \text{ FOR A SURFACE BEYOND ALL CURRENTS.}$$

WE NOW HAVE ALL THE PIECES TO CALCULATE \vec{E} :

$$\begin{aligned} \vec{E} &= \int \frac{[\rho]}{r^2} \hat{n} dvol' + \frac{1}{c} \int \frac{2([\dot{j}] \cdot \hat{n}) \hat{n} - [\dot{j}]}{r^2} dvol' + \frac{1}{c^2} \int \frac{([\ddot{j}] \cdot \hat{n}) \hat{n} - [\ddot{j}]}{r} dvol' \\ &= \int \frac{[\rho]}{r^2} \hat{n} dvol' + \frac{1}{c} \int \frac{([\dot{j}] \cdot \hat{n}) \hat{n} + ([\dot{j}] \times \hat{n}) \times \hat{n}}{r^2} dvol' + \frac{1}{c^2} \int \frac{([\ddot{j}] \times \hat{n}) \times \hat{n}}{r} dvol' \end{aligned}$$

THE FIRST TERM IS THE RETARDED VERSION OF THE STATIC FIELD; THE THIRD TERM IS THE RADIATION FIELD. WHAT IS THE SECOND TERM?

REMARK: IT'S NOT OBVIOUS, BUT FOR STATICS, WHERE $\vec{\nabla} \cdot \vec{j} = 0$, THEN $\int \frac{2([\dot{j}] \cdot \hat{n}) \hat{n} - [\dot{j}]}{r^2} dvol' = 0$. THAT IS, WE RECOVER THE CORRECT FORMS IN THE STATIC LIMIT.

NEAR ZONES:

IF WE ARE IN A SITUATION WHERE r IS SMALL OVER THE WHOLE VOLUME OF THE SOURCE, THEN $[\rho] \rightarrow \rho$, $[\dot{j}] \rightarrow \dot{j}$, AND ONLY THE $\frac{1}{r^2}$ TERMS ABOVE SURVIVE. THIS IS A VERY COMPLICATED WAY OF RECOVERING THE STATIC SOLUTIONS. I.E. $\frac{1}{r^2} \gg \frac{1}{r}$ IF r IS SMALL.

FAR ZONES:

OF GREAT INTEREST ARE THE $\frac{1}{r}$ TERMS - WHICH DOMINATE WHEN THE OBSERVER IS FAR FROM THE REGION OF THE SOURCES.

THESE ARE THE SO-CALLED FAR-ZONE OR RADIATION FIELDS

WHICH WE WISH TO ASSOCIATE WITH ELECTRO-MAGNETIC WAVES.

IT IS USEFUL TO NOTE THE EXTREME LIMIT, WHEN r IS SO LARGE THAT WE CAN REGARD IT AS A CONSTANT COMPARED TO THE VOLUME OF THE SOURCES: THEN

$$\vec{B}_{\text{RAD}} \sim \frac{1}{c^2 r} \int [\dot{j}] dvol \times \hat{n}$$

$$\vec{E}_{\text{RAD}} \sim \frac{1}{c^2 r} \left[\left[\int [\dot{j}] dvol \right] \times \hat{n} \right] \times \hat{n} = \vec{B}_{\text{RAD}} \times \hat{n}$$

FURTHER, IF WE SET $\vec{A}_{\text{RAD}} \sim \frac{1}{cr} \int [\dot{j}] dvol$,

$$\text{THEN, } \vec{B}_{\text{RAD}} \sim \frac{1}{c} \dot{\vec{A}}_{\text{RAD}} \times \hat{n} \quad \vec{E}_{\text{RAD}} \sim \frac{1}{c} (\ddot{\vec{A}}_{\text{RAD}} \times \hat{n}) \times \hat{n}$$

SO $|\vec{E}_{\text{RAD}}| = |\vec{B}_{\text{RAD}}|$ AND \vec{E}_{RAD} , \vec{B}_{RAD} AND \hat{n} ARE ALL ORTHOGONAL.

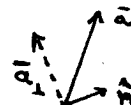
NOTE ALSO:
 $\vec{E}_{\text{RAD}} = -\vec{\nabla} \phi_{\text{RAD}} - \frac{1}{c} \dot{\vec{A}}_{\text{RAD}}$
 LORENTZ:
 $\vec{\nabla} \cdot \vec{A}_{\text{RAD}} = -\frac{1}{c} \dot{\rho}_{\text{RAD}}$
 WAVE WITH ω, \vec{k} :
 $i \vec{k} \cdot \vec{A}_{\text{RAD}} = i \frac{\omega}{c} \phi_{\text{RAD}}$
 $\phi_{\text{RAD}} = \hat{n} \cdot \vec{A}_{\text{RAD}}$

$$\begin{aligned} \vec{\nabla} \phi_{\text{RAD}} &= i k \hat{n} (\hat{n} \cdot \vec{A}_{\text{RAD}}) \\ &= i \hat{n} (\hat{n} \cdot \dot{\vec{A}}_{\text{RAD}}) \\ \vec{E}_{\text{RAD}} &= \frac{1}{c} (\hat{n} \cdot \dot{\vec{A}}_{\text{RAD}}) \hat{n} - \frac{1}{c} \ddot{\vec{A}}_{\text{RAD}} \\ &= \frac{1}{c} (\dot{\vec{A}}_{\text{RAD}} \times \hat{n}) \times \hat{n} \end{aligned}$$

SIMPLE INTERPRETATION OF THE RADIATION FIELD

WE MAY REWRITE

$$\vec{E}_{RAD} = \frac{1}{c} (\dot{\vec{A}}_{RAD} \times \hat{n}) \times \hat{n} = \frac{1}{c} [(\hat{n} \cdot \dot{\vec{A}}_{RAD}) \hat{n} - \dot{\vec{A}}_{RAD}] = -\frac{\dot{\vec{A}}_{\perp RAD}}{c}$$



WHERE $\vec{a}_{\perp} = \vec{a} - (\vec{a} \cdot \hat{n}) \hat{n}$ IS THE COMPONENT OF $\vec{a} \perp$ TO \hat{n} .

THEN, SINCE $\vec{A}_{RAD} \sim \frac{1}{cY} \int [\vec{j}] dvol$,

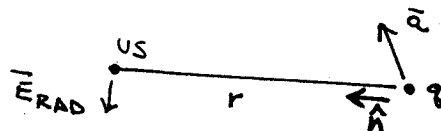
$$\vec{E}_{RAD} \sim -\frac{1}{c^2 Y} \int [\ddot{\vec{j}}]_{\perp} dvol.$$

IT IS INSTRUCTIVE TO CONSIDER THE LIMIT WHERE THE CURRENT \vec{j} IS DUE TO A SINGLE MOVING CHARGE THEN $\vec{j} dvol \rightarrow q \vec{v} = q \dot{\vec{x}}$

THEN [SO LONG AS $v \ll c$] WE HAVE

$$\vec{E}_{RAD} = -\frac{q}{c^2 Y} [\ddot{\vec{x}}_{\perp}] = -\frac{q}{c^2 Y} [\ddot{\vec{a}}_{\perp}] \quad (\ddot{\vec{a}} = \ddot{\vec{x}})$$

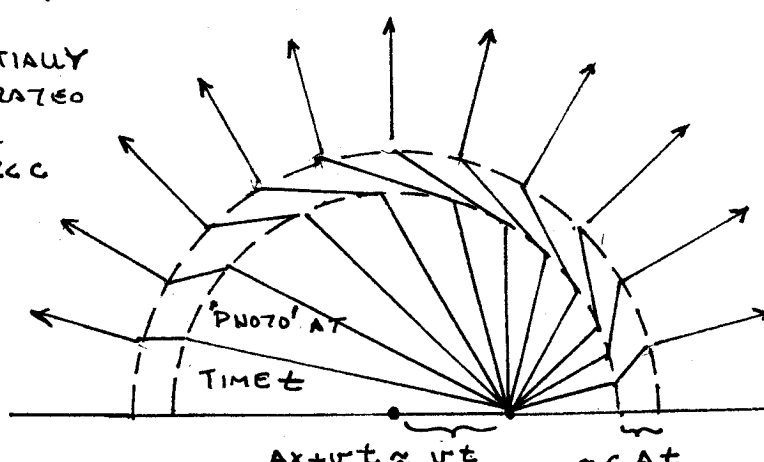
THE RADIATED ELECTRIC FIELD VARIES AS THE TRANSVERSE COMPONENT OF THE CHARGE'S ACCELERATION. THIS GIVES BOTH THE MAGNITUDE AND DIRECTION (=POLARIZATION) OF THE FIELD \vec{E} . THEN $\vec{B}_{RAD} = \hat{n} \times \vec{E}_{RAD}$.



THIS IS THE RESULT ANTICIPATED ON p141, LECTURE 12.

EXAMPLE A POINT CHARGE IS INITIALLY AT REST. THEN IT IS ACCELERATED THRU DISTANCE Δx IN TIME Δt , AFTER WHICH THE CHARGE HAS VELOCITY $v = a \Delta t \ll c$

THE TRANSITION REGION BETWEEN THE INITIAL AND FINAL ELECTROSTATIC SOLUTIONS HAS THICKNESS $c \Delta t$, AND MOVES OUTWARDS AT VELOCITY c .



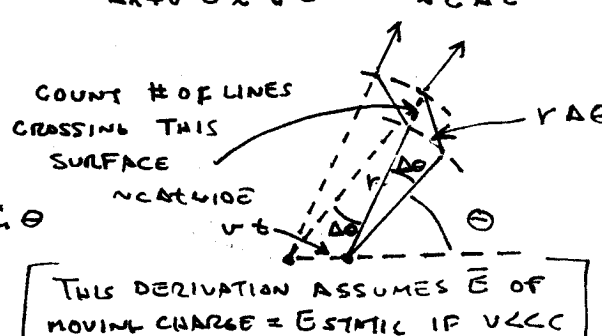
IN THIS THIN LAYER \vec{E} HAS A COMPONENT \perp TO THE LINE TO THE CHARGE.

ALL THE LINES WHICH CROSS ELEMENT $\gamma \Delta \theta$ THEN CROSS THE BAND WHICH IS $c \Delta t$ WIDE
 $\therefore E_{RAD} \approx E_{STATIC} \cdot \frac{\gamma \Delta \theta}{c \Delta t} = \frac{q \Delta \theta}{c Y \Delta t}$

BY GEOMETRY AS SHOWN, $\Delta \theta = \frac{v t \sin \theta}{r} = \frac{v}{c} \sin \theta$

SO $E_{RAD} = \frac{q a \sin^2 \theta}{c^2 Y}$

USING $v = a \Delta t$.



THIS DERIVATION ASSUMES \vec{E} OF MOVING CHARGE = E_{STATIC} IF $v \ll c$

IT IS STRIKING THAT THE RADIATION FIELDS CAN BE DERIVED ENTIRELY FROM THE VECTOR POTENTIAL, WITH NO MENTION OF THE SCALAR POTENTIAL ϕ AT ALL. OF COURSE THIS HAS TO DO WITH THE LORENTZ GAUGE RELATION $\vec{\nabla} \cdot \vec{A} = -\frac{1}{c} \frac{\partial \phi}{\partial t}$.

THE RADIATION FIELDS CARRY ENERGY AWAY FROM THE SOURCE.

$$\text{THE ENERGY DENSITY IS } U_E = \frac{E^2}{8\pi} = U_B = \frac{B^2}{8\pi}; \quad U_{\text{TOTAL}} = U_E + U_B$$

AND THE FLOW OF ENERGY IS DESCRIBED BY THE POYNTING VECTOR:

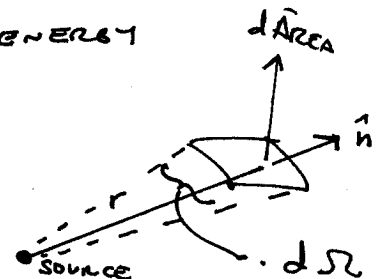
$$\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} = \frac{c}{4\pi} (\vec{B} \times \hat{n}) \times \vec{B} = \frac{c}{4\pi} B^2 \hat{n} = c U_{\text{TOTAL}} \hat{n}$$

$$\text{AND SINCE } B \sim \frac{1}{r}, \quad \vec{S} \sim \frac{\hat{n}}{r^2}.$$

THUS, THE FLOWING ENERGY OBEYS ENERGY CONSERVATION AS THE SPHERICAL WAVE EXPANDS.

A MORE DETAILED DESCRIPTION OF THE FLOW OF ENERGY CONCERNS THE POWER RADIATED INTO AN ELEMENT OF SOLID ANGLE:

$$d\Omega = \frac{\hat{n} \cdot d\vec{A}_{\text{Area}}}{r^2}$$



WE WRITE $dP = \vec{S} \cdot d\vec{A}_{\text{Area}} = \text{ENERGY CROSSING } d\vec{A}_{\text{Area}} \text{ PER UNIT TIME}$

$$= \frac{c}{4\pi} B^2 \hat{n} \cdot d\vec{A}_{\text{Area}} = \frac{c}{4\pi} B^2 r^2 d\Omega$$

$$\text{OR } \frac{dP}{d\Omega} = \frac{c}{4\pi} B^2 r^2 \quad \text{BUT SINCE } B \sim \frac{1}{r} \quad \frac{dP}{d\Omega} \text{ IS INDEPENDENT OF } r!$$

(IT DEPENDS ON THE ANGLE IN GENERAL.)

$$\text{IF WE NOTE THAT } \vec{B} \sim \frac{1}{c^2 r} \dot{\vec{A}} \times \hat{n} \sim \frac{1}{c^2 r} \int [\dot{\vec{j}}] \times \hat{n} d\omega'$$

$$\text{THEN } \frac{dP}{d\Omega} \sim \frac{1}{4\pi c^3} \left[\int [\dot{\vec{j}}] \times \hat{n} d\omega' \right]^2$$

RADIATION FROM HARMONICALLY VARYING SOURCES

A CASE OF GREAT PRACTICAL INTEREST IS WHEN THE CHARGES AND CURRENTS VARY SINUSOIDALLY:

$$\rho = \rho_0(\vec{r}) e^{-i\omega t} \quad \vec{j} = \vec{j}_0(\vec{r}) e^{-i\omega t}$$

$$\text{THEN } [\vec{j}] = \vec{j}_0 e^{-i\omega(t - r/c)} = \vec{j}_0 e^{i(\frac{\omega}{c}r - \omega t)} = \vec{j}_0 e^{i\vec{k}r - i\omega t}$$

USING $k = \omega/c$.

$$\text{THE RADIATION POTENTIAL IS THEN } \vec{A}_{\text{RAD}} = \frac{1}{c} \int \frac{e^{i(\vec{k}r - \omega t)}}{r} \vec{j}_0 d\text{vol} \approx \frac{e^{-i\omega t}}{cr} \int \vec{j}_0 d\text{vol}$$

IN WHICH WE ASSUME THAT $r \gg$ SIZE OF THE SOURCE

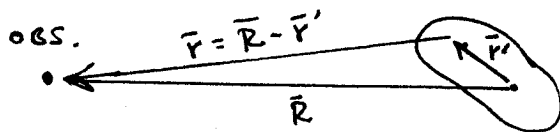
CAN WE JUSTIFY BRINGING THE FACTOR $e^{i\vec{k}r}$ OUTSIDE AS WELL?

RECALL THAT $k = \frac{2\pi}{\lambda}$ WHERE $\lambda =$ WAVELENGTH OF THE RADIATION.

SO IF THE SOURCE IS LARGE COMPARED TO λ , $e^{i\vec{k}r}$ VARIES A LOT AS WE INTEGRATE OVER THE SOURCE. HENCE WE CAN BRING OUT THE $e^{i\vec{k}r}$ ONLY IF $\lambda \gg$ SIZE OF THE SOURCE.

THIS SECOND APPROXIMATION MIGHT BE CALLED THE "DIPOLE APPROXIMATION" AS WE WOULD ONLY CALCULATE RADIATION FROM THE OSCILLATION OF THE ELECTRIC DIPOLE MOMENT OF THE SOURCE WHEN WE USE IT. (SEE THE HOMEWORK SET). HOWEVER, EVEN IN THE GENERAL CASE WE CAN PROCEED AS FOLLOWS:

$$r = |\vec{r}| \approx |\vec{R}| - \vec{r}' \cdot \hat{n}, \text{ WHERE } \hat{n} = \frac{\vec{R}}{R}$$



$$\text{SO } e^{i\vec{k}r} \approx e^{i\vec{k}R - i\vec{k}\vec{r}' \cdot \hat{n}} = e^{i\vec{k}R - i\vec{k} \cdot \vec{r}'}, \quad (\vec{k} = k\hat{n})$$

$$\text{THEN } \vec{A}_{\text{RAD}} \approx \frac{e^{i(kR - \omega t)}}{cR} \int \vec{j}_0 e^{-i\vec{k} \cdot \vec{r}'} d\text{vol}' \equiv \vec{A}_\omega e^{-i\omega t}$$

AGAIN WE NOTE THE PRESENCE OF THE SPHERICAL WAVE.

$$\text{THEN } \vec{B} = \frac{1}{c} \dot{\vec{A}}_{\text{RAD}} \times \hat{n} \Rightarrow \vec{B}_\omega = -\frac{i\omega}{c} \vec{A}_\omega \times \hat{n} = i\vec{k} \times \vec{A}_\omega$$

$$\vec{E} = \vec{B} \times \hat{n} \Rightarrow \vec{E}_\omega = i(\vec{k} \times \vec{A}_\omega) \times \hat{n}$$

THE TIME AVERAGED RADIATED POWER IS

$$\frac{dP_w}{d\Omega} = \frac{1}{4\pi c^3} \left\langle \left| \int [\dot{\mathbf{j}}] \times \hat{\mathbf{n}} dvol \right|^2 \right\rangle = \frac{1}{8\pi c^3} \omega^2 \left| \int \bar{\mathbf{j}}_w \times \hat{\mathbf{n}} e^{-i\mathbf{k}\cdot\mathbf{r}'} dvol' \right|^2$$

THESE EXPRESSIONS ARE USEFUL FOR MAKING DETAILED CALCULATIONS OF THE RADIATION FROM KNOWN CURRENT DISTRIBUTIONS - SUCH AS IN BROADCASTING ANTENNAS.

FOURIER ANALYSIS OF ARBITRARY CURRENT DISTRIBUTIONS

THE HARMONIC ANALYSIS GIVE ABOVE HAS ADDITIONAL UTILITY BECAUSE CURRENTS AND FIELDS OF ARBITRARY TIME DEPENDENCE CAN BE SYNTHESIZED OUT OF THE FOURIER COMPONENTS:

$$\bar{\mathbf{E}} = \int \bar{\mathbf{E}}_w e^{-i\omega t} d\omega \quad \bar{\mathbf{B}} = \int \bar{\mathbf{B}}_w e^{-i\omega t} d\omega, \quad \bar{\mathbf{j}} = \int \bar{\mathbf{j}}_w e^{-i\omega t} d\omega \dots$$

WE NOW FACE A SUBTLETY: DOES IT FOLLOW THAT THE RADIATED POWER FROM A GENERAL SOURCE IS JUST

$$\frac{dP(t)}{d\Omega} = \int \frac{dP_w}{d\Omega} e^{-i\omega t} d\omega \quad \text{WITH } \frac{dP_w}{d\Omega} \text{ AS ABOVE?}$$

SORRY. FOURIER ANALYSIS DOESN'T WORK QUITE THIS NICELY!

$$\text{RECALL FROM P 179 THAT } \frac{dP}{d\Omega} = \frac{1}{4\pi c^3} \left| \int [\dot{\mathbf{j}}] \times \hat{\mathbf{n}} dvol \right|^2$$

$$\text{OR } \frac{dP}{d\Omega} = f(t)^2, \text{ WHERE } f(t) \text{ CAN BE ANALYSED: } f(t) = \int f_w e^{-i\omega t} d\omega$$

$$\text{SO } \frac{dP}{d\Omega} = \iint f_w f_{w'} e^{-i(\omega + \omega')t} d\omega d\omega', \text{ WHICH IS NOT VERY USEFUL.}$$

BUT MATHEMATICS GIVES US A PARTIAL CONSOLATION. WE CAN MAKE AN INTERESTING ANALYSIS OF THE TOTAL ENERGY RADIATED OVER ALL TIMES:

$$\begin{aligned} \frac{dU}{d\Omega} &= \int \frac{dP}{d\Omega} dt = \int f(t)^2 dt = \int f dt \int f_w e^{-i\omega t} d\omega \\ &= \int f_w d\omega \int f e^{-i\omega t} dt = 2\pi \int_{-\infty}^{\infty} f_w f_w^* d\omega = 4\pi \int_0^{\infty} |f_w|^2 d\omega \\ &\quad \uparrow \\ &\quad \text{RECALL: } f_w^* = f(-\omega) \end{aligned}$$

SO IF WE DEFINE THE FREQUENCY SPECTRUM OF THE TIME-INTEGRATED INTENSITY AS

$$\frac{dU}{d\Omega} = \int_0^{\infty} \frac{dU_{\omega}}{d\Omega} d\omega$$

$$\begin{aligned} \text{Then } \frac{dU_{\omega}}{d\Omega} &= 4\pi |\dot{f}_{\omega}|^2 = \frac{4\pi}{4\pi c^3} \left[-i\omega \int \dot{\mathbf{j}}_{\omega} \times \hat{\mathbf{n}} e^{-i\mathbf{k}\cdot\mathbf{r}'} d\text{vol}' \right]^2 \\ &= \frac{\omega^2}{c^3} \left[\int \dot{\mathbf{j}}_{\omega} \times \hat{\mathbf{n}} e^{-i\mathbf{k}\cdot\mathbf{r}'} d\text{vol}' \right]^2 \end{aligned}$$

$$\text{WE CAN ALSO NOTE THAT } \dot{\mathbf{j}}_{\omega} = \frac{1}{2\pi} \int \dot{\mathbf{j}} e^{i\omega t'} dt'$$

$$\text{so } \frac{dU_{\omega}}{d\Omega} = \frac{\omega^2}{4\pi^2 c^3} \left[\int \int \dot{\mathbf{j}} \times \hat{\mathbf{n}} e^{i(\omega t' - \mathbf{k}\cdot\mathbf{r}')} dt' d\text{vol}' \right]^2$$

THIS LATTER FORM WILL PROVE TO BE USEFUL EVEN FOR CURRENTS INVOLVING CHARGES MOVING WITH $v \ll c$.

IN ANY CASE, WE HAVE ACCOMPLISHED THE TASK OF RESOLVING A PULSE OF RADIATION INTO ITS FREQUENCY COMPONENTS.

PLANE WAVES AND SPHERICAL WAVES

WE HAVE HAD OCCASION TO EMPHASIZE THE SPECIAL UTILITY OF BOTH SINUSOIDAL PLANE WAVES AND SPHERICAL WAVES. WE HAVE INTIMATED THAT A GENERAL WAVE CAN BE BUILT UP OUT OF A SUPERPOSITION OF EITHER TYPE OF WAVE.

IN PARTICULAR, A PLANE WAVE SHOULD BE EXPRESSIBLE IN TERMS OF SPHERICAL WAVES. IT TURNS OUT THAT

$$e^{i(kz - \omega t)} = e^{ikr\cos\theta} e^{-i\omega t} = \sum_{\ell} i^{\ell} (2\ell+1) j_{\ell}(kr) P_{\ell}(\cos\theta) e^{-i\omega t}$$

WHERE $j_{\ell}(kr)$ IS A SO-CALLED SPHERICAL BESSEL FUNCTION.

FOR kr LARGE, THE EXPANSION IS APPROXIMATELY

$$e^{i(kz - \omega t)} \sim \sum_{\ell} \left[\frac{e^{i(kr - \omega t)}}{r} - (-1)^{\ell} \frac{e^{-i(kr + \omega t)}}{r} \right] \frac{2\ell+1}{2ik} P_{\ell}(\cos\theta)$$

WHICH IS A SUM OF INCOMING AND OUT GOING HARMONIC SPHERICAL WAVES.

SUCH EXPANSIONS ARISE IN SCATTERING PROBLEMS IN WHICH A PLANE WAVE IS INCIDENT ON A SMALL OBJECT, LEADING TO SECONDARY SPHERICAL WAVES.

IN THIS COURSE WE WILL NOT MAKE USE OF SUCH A DETAILED EXPRESSION AS THAT GIVEN ABOVE...

EXAMPLE THE ELECTROSTATIC FIELD OF A POINT CHARGE

THE FIELD OF A POINT CHARGE HAS SPHERICAL SYMMETRY. SUPPOSE NONETHELESS WE TRY TO DESCRIBE IT IN TERMS OF PLANE WAVES!

IT'S EASIEST TO START WITH THE POTENTIAL: $\phi = \frac{q}{r} = \int f(\vec{k}) e^{i(\vec{k}\cdot\vec{r} - \omega t)} \frac{d^3\vec{k}}{(2\pi)^3}$

BY HYPOTHESIS. SINCE THE POTENTIAL IS STATIC, WE CAN ONLY DO THIS FOR ZERO FREQUENCY 'WAVES', $\omega = 0$. THEN $k \neq \omega/c$.

BUT FOR $\phi = \frac{1}{(2\pi)^3} \int f(\vec{k}) e^{i\vec{k}\cdot\vec{r}} d^3\vec{k}$, THE FOURIER INVERSION IS

$$\begin{aligned} f(\vec{k}) &= \int \phi(\vec{r}) e^{-i\vec{k}\cdot\vec{r}} d^3\vec{r} = 2\pi \int_0^\infty \int_{-1}^1 \frac{e^{-ikr\cos\theta}}{r} r^2 dr d\omega d\theta = -\frac{4\pi q}{k} \int_0^\infty \sin kr dr \\ &= -\frac{4\pi q}{k^2} \cos kr \Big|_0^\infty = \frac{4\pi q}{k^2} \quad \text{IGNORING THE OSCILLATIONS AS } r \rightarrow \infty \end{aligned}$$

[IF YOU DON'T LIKE THIS CALCULATION, NOTE THAT $\nabla^2 \phi = -4\pi q \delta^3(\vec{r})$, AND THE FOURIER COEFFICIENTS OF $\nabla^2 \phi$ ARE $-k^2$ X THOSE OF ϕ ...]

$$\begin{aligned} \text{THEN } \vec{E} &= -\vec{\nabla} \phi = -\vec{\nabla} 4\pi q \int \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2} \frac{d^3\vec{k}}{(2\pi)^3} = -4\pi i q \int \frac{\vec{k}}{k^2} e^{i\vec{k}\cdot\vec{r}} \frac{d^3\vec{k}}{(2\pi)^3} \\ &= -4\pi i q \int \hat{k} \frac{e^{i\vec{k}\cdot\vec{r}}}{k} \frac{d^3\vec{k}}{(2\pi)^3} \end{aligned}$$

THUS WE MAY SAY THAT THE STATIC COULOMB FIELD OF A POINT CHARGE IS COMPOSED OF LONGITUDINAL PLANE WAVES OF ZERO FREQUENCY.