Ph 206 LECTURE 20

RELATIVISTIC RADIATION EFFECTS [BECKER SEC 85, 69]

IN THIS AND THE FOLLOWING LECTURE WE GIVE EXAMPLES OF RADIATION EFFECTS DUE TO INDIVIDUAL CHARGES MOVING WITH VELOCITY $v \sim c$ – THE RELATIVISTIC LIMIT.

WE HAVE GIVEN EXPRESSIONS FOR $\bar{E}$ AND $\bar{B}$ DUE TO A CHARGE WITH ARBITRARY MOTION. ALL POSSIBLE ELECTROMAGNETIC QUANTITIES CAN THEN BE CALCULATED FROM THESE FIELDS. IN MANY SITUATIONS WE ARE PRIMARILY INTERESTED IN THE RADIATED <u>POWER</u> RATHER THAN THE FIELDS THEMSELVES. THIS IS ESPECIALLY TRUE FOR RELATIVISTIC RADIATION EFFECTS WHICH OFTEN SHOULD BE TREATED FROM THE POINT OF VIEW OF QUANTUM ELECTRODYNAMICS TO OBTAIN A DETAILED UNDERSTANDING. IN THIS VIEW THE PHOTON CONCEPT IS MORE IMPORTANT THAN THE FIELDS $\bar{E}$ AND $\bar{B}$, BUT THE RADIATED POWER IS SIMPLY PROPORTIONAL TO THE NUMBER OF RADIATED PHOTONS (TIMES THEIR FREQUENCY)....

WE MAY OBTAIN EXPRESSIONS FOR THE RADIATED POWER IN RELATIVISTIC SITUATIONS BY A LORENTZ TRANSFORMATION OF THE <u>LARMOR FORMULAE</u> WHICH, STRICTLY SPEAKING, HOLD ONLY FOR <u>PARTICLES</u> WHICH ARE INSTANTANEOUSLY AT REST.

RECALL
$$\frac{dP}{d\Omega} = \frac{dU}{dt\,d\Omega} = \frac{e^2 a^2 \sin^2\theta_a}{4\pi c^3} \quad \& \quad P_{tot} = \frac{dU}{dt} = \frac{2e^2a^2}{3c^2}$$

WHERE WE RE INTRODUCE $U \equiv$ ENERGY.

THE KEY INSIGHT IS THAT THE QUANTITY $\frac{dU}{dt}$ <u>IS A LORENTZ INVARIANT</u>! IF WE CAN PROVE THIS THEN WE CAN CALCULATE $\frac{dU}{dt}$ OF A MOVING CHARGE SIMPLY BY INSERTING THE CORRECT FORM OF a^2 INTO THE LARMOR FORMULA.

ONE WAY TO PROCEED IS TO NOTE THAT U IS THE TIME COMPONENT OF THE ENERGY-MOMENTUM 4-VECTOR
$$P_\mu = (U, \vec{p}c)$$
AND t IS THE TIME COMPONENT OF $x_\mu = (ct, \bar{x})$

HENCE U AND t TRANSFORM THE SAME WAY UNDER A LORENTZ TRANSFORMATION $\Rightarrow$ $\frac{dU}{dt}$ = INVARIANT.

WE MAY DEMONSTRATE THIS IN ANOTHER WAY ALSO, WHICH IS A GENERALLY INSTRUCTIVE METHOD. THE LARMOR FORMULA HOLDS IN THE PARTICULAR FRAME IN WHICH THE CHARGE IS INSTANTANEOUSLY

AT REST. IF WE CAN EXPRESS EVERY QUANTITY IN THE LARMOR FORMULA IN TERMS OF LORENTZ SCALARS, VECTORS, TENSORS ..... (SUCH THAT THE NEW FORMULA IS ALSO TRUE), THEN WE HAVE A RELATION WHICH MUST HOLD IN ANY FRAME.

TO THIS END WE REWRITE $dU = \frac{2e^2a^2}{3c^3}\,dt$

RECALL $P_\mu \equiv (U, c\bar{P})$ SO $dP_0 = \frac{2e^2a^2}{3c^4}\,dx_0$

IS IT ALSO TRUE THAT $dP_\mu = \frac{2e^2a^2}{3c^4}\,dx_\mu$ IN THIS FRAME?

YES! THE PARTICLE IS INSTANTANEOUSLY AT REST $\Rightarrow d\bar{x} = 0$,

AND $d\bar{P}$ = RADIATED MOMENTUM $= \int \frac{d\bar{P}}{d\Omega}\,d\Omega = \frac{1}{c}\int \frac{dU\,\hat{n}}{dt\,d\Omega}\,d\Omega = 0$

(LECTURE 10) THE OTHER LARMOR FORMULA

SO EVERYTHING WILL BE O.K. IF $\frac{2e^2a^2}{3c^4}$ IS A LORENTZ SCALAR. THIS IS CLEARLY TRUE FOR $\frac{2e^3}{3c^4}$, BUT WHAT ABOUT a^2?

IF a^2 IS THE SQUARE OF A 4-VECTOR, THEN IT IS INDEED A LORENTZ SCALAR. IN LECTURE 18 WE DEFINE THE 4-ACCELERATION AS $a_\mu = \frac{d^2x_\mu}{ds^2}$ AND SAW THAT $a_\mu = (0, \frac{\bar{a}^*}{c^2})$ IN THE INSTANTANEOUS REST FRAME. THUS $a_\mu a^\mu = -\frac{a^{*2}}{c^4}$ = LORENTZ SCALAR

$$\Rightarrow \quad \underline{dP_\mu = -\frac{2e^2}{3}\,a_\nu a^\nu\,dx_\mu}$$ WHICH IS A FULLY RELATIVISTIC

EXPRESSION WHICH HOLDS IN ANY (INERTIAL) FRAME. BUT IN AN ARBITRARY FRAME WE SAW THAT $a_\nu a^\nu = -\frac{\gamma^6}{c^2}\left[\dot{\beta}^2 - (\bar{\beta}\times\dot{\bar{\beta}})^2\right]$

WHERE $\dot{\bar{\beta}} = \frac{\bar{a}}{c}$ IN THAT FRAME.

HENCE
$$\underline{\frac{dU}{dt} = \frac{2e^2}{3c}\gamma^6\left[\dot{\beta}^2 - (\bar{\beta}\times\dot{\bar{\beta}})^2\right] = \frac{2e^2a^{*2}}{3c^3}}$$

IF $\bar{\beta} = 0$ $\quad \frac{dU}{dt} = \frac{2e^2}{3c}\dot{\beta}^2 = \frac{2e^2a^{*2}}{3c^3}$ WHICH CHECKS.

THE SPATIAL COMPONENT OF $dP_\mu \Rightarrow$ $\underline{\frac{d\bar{P}}{dt} = \frac{2e^2}{3c^2}\gamma^6\bar{\beta}\left[\dot{\beta}^2 - (\bar{\beta}\times\dot{\bar{\beta}})^2\right]}$ = RADIATED MOMENTUM

ON THE PROBLEM SET YOU ARE ENCOURAGED TO TRY A VARIATION ON THIS THEME TO PROVIDE A GENERAL TRANSFORMATION OF THE ANGULAR DISTRIBUTION OF THE RADIATION. [θ MEASURED W.R.T. $\bar{v}$.]

IF $\frac{dU^*}{dt^* d\Omega^*} = f(\cos\theta^*, \varphi^*)$ IN THE INSTANTANEOUS REST FRAME,

THEN $\frac{dU}{dt\, d\Omega} = \frac{1}{\gamma^4 (1-\beta\cos\theta_v)^3} f\left(\frac{\cos\theta_v - \beta}{1-\beta\cos\theta_v}, \varphi\right)$ IN AN ARBITRARY INERTIAL FRAME.

WE NOW EXPLORE THE SPECIAL CASES WHERE $v \sim c$ AND THE ACCELERATION $\bar{a}$ IS $\parallel$ OR $\perp$ TO $\bar{v}$.

BREMSSTRAHLUNG ($\bar{a} \parallel \bar{v}$)

IF $\bar{a}$ IS OPPOSITE TO $\bar{v}$, THE ACCELERATION IS LIKE THAT DUE TO STEPPING ON THE BRAKES OF A CAR. HENCE THE NAME 'BREMSSTRAHLUNG' (= BRAKING RADIATION, IN GERMAN).

BREMSSTRAHLUNG IS A VERY COMMON EFFECT SINCE ANY TIME A CHARGED PARTICLE COMES NEAR ANOTHER CHARGED PARTICLE THERE IS SOME ACCELERATION DUE TO THE COULOMB FORCE. SINCE $a = F/M$, THE EFFECT IS MUCH STRONGER FOR ELECTRONS THAN PROTONS. (TECHNICALLY, THE NAME BREMSSTRAHLUNG IS OFTEN APPLIED TO SITUATIONS OF ARBITRARY $\bar{a}$, RATHER THAN JUST $\bar{a} \parallel \bar{v}$ AS STATED HERE. ON SET 11 YOU ARE ENCOURAGED TO CONSIDER THE MORE GENERAL CASE). A DENTAL X-RAY MACHINE IS BASED ON BREMSSTRAHLUNG!

ON THE HOMEWORK SET YOU WILL DEMONSTRATE THAT FOR $\bar{a} \parallel \bar{v}$ THE RADIATION OBEYS

$$\frac{dU}{dt} = \frac{2\gamma^6 e^2 a^2}{3c^3} \quad \text{AND} \quad \frac{dU}{dt\, d\Omega} = \frac{e^2 a^2 \sin^2\theta}{4\pi c^3 (1-\beta\cos\theta)^5}$$

AFTER HAVING INTRODUCED THE ELEGANT LORENTZ TRANSFORMATION METHOD OF DERIVING THESE RESULTS, WE ALSO GIVE A 'BRUTE FORCE' DERIVATION.

WE RETURN TO OUR BASIC RESULTS OF POYNTING:

$$\frac{dU}{dt\, d\Omega} = r^2 \bar{S}\cdot\hat{n} = \frac{c r^2 E_{RAD}^2}{4\pi} = \frac{e^2}{4\pi c} \frac{\left[\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]\right]^2}{(1-\beta\cos\theta)^6} \qquad (p\ 235)$$

WITH $\bar{E}$ AS THE RADIATION FIELD DERIVED IN LECTURE 19.

FOR THE CASE THAT $\bar{a} \parallel \bar{v}$ THE $[\quad]^2 \rightarrow [\hat{n}\times(\hat{n}\times\bar{a}/c)]^2 = \frac{a^2\sin^2\theta}{c^2}$

SO $\frac{dU}{dt\, d\Omega} = \frac{e^2 a^2 \sin^2\theta}{4\pi c^3 (1-\beta\cos\theta)^6}$ WHICH DIFFERS BY $(1-\beta\cos\theta)$ FROM THE LORENTZ TRANSFORMATION RESULT!

AS WE ADVOCATE USE OF THE LORENTZ TRANSFORMATION METHOD, IT IS IMPORTANT TO CLARIFY THE PHYSICAL SIGNIFICANCE OF THE RESULTS OF THIS PROCEDURE.

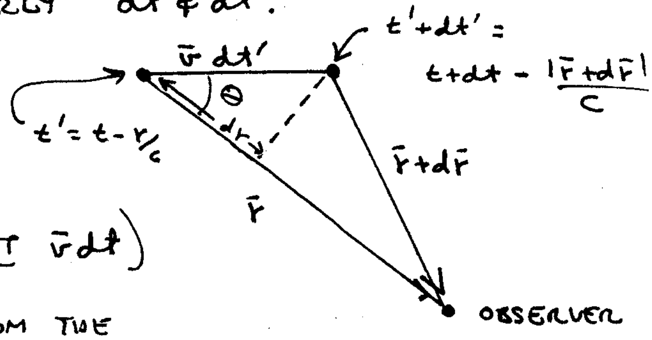
IN A FRAME IN WHICH THE CHARGE IS INSTANTANEOUSLY AT REST THERE CAN BE NO DOUBT. THE ENERGY PER SECOND DETECTED BY AN OBSERVER IS THE SAME AS THE ENERGY LOST PER SECOND BY THE CHARGE, DUE TO RADIATION.

BUT IN A FRAME IN WHICH THE CHARGE IS MOVING, SAY DIRECTLY TOWARDS THE OBSERVER, THE SITUATION IS MORE COMPLICATED. THE ENERGY LOST IN ONE SECOND BY THE MOVING CHARGE IS DECTECTED IN LESS THAN ONE SECOND BY THE OBSERVER – A KIND OF DOPPLER EFFECT.

THE CONCEPT OF RETARDED TIME ALLOWS US TO BE QUANTITATIVE.

IF t = TIME OF OBSERVATION, THEN $t' = t - r/c$ = TIME OF EMISSION OF THE RADIATION.

IF r IS VARYING WITH TIME, CLEARLY $dt \neq dt'$.



TO RELATE dt TO dt', SUPPOSE THE OBSERVER WATCHES THE PARTICLE MOVE DURING TIME dt.

THE PARTICLE MOVES DISTANCE $\bar{v}(t')dt'$ (NOT $\bar{v}\,dt$)

WRITING $|\bar{r}+d\bar{r}| = r + dr$ WE SEE FROM THE TRIANGLES THAT $dr = \bar{v}\cdot\hat{r}\,dt'$ [IF $\bar{v}$ POINTS TOWARDS THE OBSERVER, $dr<0$.]

BUT $dt' = dt - \frac{dr}{c} \Rightarrow dt = dt'\left(1 - \frac{\bar{v}\cdot\hat{r}}{c}\right) = dt'(1-\beta\cos\theta)$

HENCE
$$\underbrace{\frac{dU}{dt'd\Omega}}_{\text{RADIATION RATE AT THE CHARGE}} = \underbrace{\frac{dU}{dt\,d\Omega}}_{\text{RADIATION RATE AT THE OBSERVER}}\frac{dt}{dt'} = \frac{dU}{dt\,d\Omega}(1-\beta\cos\theta) = \frac{e^2a^2\sin^2\theta}{4\pi c^3(1-\beta\cos\theta)^5}$$

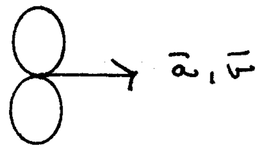
WE CONCLUDE THAT THE 'BRUTE FORCE' APPLICATION OF THE POYNTING VECTOR GIVES THE RADIATION RATE AT THE OBSERVER, WHILE <u>THE LORENTZ TRANSFORMATION METHOD GIVES THE RADIATION RATE AT THE CHARGE</u>.

BOTH RATES ARE MEASURED WITH RESPECT TO CLOCKS IN THE OBSERVER'S FRAME!

WE NOW EXPLORE SOME FEATURES OF THE RADIATION.

WHEN $\frac{v}{c} \sim 0$, THE SHAPE OF THE ANGULAR DISTRIBUTION IS

$\sin^2\theta$:



FOR $v \sim c$ THE RADIATION IS THROWN FORWARDS : THERE IS SIGNIFACANT RADIATION ONLY FOR SMALL θ

$\beta \sim 1$

AS $v \to c$ WE APPROXIMATE $1 - \frac{v}{c}\cos\theta \to 1 - \beta\left(1 - \theta^2/2\right) \sim 1 - \beta + \frac{\theta^2}{2}$

$$\sim (1-\beta)\frac{(1+\beta)}{2} + \frac{\theta^2}{2} = \frac{1}{2}\left(\frac{1}{\gamma^2} + \theta^2\right)$$

THE RATE OF RADIATION <u>AT THE CHARGE</u> IS THEN (DROPPING THE PRIME)

$$\frac{dU}{dt\,d\Omega} \to \frac{8e^2a^2}{\pi c^3}\,\frac{\gamma^8(\gamma\theta)^2}{\left[1 + (\gamma\theta)^2\right]^5}$$

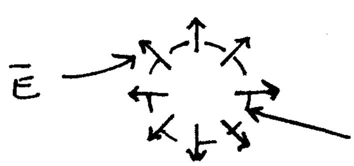
THIS DISTRIBUTION PEAKS AT $\theta_{MAX} = \frac{1}{2\gamma}$, WHICH IS AN IMPORTANT QUALITATIVE CHARACTERISTIC OF THE RADIATION.

AT θ_{MAX}, $\left.\frac{dU}{dt\,d\Omega}\right|_{MAX} \approx 10\gamma^8 \cdot \frac{e^2a^2}{4\pi c^3}$

WHICH IS A TREMENDOUS INCREASE OVER THE NON-RELATIVISTIC RESULT, IF $\gamma \gg 1$.

THE RADIATION IS <u>POLARIZED</u> AS MAY BE DETERMINED FROM THE RELATION $\bar{E} \sim \hat{r} \times (\hat{r} \times \bar{a})$ — WHICH IS RADIAL OUTWARDS FROM THE AXIS OF $\bar{v}$ AND $\bar{a}$

A HEAD-ON VIEW :



THE 'LIGHT' IS BRIGHTEST ON A CONE OF HALF-ANGLE $\theta \sim 1/2\gamma$

WE MAY ALSO ASK : WHAT COLOR IS THE RADIATION?

THAT IS, WHAT IS ITS FREQUENCY SPECTRUM, IF WE DO A FOURIER ANALYSIS OF THE RADIATION?

IT IS IMPORTANT TO NOTE THAT THE PHYSICAL ORIGIN OF BREMSSTRAHLUNG IS SUCH THAT THE ACCELERATION WILL BE NON-ZERO ONLY FOR A SHORT TIME Δt. THUS WE HAVE ONLY A SHORT PULSE OF RADIATION.

WE MAY QUICKLY ESTIMATE THE QUALITATIVE FEATURES OF THE FREQUENCY SPECTRUM BY MEANS OF THE 'UNCERTAINTY RELATIONS' OF FOURIER ANALYSIS (Ph 205, LECTURE 23): $\Delta\omega\,\Delta t \sim 1$.

A SHARP PULSE OF WIDTH Δt HAS A FREQUENCY SPECTRUM WHICH IS FLAT, UP TO A MAXIMUM FREQUENCY $\omega_{MAX} \sim \frac{1}{\Delta t}$

(SO THAT $\Delta t\,\Delta\omega \sim 1$)

IF WE SUPPOSE THE VELOCITY CHANGES BY AMOUNT Δv IN TIME Δt, THEN $a \sim \Delta v/\Delta t$, AND

$$\frac{dU}{dt} = \frac{2}{3}\frac{\gamma^6 e^2 a^2}{c^3} = \frac{2}{3}\frac{\gamma^6 e^2 \Delta v^2}{c^3\,\Delta t^2}$$

WE FURTHER IDENTIFY $U = \int_0^{\Delta t}\frac{dU}{dt}\,dt = \int_0^{\omega_{MAX}} U_\omega\,d\omega$

$$\Rightarrow \quad U_\omega = \frac{2}{3}\frac{\gamma^6 e^2 \Delta v^2}{c^3} = \text{CONSTANT}$$

$\beta = \sqrt{1 - 1/\gamma^2}$

$\Rightarrow d\beta = \frac{d\gamma}{\beta\gamma^3}$

$\Rightarrow U_\omega \approx \frac{2}{3}\frac{e^2}{c}(d\gamma)^2$ AS $\beta \to 1$

THESE QUICK CONCLUSIONS MAY BE JUSTIFIED BY A MORE DETAILED ARGUMENT.

THE ELECTRIC FIELD $E(t)$ MAY BE FOURIER ANALYSED:

$$E(t) = \int_{-\infty}^{\infty} E_\omega e^{-i\omega t}\,d\omega \quad \text{WITH} \quad E_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)\,e^{i\omega t}\,dt$$

IF $v/c \ll 1$

ROUGHLY, $E(t) \approx \frac{e\,a}{c^2 r} \sim \frac{e\,\Delta v}{c^2 r\,\Delta t}$ DURING THE INTERVAL Δt

SO $E_\omega \sim \frac{e\,\Delta v}{2\pi c^2 r}$ FOR $\omega \lesssim \frac{1}{\Delta t}$

NOW $\frac{dU}{dt} \sim \frac{c}{4\pi} r^2 E^2(t)$. AS IN LECTURE 16, WE NOTE THAT

$$U = \int \frac{dU}{dt}\,dt = \frac{c}{4\pi} r^2 \int E^2\,dt = \frac{cr^2}{4\pi}\int E\,dt \int_{-\infty}^{\infty} E_\omega e^{-i\omega t}\,d\omega = \frac{cr^2}{4\pi}\int_{-\infty}^{\infty} E_\omega\,d\omega \int E\,e^{-i\omega t}\,dt$$

$$= \frac{cr^2}{2}\int_{-\infty}^{\infty} d\omega\, E_\omega E_\omega^* = c r^2 \int_0^\infty E_\omega^2\,d\omega$$

THUS IF WE WRITE $U = \int_0^\infty U_\omega \, d\omega$ WE HAVE THAT

$$U_\omega = c r^2 E_\omega^2 = \frac{e^2 \Delta v^2}{4\pi^2 c^3} = \text{CONSTANT}, \text{ UP TO } \omega_{MAX} \sim \frac{1}{\Delta t}$$

THIS IS IN ROUGH AGREEMENT WITH OUR QUICK ARGUMENT, IF $v/c \ll 1 \Rightarrow \gamma \sim 1$.

IT IS INTERESTING TO INTERPRET THESE RESULTS IN TERMS OF PHOTONS. - THE PARTICLES OF LIGHT.

A PHOTON OF FREQUENCY ω HAS ENERGY $\hbar\omega$.

THEN $U_\omega = N_\omega \hbar\omega$ WHERE N_ω = NUMBER OF PHOTONS IN INTERVAL $d\omega$.

[STRICTLY SPEAKING: TOTAL ENERGY OBSERVED IN FREQUENCY INTERVAL $d\omega$ IS $U_\omega d\omega = N_\omega \hbar\omega \, d\omega$]

IF WE DEFINE n = NUMBER OF PHOTONS IN THE ENERGY INTERVAL $d(\hbar\omega)$ THEN

$$n \, d(\hbar\omega) = N_\omega d\omega \Rightarrow n = \frac{N_\omega}{\hbar} = \frac{U_\omega}{\hbar^2 \omega}$$

AND $$n \approx \left(\frac{e^2}{\hbar c}\right)\left(\frac{\Delta v}{c}\right)^2 \frac{1}{\hbar\omega} = \frac{1}{137}\left(\frac{\Delta v}{c}\right)^2 \frac{1}{\hbar\omega}$$

NOTING $e^2/\hbar c$ = DIMENSIONLESS CONSTANT = $1/137$

EXTREME LIMIT: $\Delta v \approx c$

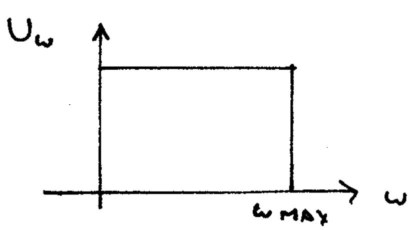
$\Rightarrow n \approx \alpha \dfrac{d\omega}{\omega}$

CLOSE VARIANTS OF THIS APPLY TO ALL RELATIVISTIC RADIATION

THE PHOTON IDEA ALSO TELLS US THAT THERE IS A MAXIMUM POSSIBLE FREQUENCY, SUCH THAT $\hbar\omega_{MAX}$ = TOTAL RADIATED ENERGY, BECAUSE IN THIS CASE THE ENTIRE PULSE OF RADIATION CONSISTS OF ONLY A SINGLE PHOTON!

$$\hbar\omega_{MAX} = U \sim \Delta t \frac{dU}{dt} \sim \frac{\gamma^6 e^2 \Delta v^2}{c^3 \Delta t} \Rightarrow \omega_{MAX} \sim \gamma^6 \left(\frac{e^2}{\hbar c}\right)\left(\frac{\Delta v}{c}\right)^2 \frac{1}{\Delta t}$$

OF COURSE, THE RADIATED ENERGY CANNOT EXCEED THE INITIAL ENERGY, γmc^2, SO $\omega_{MAX} < \frac{\gamma mc^2}{\hbar}$



AS $\omega \to 0$ THE NUMBER OF PHOTONS, N_ω DIVERGES, ALTHOUGH THE TOTAL ENERGY CARRIED BY THESE PHOTONS DOES NOT. THIS EFFECT IS CALLED THE 'INFRARED DIVERGENCE' SINCE IT OCCURS FOR LOW FREQUENCY, INFRARED RADIATION. PHYSICALLY, ANY RADIATION PROCESS TAKES PLACE IN SOME KIND OF 'BOX' WHICH LIMITS THE MAXIMUM POSSIBLE WAVELENGTH $\Rightarrow$ NO PHYSICAL DIVERGENCE.

AN AMUSING APPLICATION OF BREMSSTRAHLUNG IS THE SO-CALLED $1/f$ NOISE IN ELECTRONIC CIRCUITS.

THE MOVING ELECTRONS IN A WIRE COLLIDE WITH THE STATIONARY IONS AND EMIT BREMSSTRAHLUNG. THE NUMBER OF PHOTONS EMITTED VARIES AS $\frac{1}{\omega} \sim 1/f$.

THIS EFFECT CAN BE DETECTED AS FLUCTUATIONS IN THE CURRENT (SINCE v CHANGES). EMPIRICALLY NOISE AT FREQUENCIES AS LOW AS 1 CYCLE PER SECOND HAS BEEN OBSERVED. THE WAVE LENGTH OF A PHOTON OF THIS FREQUENCY IS $\lambda = \frac{c}{f} \sim 3 \times 10^{10}$ CM!

YET TO EXPLAIN THE PRESENCE OF THE FLUCTUATIONS WE MUST REGARD THIS WAVE AS A PARTICLE.

SYNCHROTRON RADIATION $\bar{a} \perp \bar{v}$

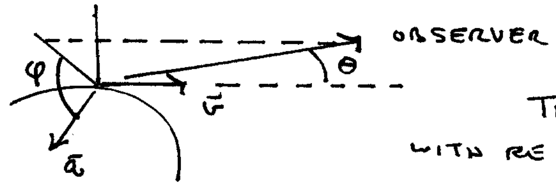
THE CASE OF $\bar{a}$ EXACTLY $\perp$ TO $\bar{v}$ ARISES PRIMARILY IN THE DEFLECTION OF CHARGED PARTICLES BY MAGNETIC FIELDS. THIS HAPPENS IN PARTICLE ACCELERATORS WITH CIRCULAR GEOMETRY, IN WHICH MAGNETIC FIELDS GUIDE THE PARTICLES IN CIRCULAR ORBITS, AND ACCELERATION IS PROVIDED ONCE A REVOLUTION VIA THE ELECTRIC FIELD INSIDE A CAVITY. A TYPE OF ACCELERATOR IS THE 'SYNCHROTRON', WHICH HAS LEANT ITS NAME TO THE RADIATION WE WILL NOW DISCUSS.

SOMETIMES THIS RADIATION IS CALLED 'MAGNETIC BREMSSTRAHLUNG', WHICH REMINDS US MORE EXPLICITLY OF THE PHYSICAL ORIGIN OF THE EFFECT.

SYNCHROTRON RADIATION IS SPECULATED TO BE THE MECHANISM OF THE VERY INTENSE SOURCES OF RADIATION WHICH EXIST IN THE DEBRIS FOLLOWING A SUPER NOVA EXPLOSION - SUCH AS THE CRAB NEBULA.

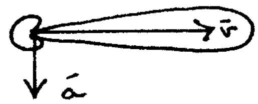
FROM OUR METHOD OF LORENTZ TRANSFORMATION (HOMEWORK SET) WE FIND THE RADIATED POWER TO BE (AT THE CHARGE)

$$\frac{dU}{dt} = \frac{2\gamma^4 e^2 a^2}{3c^3} \qquad \frac{dU}{dt\,d\Omega} = \frac{e^2 a^2}{4\pi c^3}\left[\frac{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\varphi}{(1-\beta\cos\theta)^5}\right]$$

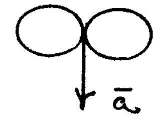


THE ANGLE φ IS MEASURED WITH RESPECT TO $\bar{a}$ IN THE PLANE $\perp$ TO $\bar{v}$.

THE RADIATION PATTERN LOOKS LIKE



WHICH IS THE TRANSFORM OF



RECALL THAT OUR LORENTZ TRANSFORM METHOD GIVES US THE RATE OF RADIATION AT THE CHARGE – WHICH IS MOVING IN A CIRCLE. HENCE θ AND φ ARE DEFINED IN A ROTATING FRAME. A FIXED OBSERVER WATCHING THE RADIATION WOULD REGARD IT AS SOMETHING LIKE THE BEAM FROM A LIGHTHOUSE.

IN THE EXTREME RELATIVISTIC LIMIT $\frac{v}{c} \to 1$, $1-\beta\cos\theta \to \frac{1}{2}\left(\frac{1}{\gamma^2}+\theta^2\right)$

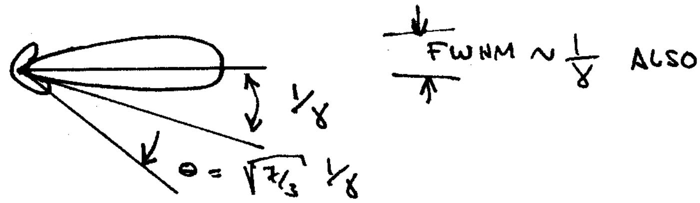
AND
$$\frac{dU}{dt\,d\Omega} \to \frac{2e^2a^2\gamma^6}{\pi c^3}\,\frac{1-2(\gamma\theta)^2(2\cos^2\varphi-1)+(\gamma\theta)^4}{\left(1+(\gamma\theta)^2\right)^5}$$

RESTRICTING TO THE PLANE OF $\bar{v}$ AND $\bar{a}$, $\cos^2\varphi = 1$

AND
$$\frac{dU}{dt\,d\Omega} \to \frac{2e^2a^2\gamma^6}{\pi c^3}\,\frac{\left[1-(\gamma\theta)^2\right]^2}{\left[1+(\gamma\theta)^2\right]^5}$$

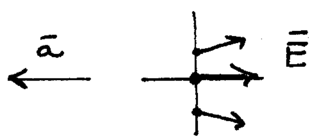
THIS DISTRIBUTION HAS MAXIMA AT $\theta = 0,\ \pm\sqrt{7/3}\,\frac{1}{\gamma}$

AND IT VANISHES AT $\theta = \pm 1/\gamma$



THE RADIATION IS POLARIZED: $\bar{E} \sim -\bar{a}_{\perp}$ (AS ALWAYS)

HEAD-ON VIEW:



$\bar{E}$ IS ESSENTIALLY POLARIZED IN THE PLANE OF THE PARTICLE'S MOTION.

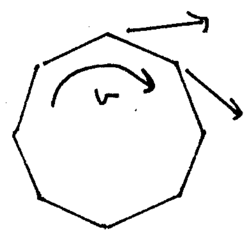
FOR RELATIVISTIC MOTION IN A CIRCLE OF RADIUS R THE ACCELERATION IS $a = v^2/R \sim c^2/R$

THE <u>TOTAL RADIATED POWER</u> IS THEN
$$\frac{dU}{dt} = \frac{2\gamma^4e^2a^2}{3c^3} = \frac{2\gamma^4e^2c}{3R^2} = \frac{2}{3}\frac{e^2U^4}{m_0^4c^7R^2}$$

THE RADIATION IS INTENSE UNLESS R IS VERY LARGE. $(U = \gamma m_0c^2)$

FROM THE POINT OF VIEW OF PARTICLE ACCELERATORS, THIS RADIATION IS A LOSS, AND REQUIRES THE ACCELERATORS TO BE VERY BIG TO AVOID IT. THE LIMITING CASE IS CLEARLY $R \to \infty$, $\Rightarrow$ LINEAR ACCELERATOR. WHEN ACCELERATING ELECTRONS, WHOSE MASS IS SMALL, THE SYNCHROTRON RADIATION LOSS IS A SERIOUS POWER DRAIN. THIS LED TO THE CONSTRUCTION OF THE STANFORD LINEAR ACCELERATOR - A DEVICE CONSISTING OF RESONANT CAVITIES 2 MILES LONG. WITH IT ELECTRONS CAN BE ACCELERATED UNTIL $\gamma \sim 10^5$.

ON THE OTHER HAND, SOME PEOPLE LIKE SYNCHROTRON RADIATION AS A SOURCE OF INTENSE X-RAYS. THE RADIATION INTENSITY CAN BE ENHANCED BY DECREASING THE RADIUS OF CURVATURE OF THE ELECTRONS PATH. TYPICALLY ONE WANTS THE RADIATION ONLY AT CERTAIN PLACES (FOR X-RAY EXPERIMENT), SO ONE BUILDS ELECTRON ACCELERATORS IN THE FORM OF POLYGONS



INTENSE RADIATION AT THE 'CORNERS'

WE FINISH WITH SOME REMARKS ABOUT THE <u>FREQUENCY</u> <u>DISTRIBUTION</u> OF THE SYNCHROTRON RADIATION.

AGAIN WE USE THE 'UNCERTAINTY RELATIONS' $\Delta\omega\, \Delta t \sim 1$ TO GET A QUICK QUALITATIVE SENSE OF THE RESULT.

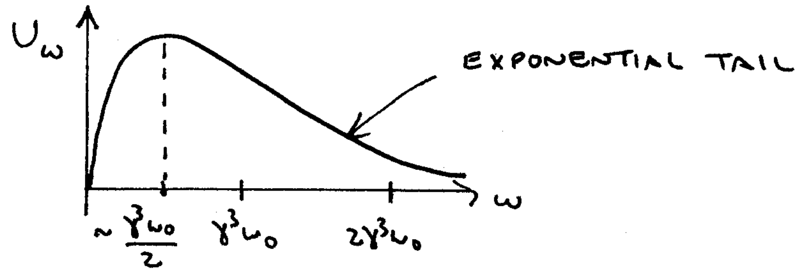
SUPPOSE THE CHARGE MOVES IN A CIRCLE WITH ANGULAR VELOCITY $\omega_0 = v/R \sim c/R$. THE 'SEARCHLIGHT' BEAM OF SYNCHROTRON RADIATION HAS AN ANGULAR WIDTH $\Delta\theta \sim 1/\gamma$. THEREFORE IT SWEEPS ACROSS AN OBSERVER AT FIXED ANGLE IN TIME $\Delta t' = \frac{\Delta\theta}{\omega_0} \sim \frac{1}{\gamma\omega_0}$.

NOTE THAT WE MUST AGAIN LABEL THE TIME AT THE CHARGE AT t', WHICH IS THE <u>RETARDED TIME</u> FROM THE POINT OF VIEW OF THE OBSERVER. THE FREQUENCY SPECTRUM SEEN BY THE OBSERVER WILL DEPEND ON THE DURATION OF THE PULSE Δt, NOT $\Delta t'$.

AS ON p239
$$\Delta t = \Delta t'\left(1 - \frac{\bar{v}\cdot\hat{r}}{c}\right) \sim \Delta t'(1-\beta) \sim \Delta t'(1-\beta)\frac{1+\beta}{2} \sim \frac{\Delta t'}{2\gamma^2}$$

SO $\Delta t \sim \frac{1}{2\gamma^3\omega_0} \Rightarrow \Delta\omega \sim 2\gamma^3\omega_0$

THE RESULT OF AN EXACT CALCULATION OF THE FREQUENCY SPECTRUM (OF THE INTENSITY) LOOKS LIKE



SO OUR QUICK ARGUMENT INDEED GIVES THE MAIN FEATURES OF THE SPECTRUM.

EXAMPLE A SYNCHROTRON LIGHT SOURCE HAS RADIUS $R = 3$ METERS AND CIRCULATES ELECTRONS OF ENERGY 500 MeV/c^2
(BROOKHAVEN NATIONAL LAB, LONG ISLAND)

THEN $\gamma \sim 10^3$ $\quad \omega_0 \sim \frac{c}{R} \sim 10^8$

$\omega \sim \gamma^3 \omega_0 \sim 10^{17}$

GREEN LIGHT ($\lambda \sim 6 \times 10^{-5}$ cm) HAS $\omega \sim 3 \times 10^{15}$

SO THE SYNCHROTRON RADIATION SPECTRUM EXTENDS FROM THE FAR ULTRAVIOLET INTO THE X-RAY REGION. THE RADIATION BECOMES 'HARDER' QUICKLY WITH INCREASING ELECTRON ENERGY, WHICH MOTIVATES THE RECENT CONSTRUCTION OF LIGHT SOURCES OF 3-5 GeV/c^2 ENERGY.

ANOTHER VIEW OF SYNCHROTRON RADIATION

(SEE ALSO FEYNMAN VOL I, CHAPTER 34)

INSTEAD OF EMPHASIZING THE ANGULAR DISTRIBUTION OF THE RADIATION AT THE MOVING CHARGE, WE CAN GIVE PROMINENCE TO THE TIME DEPENDENCE OF THE RADIATION SEEN BY A FIXED OBSERVER.

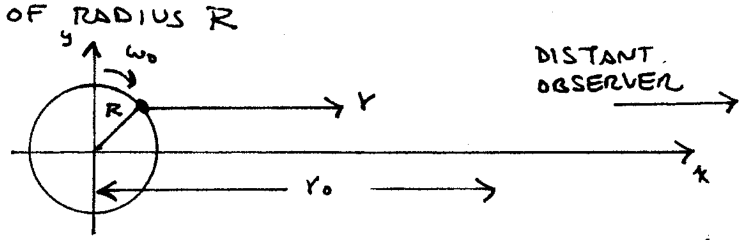
FEYNMAN HAS GIVEN ALTERNATIVE EXPRESSIONS FOR THE FIELDS DUE TO A RADIATING CHARGE, COMPARED TO OUR VERSION (p 234) OBTAINED BY DIFFERENTIATING THE LIENARD-WIECHERT POTENTIALS. WITHOUT DERIVING FEYNMAN'S RESULT IN FULL, WE CAN GIVE THE MAIN SENSE OF HIS APPROACH BY RETURNING TO OUR EXPRESSIONS FOR RADIATION FROM CONTINUOUS CHARGE DISTRIBUTIONS (LECTURE 16). [SEE ALSO, HEAVISIDE, PP 438-445, VOL III, ELECTROMAGNETIC THEORY (1912).]

THERE WE SAW THAT THE RADIATION FROM THE ELECTRIC DIPOLE MOMENT OF A CHARGE DISTRIBUTION IS

$$\bar{E}_{RAD} = \frac{([\ddot{\bar{p}}] \times \hat{n}) \times \hat{n}}{c^2 r} \qquad \text{WHERE } [\bar{p}] = \bar{p}(t' = t - r/c) = \text{RETARDED DIPOLE MOMENT}$$

THEN $[\ddot{\bar{p}}] \equiv \frac{d^2 \bar{p}(t')}{dt^2}$, TO MAKE A CHANGE OF NOTATION.

FOR SYNCHROTRON RADIATION, WE CONSIDER A CHARGE e MOVING WITH ANGULAR VELOCITY ω_0 ON A CIRCLE OF RADIUS R



THE MOTION OF THE CHARGE IS

$$x = R \sin \omega_0 t'$$
$$y = R \cos \omega_0 t'$$

WHERE THE USE OF t' REMINDS US THAT THESE RELATIONS HOLD FOR TIME MEASURED AT THE CHARGE.

FOR AN OBSERVER AT DISTANCE r_0 ALONG THE X-AXIS, $\hat{n} = \hat{x}$

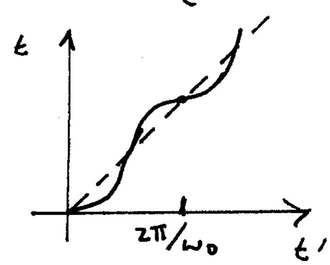
$$\text{so} \quad \bar{E}_{RAD} = -\frac{1}{c^2 r}\frac{d^2 p_y}{dt^2}\hat{y} \sim \frac{-e}{c^2 r_0}\frac{d^2 y}{dt^2}\hat{y} = -\frac{eR\omega_0^2}{c^2 r_0}\frac{d^2}{dt^2}\cos\omega_0 t' \,\hat{y}$$

WE WISH TO RE-EXPRESS THIS IN TERMS OF TIME t AT THE OBSERVER.

$$\text{NOW} \quad t' = t - \frac{r}{c} = t - \frac{r_0}{c} + \frac{x}{c} \Rightarrow \underline{t = \frac{r_0}{c} + t' - \frac{R}{c}\sin\omega_0 t'}$$

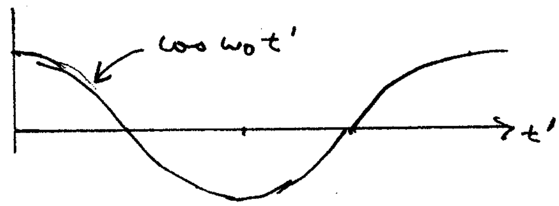
NOW IF $v = \omega_0 R \sim c$ THEN FOR SMALL t', $t \approx \frac{r_0}{c} + t'\left(1 - \frac{\omega_0 R}{c}\right)$

SO THAT t HARDLY CHANGES AT ALL WHILE t' INCREASES.

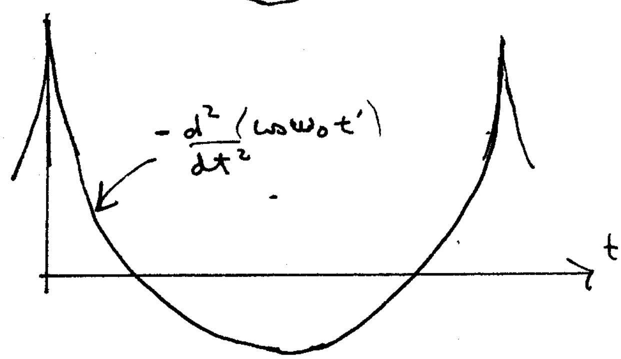


AT t' SMALL $\Rightarrow x \sim 0$, $y \sim R$, AND THE CHARGE APPROACHES THE OBSERVER. IN THIS CONFIGURATION, LARGE APPARENT MOVEMENT AT THE SOURCE OCCURS DURING VERY LITTLE CHANGE OF TIME AT THE OBSERVER $\Rightarrow$ BIG RADIATION EFFECTS.

DERIVATIVES WITH RESPECT TO t ARE VERY MUCH ENHANCED WHENEVER $y(t') \sim R$, AS SKETCHED AT THE RIGHT.



THESE GIANT DERIVATIVES REPRESENT THE LARGE $\bar{E}$ FIELD SEEN BY THE OBSERVER WHENEVER THE SOURCE IS HEADING DIRECTLY TOWARDS THE OBSERVER.



ON THE HOMEWORK SET WE ENCOURAGE YOU TO SOLVE FOR $E_{RAD}(t)$, AND MAKE A FOURIER ANALYSIS OF THIS, TO CONFIRM OUR QUALITATIVE ANALYSIS OF THE FREQUENCY SPECTRUM GIVEN EARLIER.

AN INTERESTING ARTICLE COMMEMORATING THE FIRST OBSERVATION OF SYNCHROTRON RADIATION APPEARED IN AM. J. PHYS. 51, 278 (1983), BY H.C. POLLOCK.

ELECTRICAL PAPERS

BY

OLIVER HEAVISIDE

IN TWO VOLUMES

VOL. II.

OLIVER HEAVISIDE'S SCIENTIFIC WRITINGS ARE CONTAINED IN FIVE VOLUMES: ELECTRICAL PAPERS, PUBLISHED IN TWO VOLUMES, AND ELECTROMAGNETIC THEORY, PUBLISHED IN THREE VOLUMES. THE PRESENT, SECOND EDITION OF THE ELECTRICAL PAPERS IS A REPRINT, IN TWO VOLUMES, OF THE FIRST EDITION, PUBLISHED AT LONDON IN 1892 AND IS TEXTUALLY UNALTERED, EXCEPT FOR CORRECTION OF ERRATA. IT IS PRINTED ON ALKALINE PAPER, AND IS PUBLISHED AT NEW YORK, N. Y., 1970.

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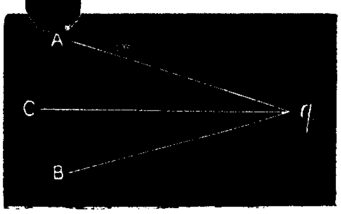
imposition of the elementary solutions, or by solving a bidimensional problem in a well-known manner.

Those who are acquainted with my papers in this journal will recognise that what we have arrived at is simply the elementary plane wave travelling along a distortionless circuit. All roads lead to Rome!

Returning to the case of a charge q at a point moving through a dielectric, if the speed of motion exceeds that of light, the disturbances are wholly left behind the charge, and are confined within a cone, AqB. The charge is at the apex, moving from left to right along Cq. The semi-angle, θ, of the cone, or the angle AqC, is given by



$$\sin\theta = v/u,$$

where v is the speed of light, and u that of the charge. The magnetic lines are circles round the axis, or line of motion. The displacement is away from q, of course, and of total amount q, but not uniformly distributed within the cone. The electric current is towards q in the inner part of the cone, and away from q in the outer.

It will be seen that the electric stress tends to pull the charge back. Therefore, applied force on q in direction Cq is required to keep up the motion. Its activity is accounted for by the continuous addition at a uniform rate which is being made to the electric and magnetic energies at q. For the motion at the wave-front, at any point on Aq or Bq, is perpendicularly outward, not towards q. Whilst the cone is thus expanding all over, the forward motion of q continually renews the apex, and keeps the shape unchanged.

Steady motion alone is assumed.

To avoid misconception I should remark that this is not in any way an account of what would happen if a charge were impelled to move through the ether at a speed several times that of light, about which I know nothing; but an account of what would happen if Maxwell's theory of the dielectric kept true under the circumstances, and if I have not misinterpreted it. [See footnote on p. 516, later.]

Nov. 18, 1888.

PART III.

[illegible] disturbances being propagated through the dielectric ether at the speed of light, when, therefore, a charge is in motion through the medium, the discussion of the effects produced naturally involves the consideration of three cases, those in which the speed u of the charge is less than, or equal to, or greater than v, that of light.

In a previous communication [Part II. above], I gave the complete and very simple solution of the intermediate case of equality of speeds. A formal demonstration is unnecessary, as the satisfaction of the necessary conditions may be immediately tested.

But I was not then aware that the case $u<v$ admitted of being presented in a nearly equally-simple form. That such is the fact is rather surprising, for it is very exceptional to arrive at simple results, and these now in question are sufficiently free from complexity to take a place in text-books of electricity.

Let the axis of z be the line of motion of the charge q at speed u. Everything is symmetrical with respect to this axis. The lines of electric force are radial out from the charge. Those of magnetic force are circles about the axis. The two forces are perpendicular. Having thus settled the directions, it only remains to specify their intensities at any point P distant r from the charge, the line $\mathbf{r}$ making an angle θ with the axis. Let E be the intensity of the electric, and H of the magnetic force. Then, if c is the permittivity and μ the inductivity, such that $\mu c v^2 = 1$, we have

$$(u<v)\left\{\begin{array}{ll} cE = \dfrac{\dfrac{q}{r^2}\left(1-\dfrac{u^2}{v^2}\right)}{\left(1-\dfrac{u^2}{v^2}\sin^2\theta\right)^{\frac{3}{2}}}, & \qquad\text{(A)} \\ H = cEu\sin\theta. & \qquad\text{(B)} \end{array}\right.$$

That (A), (B) represent the complete solution may be proved by subjecting them to the proper tests. Premising that the whole system is in steady motion at speed u, we have to satisfy the two fundamental laws of electromagnetism:—

(1). (Faraday's law). The electromotive force of the field [or voltage] in any circuit equals the rate of decrease of the induction through the circuit (or the magnetic current $\times -4\pi$).

(2). (Maxwell's law). The magnetomotive force of the field [or gaussage] in any circuit equals the electric current $\times 4\pi$ through the circuit.

Besides these, there is continuity of the displacement to be attended to. Thus:—

(3). (Maxwell). The displacement outward through any surface equals the enclosed charge.

Since (A) and (B) satisfy these tests, they are correct. And since no unrealities are involved, there is no room for misinterpretation.

When u/v is very small, we have, approximately,

$$cE = \frac{q}{r^2}, \qquad H = \frac{qu}{r^2}\sin\theta,$$

representing Prof. J. J. Thomson's solution—that is, the lines of displacement radiate uniformly from the charge, and the magnetic force is that of the corresponding displacement-currents together with the moving charge regarded as a current-element of moment qu. Instantaneous action through the medium is involved—that is, to make the solution quite correct.

That the lines of electric force should remain straight as the speed of the charge is increased is itself a rather remarkable result. Examining

ELECTROMAGNETIC THEORY

BY

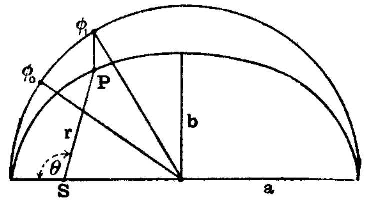
OLIVER HEAVISIDE

VOL. III

THE PRESENT WORK IS AN UNABRIDGED EDITION, IN THREE VOLUMES, OF A WORK ORIGINALLY PUBLISHED AT LONDON IN 1893, 1899, AND 1912, RESPECTIVELY, TO WHICH HAS BEEN ADDED A FOREWORD BY SIR EDMUND WHITTAKER, 'OLIVER HEAVISIDE, AN HISTORICAL FOREWORD,' AND THREE APPENDICES, AS FOLLOWS. TO THE FIRST VOLUME, 'THE WORK OF OLIVER HEAVISIDE,' BY B. A. BEHREND. TO THE THIRD VOLUME, 'SOME UNPUBLISHED NOTES OF OLIVER HEAVISIDE,' BY H. J. JOSEPHS AND IN ADDITION 'THE HEAVISIDE PAPERS FOUND AT PAIGNTON,' BY H. J. JOSEPHS. THE FIRST EDITION OF THE PRESENT WORK WAS REISSUED IN 1922 AND AGAIN IN 1925, AT LONDON. A SECOND EDITION, WITH AN INTRODUCTION BY E. WEBER, WAS PUBLISHED AT NEW YORK IN 1950. THE PRESENT EDITION IS PUBLISHED AT NEW YORK, IN 1971. IT IS PRINTED ON SPECIAL 'PERMANENT' ALKALINE PAPER.

These results, (88), (89), used in (78), (79), complete the explicit solutions.

(16). This brings us to Bessel's expansions of the formulas for the mean and eccentric anomaly in a planetary orbit. Very few electronic investigators know much about this old subject, so for comparison with the electromagnetic problem, I add a few notes thereon. The figure shows part of the elliptic orbit, and part of a



concentric circle. The polar co-ordinates of a point on the ellipse are r, θ, with the focus as origin; the circle has radius a, the major semi-axis. The minor one is b. Then

$$r=a(1-e\cos\phi_1)=\frac{a(1-e^2)}{1+e\cos\theta},\qquad e^2=1-\frac{b^2}{a^2},\tag{90}$$

is the polar equation referred to focus. Draw a perpendicular from the end of r to the circle at ϕ_1, and let $\phi_0=nt$. Then the angle ϕ_0 is called the mean anomally, ϕ_1 the eccentric anomaly, and θ the true anomaly, when a planet is at P, and the sun at S. Let ϕ_0 revolve uniformly in the circle. We have, by dynamics of particles,

$$\phi_1=\phi+e\sin\phi_1,\qquad \tan\tfrac{1}{2}\theta=\left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\phi_1.\tag{91}$$

The uniform revolution of ϕ_0 is accompanied by a continuously variable revolution of ϕ_1. Bessel showed that

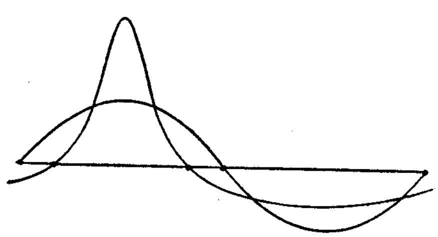
$$\phi_1=\phi_0+2\{J_1(e)\sin\phi_0+\tfrac{1}{2}J_2(2e)\sin 2\phi_0+\tfrac{1}{3}J_3(3e)\sin 3\phi_0+\ldots\}$$
$$\cos\phi_1=-\tfrac{1}{2}e+(J_0-J_2)(e)\cos\phi_0+\tfrac{1}{2}(J_1-J_3)(2e)\cos 2\phi_0 + \tfrac{1}{3}(J_2-J_4)(3e)\cos 3\phi_0+\ldots$$
$$\sin\phi_1=(J_0+J_2)(e)\sin\phi_0+\tfrac{1}{2}(J_1+J_3)(2e)\cos 2\phi_0+\tfrac{1}{3}(J_2+J_4)(3e)\sin 3\phi_0+\ldots$$

If in these, we put $e=$ my β, and reduce ϕ_0 and ϕ_1 by $\frac{1}{2}\pi$ each, we make $\phi=\phi_1+\beta\cos\phi_1$, and the results will correspond to some in my electromagnetic problem above, although there is no dynamical resemblance between the two problems.

(17). But it is not from these expansions, however useful for minute calculation, that we can see what the resultant effect is. For that we should examine the unexpanded **H** formulas themselves, giving special values to ϕ_1 and ϕ_0 in a complete revolution.

There is a great difference between the results on the axis and in the equatorial plane, where the Doppler effect is fully developed. In speaking of dopplerisation, it should be understood in a wider sense than in the old Doppler effect. The distortional effects due to the motion of a source of disturbance should be understood. In the uniform straight motion of a charge, for instance, the effect is equatorial compression of the tubes of displacement, and this is the same fore and aft. This is different from the old Doppler effect, which is a compression in front of the moving source of light, causing an increase of frequency in light of any one sort. Behind effects are ignored; but in plane waves the effect is the opposite, a lengthening of the waves and a lowering of frequency.

Now when a charge revolves in a fixed circle, and the point of observation is in its plane, the charge is sometimes moving towards, and sometimes moving away from the observing point. There is compression and increase of intensity in the first case, and expansion and decrease of intensity in the second case. So the sine wave is distorted, with a shifting closer of the nodes, an increase of intensity



between them, and a decrease outside. The main effect at the observing point of increasing the speed gradually from 0 up to v is to gradually squeeze the nodes together once per revolution, with a great concentration of the disturbance. During the rest of the revolution, the disturbance, of the opposite sign, is widespread and weak. The final result at $u/v=1$, is mere pulses, one per revolution, together with the spread weak disturbance of the opposite sign. The figure will show the beginning of this process.

In the plane of the orbit, $\theta=\frac{1}{2}\pi$, $H_{2\phi}=0$, and

$$\mathbf{H}_{2\theta}=\frac{Qnu}{4\pi rv}\,\frac{\sin\phi_1-u/v}{\{1-(u/v)\sin\phi_1\}^3},\qquad \begin{aligned}\phi_1&=\phi-nt_1\\&=\phi_0-\frac{u}{v}\cos\phi_1.\end{aligned}\tag{92}$$

The position of the nodes is given by

$$\sin\phi_1=\frac{u}{v},\qquad \phi_1=\phi_0-\frac{u}{v}\left(1-\frac{u^2}{v^2}\right)^{\frac{1}{2}}.\tag{93}$$

Say $u/v=\frac{1}{2}$, then $\phi_1=\pi/6$, and $\phi_0=\pi/6\pm\frac{1}{2}(\frac{3}{4})^{\frac{1}{2}}$.

Ph 206 Lecture 20

The Photon Spectrum in Synchrotron Radiation

There are two useful forms of the photon spectrum that we can display.

From p. 244 $$\frac{dU}{dt} \sim \frac{\gamma^4 e^2 c}{R^2}$$

From p. 245, we say that the radiation occupies $\Delta\omega \sim \gamma^3 \omega_0 \sim \frac{\gamma^3 c}{R}$

so, $$dU \sim \frac{\gamma e^2 \Delta\omega}{R} \Delta t$$

In terms of photons, $dU = \hbar\omega\, dn$, so $$dn \sim \gamma \underbrace{\frac{e^2}{\hbar c}}_{\alpha} \frac{c\Delta t}{R} \frac{\Delta\omega}{\omega}$$

If the radiation results from the passage of the particle thru a magnetic field of length L, then $L = c\Delta t$

Thus $$\underline{dn = \frac{\gamma L}{R} \alpha \frac{d\omega}{\omega}}$$ (compare p. 242 for Bremsstrahlung)

Now, $\frac{L}{R} = \Delta\Theta =$ bend angle of the particle's trajectory

i.e. $$dn = \gamma \Delta\Theta\, \alpha \frac{d\omega}{\omega}$$

The total number of radiated photons is estimated noting that $\int \frac{d\omega}{\omega} \sim 1$ for the synchrotron radiation spectrum

$$\Rightarrow \quad \underline{n \sim \frac{\gamma L}{R} \alpha \sim \gamma \Delta\Theta\, \alpha}$$

While the central frequency of the spectrum, $\gamma^3 \omega_0$, depends on the magnetic field strength, the total number of photons radiated depends only on the bend angle (times γ).

An observer at a particular location sees only part of the radiation. Indeed, since the angular spread of the radiation is $\Delta\Theta \sim 1/\gamma$, the observer detects radiation only for a bend of $\sim 1/\gamma$

$$\Rightarrow \quad \underline{dn_{obs} \sim \alpha \frac{d\omega}{\omega}}$$

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Ph 206 Lecture 20

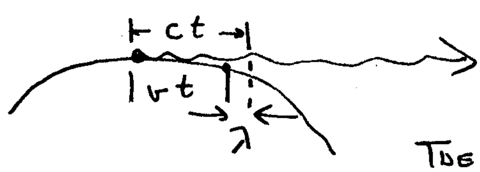
The Formation Length

Since the synchrotron radiation is produced by a particle moving close to the speed of light, the radiation remains close to the particle for quite a while.

We ask: when does the radiation really become radiation?

Roughly: when it leaves the 'near zone' and enters the 'far zone'.

i.e. when the radiation pulls ahead of the particle by 1 wavelength.

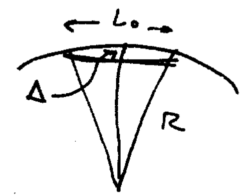


$$\lambda = ct - vt = ct\left(1 - \frac{v}{c}\right) = L_0 \frac{1-\beta^2}{1+\beta} \sim \frac{L_0}{2\gamma^2}$$

The formation length, $L_0 \sim 2\gamma^2 \lambda$ is the distance the particle travels while the light pulls ahead 1 wave. This is the effective length of the source region for a single photon.

Because the particle's trajectory is curved, the transverse size of the source region is $\Delta \sim R\left(1 - \cos\frac{L_0}{2R}\right) \sim \frac{L_0^2}{R}$,

$$\sim \frac{\gamma^4 \lambda^2}{R}$$



Now, the typical photon has frequency $\omega \sim \gamma^3 \omega_0 \sim \frac{\gamma^3 c}{R}$, ($\omega$: c/λ)

so $\frac{\gamma^3 \lambda}{R} \approx 1 \Rightarrow \underline{\underline{\Delta \sim \gamma \lambda}}$

This is consistent with the laws of diffraction:

A source of transverse size Δ can emit a beam of light of angle $\Theta \sim \frac{\lambda}{\Delta} \sim \frac{1}{\gamma}$.

And this is exactly the characteristic angular size of the synchrotron radiation pattern that we found by other arguments.

Ph 206 Lecture 20

Fluctuations in Syncrotron Radiation

All radiation processes are subject to fluctuations. We use the example of syncrotron radiation to make a few comments on this topic.

Fluctuations arise for at least 2 reasons:

1. Quantum fluctuations in the radiation of a single charge; the energy is radiated in quanta
2. Classical fluctuations in the radiation by a collection of N charges; fluctuations in the charge density lead to fluctuations in the radiated fields & power.

Quantum Fluctuations: Important results in this complicated subject can be obtained quickly via a remarkable insight of Hawking & Unruh (1976)

Hawking: An observer in a gravitational field of local acceleration g ~~feed~~ experiences effects of electromagnetic quantum fluctuations equivalent to being immersed in a blackbody radiation spectrum of temperature

$$T = \frac{\hbar g}{2\pi c K} \qquad K = \text{Planck's constant}$$

Unruh: Equivalence principle $\Rightarrow$ an accelerated observer experiences a radiation bath of temperature $T = \dfrac{\hbar a}{2\pi c k}$, a measured in inst. rest frame.

Applications: physics of electron storage rings..

Classical Density Fluctuations. The famous example is the argument of Smoluchowski (1908) & Einstein (1910) that the sky is blue only because of fluctuations in the molecular density in the atmosphere. If the sky were a crystal, there would be essentially complete destructive interfence among the light scattered by the molecules.

Application to synchrotron radiation: If the electrons were uniformly spaced around the ring, the synchrotron radiation would be zero - due to destructive interference.

A subtle result: If the electrons occupy only a small region (a "bunch"), the fluctuations in their synchrotron radiation can be used to determine the bunch length!

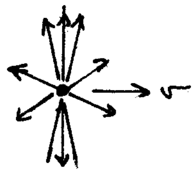
Ph 206 Lecture 23

Bremsstrahlung Revisited – The Method of Virtual Quanta

An interesting technique of Fermi (1925), popularized by Weizsäcker & Williams in 1935, can be used to relate scatter process involving charged particles to one involving E-M waves (photons)

The basic example related bremsstrahlung (pp. 238-243) to Thomson scattering.

The E-M fields of a fast-moving charged particle are strong only for angles nearly 90° to the velocity. Also $\bar{B} = \frac{\bar{v}}{c} \times \bar{E}$.



As $v \to c$ the fields are very much like a pulse of radiation – sometimes called the equivalent pulse of virtual quanta

If the particle scatters off another charged particle, the result is nearly equivalent to the scattering of the equivalent pulse off the second particle. The scattered 'virtual quanta' then emerge as the real photons previously described as bremsstrahlung.

We begin the detailed analysis by considering a charged particle of (rest) mass m, charge e and velocity $v \approx c$ that passes by a second charge particle of charge Ze with 'impact parameter' b (= distance of closest approach). The probability that impact parameter b lies in interval db is $2\pi b\, db$, supposing the first particle has uniform probability of being anywhere in a $1\,\mathrm{cm}^2$ (cgs units) area. Thus we can sum our results over the possible locations of the first particle by integrating over b with weight $2\pi b$.

At transverse distance b from the first particle the flux of energy (energy/area/time) is approximately the Poynting vector for the transverse fields:

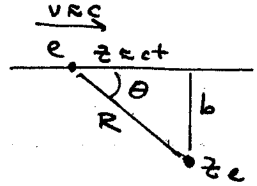
$$S(b) = \frac{dU(b)}{dA\,dt} = \frac{c}{4\pi} E_\perp^2(b)$$

We will integrate over time to characterize the total effect of the interaction between the two particles:

$$\frac{dU(b)}{dA} = \frac{c}{4\pi}\int E_\perp^2(b)\,dt = c\int_0^\infty d\omega\, E_\omega^2(b) \quad \text{where} \quad E_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_\perp(b,t)\, e^{i\omega t}\,dt$$

Recalling the argument of p. 248. From p. 234 we recall the form of the field of a rapidly moving charge:

$$\bar{E} = \frac{e\,\bar{R}}{\gamma^2 R^3 (1-\beta^2\sin^2\theta)^{3/2}} \quad \text{so} \quad E_\perp = \frac{e\,b}{\gamma^2 (b^2+c^2t^2)^{3/2}\left(1-\frac{\beta^2 b^2}{b^2+c^2t^2}\right)^{3/2}} = \frac{e\gamma b}{(b^2+\gamma^2 c^2 t^2)^{3/2}}.$$



HENCE $$E_\omega(b) = \frac{e\gamma b}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}\, dt}{b^2 + \gamma^2 c^2 t^2} \underset{s = \frac{\gamma c t}{b}}{\overset{\uparrow}{=}} \frac{e}{2\pi b c}\int_{-\infty}^{\infty} \frac{e^{\frac{i\omega b s}{\gamma c}}\, ds}{(s^2+1)^{3/2}}$$

THIS INTEGRAL RESULTS IN A MODIFIED BESSEL FUNCTION. HOWEVER, THE ESSENTIAL RESULT FOLLOWS FROM A SIMPLE APPROXIMATION.

IF $\frac{\omega b}{\gamma c} < 1$ THEN $e^{\frac{i\omega b s}{\gamma c}} \approx 1$ IN THE REGION NEAR $s = 0$ WHERE THE DENOMINATOR IS SMALL. BUT WHEN $\frac{\omega b}{\gamma c} > 1$ THE FACTOR $e^{i\frac{\omega b s}{\gamma c}}$ OSCILLATES NEAR $s=0$ AND THE INTEGRAL IS VERY SMALL.

HENCE $$E_\omega^{(b)} \approx \begin{cases} \frac{e}{2\pi b c}\int_{-\infty}^{\infty}\frac{ds}{(s^2+1)^{3/2}} = \frac{e}{\pi b c}, & \omega < \frac{\gamma c}{b} \iff b < \frac{\gamma c}{\omega} \\ 0 & \omega > \frac{\gamma c}{b} \end{cases}$$

AND $$\frac{dU(b)}{dA} = \int_0^\infty d\omega\; c E_\omega^2(b) = \int_0^{\gamma c/b} d\omega \cdot \frac{e^2}{\pi^2 b^2 c} \equiv \int_0 d\omega\, \frac{dU_\omega(b)}{dA}$$

THUS $$\frac{dU_\omega(b)}{dA} = \frac{e^2}{\pi^2 b^2 c} \qquad \left(\omega < \frac{\gamma c}{b}\right)$$ IS THE ENERGY PER AREA PER FREQUENCY INTERVAL IN THE FIELD OF THE FIRST PARTICLE AS OBSERVED BY THE SECOND PARTICLE AT IMPACT PARAMETER b.

WE CAN SUM OVER IMPACT PARAMETER BY WEIGHTING BY $2\pi b$ AND INTEGRATING:

$$U_\omega = \int_{b_{MIN}}^{b_{MAX}} 2\pi b\, db\, \frac{dU_\omega(b)}{dA} = \frac{2}{\pi}\frac{e^2}{c}\ln\frac{b_{MAX}}{b_{MIN}}$$

THIS ANALYSIS HOLDS ONLY FOR FREQUENCIES SUCH THAT $\omega \lesssim \frac{\gamma c}{b}$, SO $b_{MAX} \approx \frac{\gamma c}{\omega}$.

b_{MIN} IS NOT READILY OBTAINED BY A CLASSICAL ARGUMENT.

FROM QUANTUM MECHANICS, $b_{MIN} \sim$ UNCERTAINTY IN POSITION OF PARTICLE IN $x = \frac{\hbar}{mc}$.

THUS $$U_\omega = \frac{2}{\pi}\frac{e^2}{c}\ln\frac{\gamma m c^2}{\hbar\omega}.$$

THIS VANISHES WHEN $\hbar\omega = \gamma m c^2$, I.E. WHEN A SINGLE PHOTON WOULD CARRY 100 % OF THE INITIAL ENERGY

IT IS CUSTOMARY TO CONVERT THIS CLASSICAL ENERGY SPECTRUM TO A SPECTRUM OF VIRTUAL QUANTA OF ENERGY $u = \hbar\omega$ VIA

$$n \cdot u \cdot du = U_\omega\, d\omega.$$

THUS $n = \frac{2}{\pi}\frac{\alpha}{\omega}\ln\frac{\gamma m c^2}{\omega}$ = ENERGY SPECTRUM OF THE VIRTUAL QUANTA OF THE FIELD OF A FAST-MOVING CHARGED PARTICLE (SUMMED OVER IMPACT PARAMETER).

THE EMISSION OF RADIATION IN THE COLLISION OF TWO CHARGED PARTICLES CAN NOW BE THOUGHT OF AS THE SCATTERING OF THE VIRTUAL QUANTA ASSOCIATED WITH THE FIRST PARTICLE BY THE SECOND PARTICLE (OR VICE VERSA!).

WE REPORT THE RESULT IN THE LANGUAGE OF SCATTERING CROSS SECTIONS.

FOR EXAMPLE IF N_1 PARTICLES OF TYPE 1 COLLIDE WITH N_2 PARTICLES OF TYPE 2 EACH EACH SET OF PARTICLES OCCUPIES AREA A ⊥ TO THEIR DIRECTION OF MOTION, THEN WE WRITE

$$\frac{dN}{d\omega} = \frac{N_1 N_2}{A}\frac{d\sigma}{d\omega}$$

WHERE $dN/d\omega$ = # OF PHOTONS EMITTED IN ENERGY INTERVAL $d\omega$, AND

$d\sigma/d\omega$ = SCATTERING CROSS SECTION : DIMENSIONS OF AREA/ENERGY.

[VARIATION: IF PARTICLES 2 ARE IN A SOLID OF NUMBER DENSITY ρ_2 AND LENGTH L_2 THEN $N_2 = \rho_2 L_2 A$, AND $\frac{dN}{d\omega} = N_1 \rho_2 L_2 \frac{d\sigma}{d\omega}$.

ALSO ρ_2 (NUMBER DENSITY) = $\frac{N_0 \rho_2 \text{ (MASS DENSITY)}}{A_2 \text{ (ATOMIC \#)}}$, N_0 = AVAGADRO'S #]

THE WEIZSÄCKER-WILLIAMS APPROXIMATION IS THAT $\frac{d\sigma_{\text{BREMS}}}{d\omega} \approx n\,\sigma_{\text{THOMSON}}$

WHERE n IS THE SPECTRUM AT THE TOP OF THE PAGE, AND σ_{THOMSON} DESCRIBES THE SCATTERING OF ELECTRONS AND PHOTONS.

IF PARTICLE 2 HAS CHARGE Ze RATHER THAN e WE SHOULD USE $\sigma_T = \frac{8\pi}{3} Z^2 r_0^2$.

THEN $\frac{d\sigma_{\text{BREMS}}}{d\omega} \approx \frac{16}{3} Z^2 \frac{\alpha r_0^2}{\omega}\ln\frac{\gamma m c^2}{\omega} \approx \frac{1}{\omega}$ AS ON p. 242.

WHICH IS QUITE ACCURATE FOR SMALL γ. FOR $\gamma \gg 1$ THE RESULT WOULD BE IMPROVED BY USING THE 'KLEIN-NISHINA' CROSS SECTION RATHER THAN THE THOMSON CROSS SECTION FOR SCATTERING OF ELECTRONS AND PHOTONS.

IN THE CASE OF SCATTERING ELECTRONS OFF NUCLEI WE CAN JUSTIFY USE OF $r_0 = e^2/m$ WITH $m = m$(ELECTRON) BY TRANSFORMING TO A FRAME IN WHICH THE ELECTRON IS AT REST, AND SO IS THE 'TARGET'. IN THIS FRAME THE THOMSON CROSS SECTION HOLDS WITH $r_0 = e^2/m$, AND $b_{MIN} = \hbar/mc$. THE FIELD OF THE MOVING NUCLEUS DEPENDS ON Ze RATHER THAN e, SO THE SPECTRUM OF VIRTUAL QUANTA IS Z^2 TIME THE FORM AT THE TOP OF THE PAGE. FINALLY, TRANSFORM BACK TO THE LAB FRAME, RECALLING THAT TRANSVERSE AREAS ARE INVARIANT.