TWO DIMENSIONAL POTENTIAL PROBLEMS WITH CYLINDRICAL OR RADIAL BOUNDARIES

SUCH PROBLEMS SUGGEST THE USE OF POLAR COORDINATES.

$$\phi = \phi(r,\theta) \qquad \nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \qquad \text{(BECKER 13.4a)}$$

WE CONSIDER SOLUTIONS TO LAPLACE'S EQUATION $\nabla^2\phi = 0$.

AGAIN WE TRY SEPARATION OF VARIABLES: $\phi = R(r)\,\Theta(\theta)$

$$\text{so} \qquad \underbrace{\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)}_{=\,+k_n^2} + \underbrace{\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2}}_{=\,-k_n^2} = 0$$

$\Leftarrow$ SEPARATION CONSTANTS

$$\frac{d^2\Theta}{d\theta^2} + k_n^2\,\Theta = 0 \Rightarrow \Theta_n = A_n\cos k_n\theta + B_n\sin k_n\theta$$

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) - k_n^2 R = 0$$

WE TRY A POWER SOLUTION: $R = r^\alpha$

$$r\frac{d}{dr}\left(r\frac{d r^\alpha}{dr}\right) = r\frac{d}{dr}\alpha r^\alpha = \alpha^2 r^\alpha$$

SO IF $\alpha = \pm k_n$ IT WORKS!

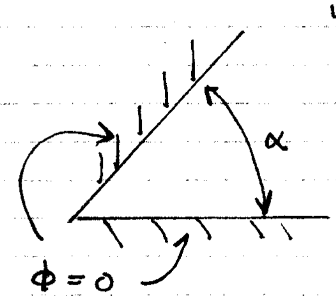
ie $R = r^{k_n}$ OR $\frac{1}{r^{k_n}}$

SPECIAL CASE: $k_n^2 = 0 \Rightarrow \Theta = a + b\theta\,;\ R = c + d\ln r$

THE GENERAL SOLUTION IS THEN

$$\phi = (E + F\theta)(G + H\ln r) + \sum_n (A_n\cos k_n\theta + B_n\sin k_n\theta)\left(C_n r^{k_n} + \frac{D_n}{r^{k_n}}\right)$$

EXAMPLE WEDGED SHAPED CORNER WITH GROUNDED CONDUCTING WALLS



SUPPOSE FOR THE MOMENT THAT ALL OTHER CHARGES ARE AT LARGE r.

THEN $\phi(r=0)=0 \Rightarrow \phi = \sum_n (A_n \cos k_n\theta + B_n \sin k_n \theta)\, r^{k_n}$

$$\phi(\theta=0)=0 \Rightarrow A_n = 0$$

$$\phi(\theta=\alpha)=0 \Rightarrow k_n = \frac{n\pi}{\alpha}$$

AND $\phi = \sum_n B_n \sin \frac{n\pi\theta}{\alpha}\, r^{\frac{n\pi}{\alpha}}$.

TO SOLVE FOR THE B_n WE MUST HAVE SOME ADDITIONAL INFORMATION ABOUT ϕ FOR SOME $r>0$. HOWEVER THE MOST PROMINENT FEATURES OF THE BEHAVIOR OF ϕ NEAR THE CORNER ARE INDEPENDENT OF ϕ AT LARGE r (SO LONG AS ϕ IS NOT TOO PATHOLOGICAL).

FOR AS $r \to 0$, THE FIRST TERM IN THE SERIES CLEARLY DOMINATES

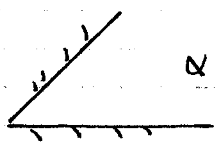
$$\phi(r \sim 0) \sim B \sin \frac{\pi\theta}{\alpha}\, r^{\pi/\alpha}$$

THE ELECTRIC FIELD IS GIVEN BY $\bar{E} = -\bar{\nabla}\phi$

$$E_r = -\frac{\partial \phi}{\partial r} = -\frac{\pi B}{\alpha} \sin\frac{\pi\theta}{\alpha}\, r^{\pi/\alpha - 1}$$

$$E_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\pi B}{\alpha}\cos\frac{\pi\theta}{\alpha}\, r^{\pi/\alpha - 1}$$

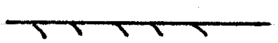
THE INDUCED SURFACE CHARGE DENSITY IS $\sigma = \frac{E_\perp}{4\pi} = \frac{E_\theta(0)}{4\pi} = -\frac{B}{4\alpha}\, r^{\pi/\alpha - 1}$



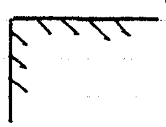
$\alpha = \pi/4$ $\quad E$ & $\sigma \sim r^3$

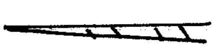
$\alpha = \pi/2$ $\quad E$ & $\sigma \sim r$

$\alpha = \pi$ $\quad E$ & $\sigma \sim$ CONSTANT



$\alpha = \frac{3\pi}{2}$ $\quad E$ & $\sigma \sim \frac{1}{\sqrt[3]{r}}$





$\alpha \sim 2\pi$ $\quad E$ & $\sigma \sim \frac{1}{\sqrt{r}}$

HENCE WE EXPECT STRONG FIELDS NEAR SHARP CORNERS! ALSO CHARGE ON A CONDUCTOR COLLECTS WHERE THE CURVATURE IS THE GREATEST.

IF WE WISH TO AVOID HIGH VOLTAGE BREAKDOWNS, THE CONDUCTORS MUST BE ROUNDED.
CONVERSELY, IF WE WISH TO ENHANCE BREAKDOWN - AS WITH A LIGHTNING ROD, USE A POINTED CONDUCTOR!

<u>EXAMPLE</u> WE CONSIDER THE SOMEWHAT ACADEMIC CASE THAT $\phi = V$ WHEN $r = a$, TO ILLUSTRATE THE COMPLETION OF A WEDGE PROBLEM.

THEN $$V = \sum_n B_n\, a^{\frac{n\pi}{\alpha}} \sin\frac{n\pi\theta}{\alpha}$$

$$\int_0^\alpha V \sin\frac{n\pi\theta}{\alpha}\, d\theta = B_n\, a^{\frac{n\pi}{\alpha}} \int_0^\alpha \sin^2\frac{n\pi\theta}{\alpha}\, d\theta = \frac{\alpha}{2} B_n\, a^{\frac{n\pi}{\alpha}}$$

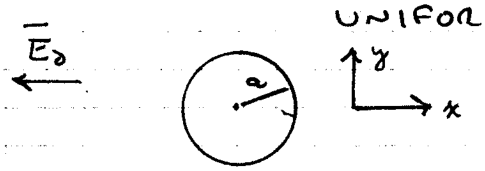
$$\parallel$$

$$\frac{\alpha V}{n\pi}\begin{cases} 0 & \text{IF } n \text{ EVEN} \\ 2 & \text{IF } n \text{ ODD}\end{cases} \quad \Rightarrow \quad B_n = \frac{4V}{n\pi\, a^{n\pi/\alpha}} \qquad (n \text{ ODD})$$

AND $$\phi = \frac{4V}{\pi} \sum_{n\ \text{ODD}} \frac{1}{n}\left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}} \sin\frac{n\pi\theta}{\alpha} = \frac{2V}{\pi} \tan^{-1}\left(\frac{2\sin\frac{\pi\theta}{\alpha}}{\left(\frac{a}{r}\right)^{\pi/\alpha} - \left(\frac{r}{a}\right)^{\pi/\alpha}}\right)$$

USING A VARIATION ON OUR METHOD OF SUMMING SUCH SERIES.

<u>EXAMPLE</u> CONDUCTING OR DIELECTRIC CYLINDER IN A UNIFORM ELECTRIC FIELD.



(THIS COULD BE SOLVED BY THE IMAGE METHOD.)

NOW $\bar{E}_0 = -E_0\hat{x} \Rightarrow \phi_0 = E_0 x = E_0 r\cos\theta$

FOR $r > a$ WE EXPECT THE SOLUTION FOR ϕ WILL BE

ϕ_0 + TERMS WHICH VANISH AS $r \to \infty$

IE $$\phi = \phi_0 + \sum_n \left(A_n \cos k_n\theta + B_n \sin k_n\theta\right)\frac{1}{r^{k_n}}$$

NOW $\phi(\theta) = \phi(\theta + 2\pi) \Rightarrow k_n = n$

WE ALSO EXPECT THE SYMMETRY $\phi(\theta) = \phi(-\theta) \Rightarrow B_n = 0$

SYMMETRY EXPERTS WILL NOTE THAT WE ALSO EXPECT $\phi(\pi-\theta) = -\phi(\theta) \Rightarrow n$ ODD

BUT EVEN WITHOUT THIS LAST RESTRICTION $\phi = \phi_0 + \sum_n \frac{A_n}{r^n} \cos n\theta \quad (r>a)$

FOR A <u>GROUNDED</u> <u>CONDUCTOR</u>, $\phi(r=a) = 0$

or $$0 = E_0 a \cos\theta + \sum_n \frac{A_n}{a^n} \cos n\theta$$

CLEARLY $A_n = 0$ FOR $n \neq 1$, AND $A_1 = -E_0 a^2$

so $$\phi = E_0\left(r - \frac{a^2}{r}\right)\cos\theta \qquad (r>a)$$

FOR A <u>DIELECTRIC</u> <u>CYLINDER</u> THE POTENTIAL VARIES FOR $r<a$.
BUT IT MUST NOT BLOW UP $\Rightarrow \phi = \sum_n B_n r^n \cos n\theta \qquad r<a$

(INSIDE THE SURFACE $r=a$ WE DON'T NEED TO TAKE ACCOUNT OF THE SPECIAL CONDITIONS AT ∞ WHICH ESTABLISH E_0 .... ALTHO WE COULD....)

AT $r=0$ WE SET $\phi = 0$ (BY DEF.) $\Rightarrow B_0 = 0$

THE CONNECTION WITH THE SOLUTION FOR $r>a$ IS ESTABLISHED BY RECALLING THAT $D_\perp$ IS CONTINUOUS AT A CHARGE FREE DIELECTRIC BOUNDARY.

i.e. $$\left.\frac{\partial\phi}{\partial r}\right|_{r>a} = \epsilon \left.\frac{\partial \phi}{\partial r}\right|_{r<a} \qquad (\epsilon = \text{DIELECTRIC CONSTANT})$$

ALSO WE EXPECT ϕ TO BE CONTINUOUS AT $r=a$ $\qquad (\bar{\nabla}\times\bar{E} = 0)$

ϕ CONTINUOUS $\Rightarrow$ $$E_0 a + \frac{A_1}{a} = B_1 a \qquad (n=1)$$
$$\frac{A_n}{a^n} = B_n a^n \qquad (n>1)$$

$D_\perp$ CONTINUOUS $\Rightarrow$ $$E_0 - \frac{A_1}{a^2} = \epsilon B_1 \qquad (n=1)$$
$$-\frac{nA_n}{a^{n+1}} = \epsilon n B_n a^{n-1} \qquad (n>1)$$

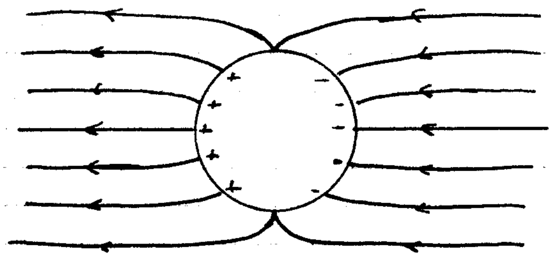
$$\Rightarrow A_n = B_n = 0 \text{ for } n>1 \; ; \; A_1 = -a^2 E_0 \frac{\epsilon-1}{\epsilon+1} \qquad B_1 = \frac{2E_0}{\epsilon+1}$$

$$\phi = E_0\left(r - \frac{\epsilon-1}{\epsilon+1}\frac{a^2}{r}\right)\cos\theta \quad (r>a) \quad \Bigg| \quad \phi = \frac{2E_0 r\cos\theta}{\epsilon+1} = \frac{2E_0 x}{\epsilon+1} \quad (r<a)$$

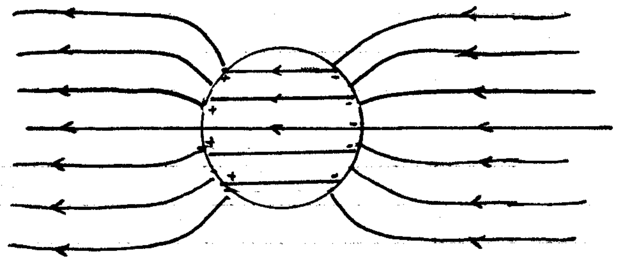
FOR $r<a$, $\bar{E} = -\nabla\phi =$ <u>CONSTANT</u> INSIDE THE DIELECTRIC!

THE SOLUTION FOR THE CONDUCTOR IS OBTAINED IN THE LIMIT $\epsilon \to \infty$.

THE FIELD LINES LOOK SOMETHING LIKE



CONDUCTOR
(WITH INDUCED SURFACE CHARGE)



DIELECTRIC
(WITH INDUCED POLARIZATION CHARGE AT SURFACE)

LAPLACE'S EQUATION IN SPHERICAL COORDINATES

AN IMPORTANT SET OF POTENTIAL PROBLEMS ARE THOSE SUITED FOR ANALYSIS IN SPHERICAL COORDINATES.

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

A COMMON NOTATIONAL DEVICE IS TO REPLACE θ BY $\mu = \cos\theta$

$$\Rightarrow \quad \frac{\partial}{\partial \theta} = -\sin\theta \frac{\partial}{\partial \mu}$$

$$\nabla^2 \phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \phi}{\partial \mu}\right] + \frac{1}{r^2(1-\mu^2)}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

WE TRY THE SEPARATED SOLUTION $\phi = R(r)\,\Theta(\theta)\,\Phi(\varphi)$

THEN $\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) - kR = 0$ WITH k = 1ST SEPARATION CONSTANT

AND $\frac{1}{\Theta}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \Theta}{\partial \mu}\right] + \frac{1}{1-\mu^2}\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} + k = 0$

FOR THE SECOND SEPARATION, WE SET $\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} = -m^2$

SO WE HAVE SIMPLE EXPRESSIONS FOR $\Phi = e^{\pm i m \varphi}$

WE CERTAINLY EXPECT $\Phi(\varphi) = \Phi(\varphi + 2\pi) \Rightarrow m = 0, 1, 2 \ldots\ldots$

[FOR PROBLEMS WITH NO φ BOUNDARY]

THIS LEAVES THE θ (OR μ) EQUATION AS

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + \left[k - \frac{m^2}{1-\mu^2}\right]\Theta = 0$$

WE CONSIDER THE RADIAL EQUATION NEXT: $\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) - KR = 0$

AS BEFORE, WE TRY A POWER SOLUTION : $R = r^n$

THEN $\frac{d}{dr} r^2 \frac{dR}{dr} = n(n+1) r^n = n(n+1) R = KR$

SO IT WORKS IF $K = n(n+1)$. WE EXPECT A SECOND SOLUTION TO A 2ND ORDER DIFFERENTIAL EQUATION. WE ACTUALLY HAVE IT ALREADY, BECAUSE IF $n' = -(n+1)$ THEN $n'(n'+1) = -(n+1)(-n-1+1) = n(n+1)$

SO FOR EACH VALUE OF K WE HAVE TWO RADIAL SOLUTIONS: r^n AND $\frac{1}{r^{n+1}}$ WHERE $K = n(n+1)$.

RETURNING TO THE Θ EQUATION, WE NOTE THAT EACH SOLUTION HAS TWO LABELS K (OR EQUIVALENTLY n) AND m.

THE EQUATION IS 2ND ORDER, SO THERE ARE ACTUALLY 2 SOLUTIONS.

THESE ARE OFTEN LABELLED $P_n^m(\mu)$ AND $Q_n^m(\mu)$

AND HAVE SERIES EXPANSIONS $P_n^m = \sum_\ell A_{n,\ell}^m \mu^\ell$ ETC.

WHICH YOU CAN EXPLORE BY DIRECT SUBSTITUTION INTO THE DIFFERENTIAL EQUATION, P_n^m IS WELL BEHAVED, BUT THE Q_n^m ARE SINGULAR AT $\mu = \pm 1$, AND DON'T OFTEN APPEAR IN PHYSICS PROBLEMS.

POTENTIAL PROBLEMS WITH AXIAL SYMMETRY

WE SHALL RESTRICT OURSELVES TO PROBLEMS WITH ROTATIONAL SYMMETRY ABOUT THE AXIS $\theta = 0$.

THEN WE MUST HAVE THE AXIAL FUNCTION $\Phi(\varphi) = \text{CONST} \Rightarrow m = 0$

THE μ EQUATION IS NOW, WRITING $K = n(n+1)$

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{d\Theta}{d\mu}\right) + n(n+1)\,\Theta = 0$$

THE SOLUTION TO THIS EQUATION ARE THE LEGRENDRE POLYNOMIALS

$P_n(\mu)$ WHICH WE HAVE ALREADY MET ON OCCASION

(AGAIN WE IGNORE THE $Q_n(\mu)$ WHICH ARE SINGULAR AT $\mu = \pm 1$)

$P_0 = 1$; $P_1 = \mu$; $P_2 = \frac{3\mu^2 - 1}{2}$ .....

THE LEGENDRE POLYNOMIALS HAVE MANY NOTABLE PROPERTIES.

THEY CAN BE WRITTEN $P_n(\mu) = \frac{1}{2^n n!} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n$

$$P_n(-\mu) = (-1)^n P_n(\mu)$$

$P_n(1) = 1$ FOR ALL l WHILE $P_n(-1) = (-1)^n$; $P_n(0) = 0$ FOR n ODD

$$\int_{-1}^{1} P_m(\mu) P_n(\mu)\, d\mu = \frac{2}{2n+1} \delta_{mn}$$

SO IF $f(\mu) = \sum_n A_n P_n(\mu)$ THEN $A_n = \frac{2n+1}{2} \int_{-1}^{1} f(\mu) P_n(\mu)\, d\mu$

OUR GENERAL SOLUTION TO LAPLACE'S EQUATION FOR PROBLEMS WITH AXIAL SYMMETRY IS THEN

$$\phi = \sum_n \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)$$

<u>EXAMPLE</u> CONDUCTING SPHERE HELD AT POTENTIAL V

THE SOLUTION TO THIS IS ELEMENTARY: $\phi = V\left(\frac{a}{r}\right)$ $r > a$

WHERE a = RADIUS OF SPHERE. CAN WE VERIFY THIS WITH OUR HIGH-POWERED TECHNIQUE?

WE WANT ϕ FOR $r > a$, AND AS $r \to \infty$ ϕ SHOULD NOT BLOW UP

SO $\phi = \sum_n \frac{B_n}{r^{n+1}} P_n(\cos\theta)$

AT $r = a$, $\phi = V = \sum_n \frac{B_n}{a^{n+1}} P_n(\cos\theta)$

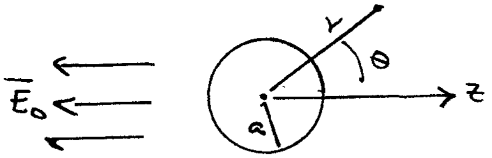
THEN $\frac{B_n}{a^{n+1}} = \frac{2n+1}{2} \int_{-1}^{1} V P_n(\cos\theta)\, d\cos\theta$

NOW $P_0 = 1 = $ const, SO OUR INTEGRAL IS ALSO $\frac{(2n+1)V}{2} \int_{-1}^{1} P_0 P_n = V \delta_{n0}$

$B_0 = Va$ & $B_n = 0$ $n \neq 0$

FINALLY $\phi = V \frac{a}{r}$ AS EXPECTED.

EXAMPLE: CONDUCTING OR DIELECTRIC SPHERE IN A UNIFORM EXTERNAL FIELD.



(SEE PROBLEM ⑧, SET 2 FOR THE SOLUTION BY IMAGES)

WE PROCEED SIMILARLY TO OUR SOLUTION OF THE CYLINDER IN A UNIFORM EXTERNAL FIELD.

$$\bar{E}_0 = -E_0\hat{z} \Rightarrow \phi_0 = E_0 z = E_0 r\cos\theta = E_0 r\, P_1(\cos\theta)$$

HENCE FOR $r > a$ $\quad \phi = E_0 r P_1 + \sum_n \frac{A_n}{r^{n+1}} P_n(\cos\theta)$

WHILE FOR $r < a$ $\quad \phi = E_0 r P_1 + \sum_n B_n r^n P_n(\cos\theta)$

(THIS TIME WE EXHIBIT ϕ_0 EXPLICITLY FOR $r<a$. IT REALLY DOESN'T MATTER WHETHER WE DO THIS OR NOT.)

AT $r=a$ THE POTENTIAL SHOULD BE CONTINUOUS $\Rightarrow B_n a^n = A_n / a^{n+1}$

GROUNDED CONDUCTING SPHERE $\quad \bar{E} = 0$ FOR $r<a$. $\phi = 0$ BY DEF.

$$\therefore \; B_1 = -E_0, \quad B_n = 0 \quad n \neq 1$$

AND $\quad A_1 = a^3 B_1 = -a^3 E_0 \qquad A_n = 0 \quad n \neq 1$

SO FOR $r > a$ $\quad \phi = E_0\left(r - \frac{a^3}{r^2}\right) P_1 \cos\theta = E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$

DIELECTRIC SPHERE $\quad D_\perp$ CONTINUOUS AT $r=a \Rightarrow \left.\frac{\partial\phi}{\partial r}\right|_{r>a} = \epsilon \left.\frac{\partial \phi}{\partial r}\right|_{r<a}$

ϵ = DIELECTRIC CONSTANT.

SO $\quad E_0 P_1 - \sum_n \frac{(n+1)A_n}{a^{n+2}} P_n = \epsilon\left(E_0 P_1 + \sum_n n \underbrace{B_n a^{n-1}}_{A_n/a^{n+2}} P_n\right)$ FROM ABOVE

SINCE THE P_n ARE ORTHOGONAL, THE COEFICIENTS MUST BE SEPARATELY EQUAL FOR EACH n.

$\therefore A_n = 0$ FOR $n \neq 1$, AND $\quad -\frac{\epsilon+2}{a^3} A_1 = (\epsilon - 1) E_0$

OR $\quad A_1 = -\frac{\epsilon-1}{\epsilon+2} a^3 E_0 \; ; \; B_1 = -\frac{\epsilon-1}{\epsilon+2} E_0$

FOR $r > a$, $\phi = E_0\left(r - \frac{\epsilon-1}{\epsilon+2}\frac{a^3}{r^2}\right)P_1$

FOR $r < a$ $\phi = E_0\left(r - \left(\frac{\epsilon-1}{\epsilon+2}\right)r\right)P_1$

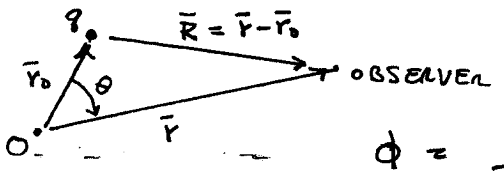
$$= \frac{3}{\epsilon+2}E_0 r\cos\theta = \frac{3E_0 z}{\epsilon+2}$$

FOR $r<a$ $\bar{E} = -\bar{\nabla}\phi = -\frac{3E_0}{\epsilon+2}\hat{z}$ = CONSTANT;

FOR $r>a$ ϕ IS THAT OF A CONSTANT FIELD, PLUS A DIPOLE OF STRENGTH $P = -\frac{\epsilon-1}{\epsilon+2}a^3E_0$ $\Longleftrightarrow$ IMAGE METHOD!

AGAIN WE NOTE THAT AS $\epsilon\to\infty$ THE DIELECTRIC SOLUTION BECOMES THAT OF THE CONDUCTOR.

EXAMPLE POTENTIAL OF A POINT CHARGE NOT AT THE ORIGIN



THIS IS ESSENTIALLY THE SITUATION WHICH LED TO THE MULTIPOLE EXPANSION (p. 11)

$$\phi = \frac{q}{R} = \frac{q}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}}$$

IF WE EXPAND THE DENOMINATOR IN A TAYLOR SERIES, WE FIND

$$\phi = \frac{q}{r_0}\sum_n\left(\frac{r}{r_0}\right)^n P_n(\cos\theta) \quad \text{FOR } r<r_0$$

$$\phi = \frac{q}{r}\sum_n\left(\frac{r_0}{r}\right)^n P_n(\cos\theta) \quad \text{FOR } r>r_0 \quad \text{(THE FORM ON p. 11)}$$

CAN WE CONFIRM THIS VIA OUR SEPARATION OF VARIABLES APPROACH?

WE DIVIDE THE PROBLEM INTO 2 PARTS, WITH THE SPHERE OF RADIUS r_0 AS THE "BOUNDARY"

THEN FOR $r<r_0$ WE EXPECT $\phi = \sum_n b_n\left(\frac{r}{r_0}\right)^n P_n(\cos\theta)$

WHILE FOR $r>r_0$ $\phi = \sum_n a_n\left(\frac{r_0}{r}\right)^{n+1}P_n(\cos\theta)$

WE HAVE DEFINED $A_n = a_n r_0^{n+1}$ AND $B_n = \frac{b_n}{r_0^n}$

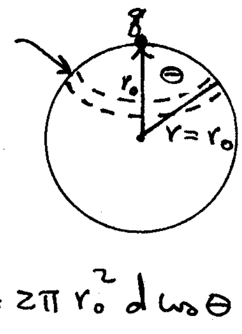
AS THESE POWERS OF r_0 WILL SURELY APPEAR ONCE WE CONSIDER THE BOUNDARY CONDITIONS AT $r=r_0$.

NOW WE EXPECT ϕ TO BE CONTINUOUS AT THE BOUNDARY

$\Rightarrow a_n = b_n$ (WHICH SHOWS A FURTHER ADVANTAGE OF USING a_n RATHER THAN A_n)

TO GET ANOTHER CONDITION, WE LOOK AT THE ELECTRIC FIELD AT $r = r_0$. THIS WILL BE CONTINUOUS EXCEPT WHEN VECTORS $\bar{r}$ AND $\bar{r}_0$ LINE UP, I.E. WHEN $\cos\theta = 1$.

THEN WE USE GAUSS LAW FOR A "PILLBOX" OF AREA dS IN THE FORM OF A BAND AROUND THE SPHERE OF RADIUS r_0 AT ANGLE θ, TO SEE THAT



$$\int dS \left(E_r\big|_{r>r_0} - E_r\big|_{r<r_0} \right) = 4\pi q_{\text{INSIDE}}$$

IN TERMS OF ANGLE θ, WE CAN WRITE $dS = r_0^2 d\Omega = 2\pi r_0^2 \, d\cos\theta$

HENCE
$$E_r\big|_{r>r_0} - E_r\big|_{r<r_0} = \frac{2q}{r_0^2}\,\delta(\cos\theta - 1)$$
$$\parallel$$
$$-\frac{\partial\phi}{\partial r}\bigg|_{r>r_0} + \frac{\partial\phi}{\partial r}\bigg|_{r<r_0}$$

HENCE
$$\sum_n a_n \frac{(n+1)}{r_0} P_n + \sum_n a_n \frac{n}{r_0} P_n = \frac{2q}{r_0^2}\,\delta(\cos\theta - 1)$$

$$\sum_n (2n+1)\, a_n P_n = \frac{2q}{r_0}\,\delta(\cos\theta - 1)$$

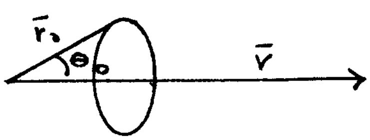
$$\therefore \quad (2n+1)\, a_n = \frac{2q}{r_0}\,\frac{2n+1}{2} \underbrace{\int_{-1}^{1} P_n\, \delta(\cos\theta - 1)\, d\cos\theta}_{1}$$ SINCE $P_n(1) = 1$

AND $a_n = b_n = \dfrac{q}{r_0}$ FOR ALL n

LEADING TO THE EXPRESSIONS DERIVED VIA THE TAYLOR SERIES.

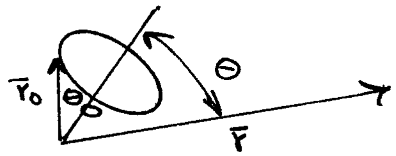
<u>EXAMPLE</u> POTENTIAL DUE TO A RING OF CHARGE.

WE FIRST SUPPOSE THE OBSERVER IS ALONG THE AXIS OF THE RING



CLEARLY THE POTENTIAL IS JUST THE SAME AS THAT FOR A POINT CHARGE AS FOUND ABOVE, WITH $\theta = \theta_0$.

A HARDER CASE IS WHEN THE OBSERVER IS OFF AXIS



NOTE THAT THE RING LIES ON A SPHERE OF RADIUS r_0, SO WE STILL EXPECT A MANAGEABLE SOLUTION IN SPHERICAL COORDINATES.

OUR GENERAL SOLUTION FOR THE POTENTIAL IS

$$\phi = \sum_n \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)$$

THE COEFFICIENTS IN THE ( ) ARE INDEPENDENT OF $\cos\theta$.

SO WE MAY EVALUATE THEM WHEN $\cos\theta = 1$ ($\Rightarrow P_n = 1$) IF WE WISH. BUT THEN THE PROBLEM REDUCES TO THE ON AXIS CASE WE JUST SOLVED.

THUS $A_n = \frac{q}{r_0^{n+1}} P_n(\cos\theta_0)$ FOR $r < r_0$ ($B_n = 0$ THERE)

$B_n = q\, r_0^n P_n(\cos\theta_0)$ FOR $r > r_0$ ($A_n = 0$ THERE)

BY COMPARING WITH p 57, AND SETTING $\theta = \theta_0$ THERE

$$\phi = \frac{q}{r_0} \sum_n \left(\frac{r}{r_0}\right)^n P_n(\cos\theta_0) P_n(\cos\theta) \qquad r < r_0$$

$$\phi = \frac{q}{r} \sum_n \left(\frac{r_0}{r}\right)^n P_n(\cos\theta_0) P_n(\cos\theta) \qquad r > r_0$$

IF $\theta_0 = 90°$ AND $r_0 = a$ THEN $\phi\Big|_{r>a} = \frac{q}{r} - \frac{q a^2}{4 r^3}(3\cos^2\theta - 1) + \cdots$

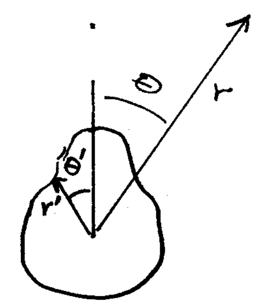
NOTING THAT $P_n(0) = 0$ FOR n ODD. COMPARE PROB 3, SET 1.

<u>EXAMPLE</u> MULTIPOLE EXPANSION OF A ROTATIONALLY SYMMETRIC CHARGE DISTRIBUTION

SUCH A CHARGE DISTRIBUTION CAN BE THOUGHT OF AS BEING MADE UP OUT OF A SERIES OF CHARGED RINGS WITH A COMMON AXIS.

HENCE FOR r OUTSIDE THE CHARGE DISTRIBUTION,

$$\phi = \sum_n \frac{P_n(\cos\theta)}{r^{n+1}} \int r'^n \rho\, P_n(\cos\theta')\, d\mathrm{vol}'$$



WE CAN DEFINE THE nTH MULTIPOLE MOMENT AS

$$Q_n = \int r'^n \rho P_n(\cos\theta')\, dvol'$$

$$Q_0 = \int \rho\, dvol' = \text{CHARGE}$$

$$Q_1 = \int r' \rho \cos\theta'\, dvol' = \int z' \rho\, dvol' = P_z = \text{DIPOLE MOMENT}$$

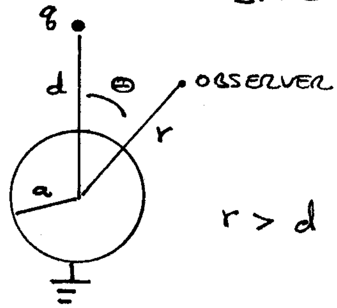
$$Q_2 = \int r'^2 \rho \left(\frac{3\cos^2\theta' - 1}{2}\right) dvol = \int \rho \left(\frac{3z'^2 - r'^2}{2}\right) dvol'$$

WHICH DIFFERS BY A FACTOR OF TWO FROM OUR PREVIOUS DEFINITION OF THE QUADRUPOLE MOMENT OF AN AXIALLY SYMMETRIC DISTRIBUTION (p 13)

THEN $\phi = \sum_n \frac{Q_n P_n(\cos\theta)}{r^{n+1}}$ DEMONSTRATES VERY COMPACTLY THE r AND θ DEPENDENCE OF THE MULTIPOLE EXPANSION TO ARBITRARY ORDER.

WE COULD ALSO CONSTRUCT A MULTIPOLE EXPANSION FOR r COMPLETELY INSIDE THE CHARGE DISTRIBUTION.

- EXAMPLE POINT CHARGE OUTSIDE A GROUNDED CONDUCTING SPHERE



THE POTENTIAL IS THAT DUE TO THE POINT CHARGE q AND THE UNKNOWN CHARGE DISTRIBUTION ON THE SPHERE.

$r > d$ $$\phi = \underbrace{\frac{q}{r}\sum_n \left(\frac{d}{r}\right)^n P_n(\cos\theta)}_{\phi \text{ DUE TO } q} + \underbrace{\frac{1}{r}\sum_n a_n \left(\frac{d}{r}\right)^n P_n(\cos\theta)}_{\phi \text{ DUE TO SPHERE}}$$

$a < r < d$ $$\phi = \frac{q}{d}\sum_n \left(\frac{r}{d}\right)^n P_n(\cos\theta) + \frac{1}{r}\sum_n a_n \left(\frac{d}{r}\right)^n P_n(\cos\theta)$$

$r < a$ $$\phi = 0$$

THE CONTINUITY OF ϕ AT $r = a$ IMPLIES

$$0 = \frac{q}{d}\sum_n \left(\frac{a}{d}\right)^n P_n + \frac{1}{a}\sum_n a_n \left(\frac{d}{a}\right)^n P_n$$

HENCE $$a_n = -q\left(\frac{a}{d}\right)^{2n+1}$$

THUS THE POTENTIAL DUE TO THE CHARGE ON THE SPHERE IS

$$-\frac{q}{r}\sum_n \frac{a^{2n+1}}{d^{2n+1}}\left(\frac{d}{r}\right)^n P_n(\cos\theta) = -\left(\frac{q a}{d}\right)\frac{1}{r}\sum_n\left(\frac{a^2}{d r}\right)^n P_n(\cos\theta)$$

WHICH IS EXACTLY THE POTENTIAL DUE TO A

CHARGE $q' = -\dfrac{q a}{d}$ LOCATED AT $c = \dfrac{a^2}{d}$ ON THE

LINE FROM THE CENTER OF THE SPHERE TO THE POINT CHARGE.

THIS IS EXACTLY THE PRESCRIPTION OF THE IMAGE METHOD FOUND IN LECTURE 4!

EXAMPLE THE POTENTIAL INSIDE A UNIFORM SPHERE OF CHARGE

THIS MAY, OF COURSE, BE SOLVED BY ELEMENTARY METHODS.

THE SPHERE HAS RADIUS a AND TOTAL CHARGE Q.

THEN $\phi(r) = \phi_{r'<r} + \phi_{r'>r}$

WHERE r' IS THE RADIUS TO SOME ELEMENT OF CHARGE, AND $r<a$.

FROM p 59 $(r' \Leftrightarrow r_0)$

$$\phi_{r'<r} = \sum_n \frac{P_n(\cos\theta)}{r^{n+1}}\int_0^r 2\pi r'^2 dr' \underbrace{\int_{-1}^{1} d\cos\theta' \, \rho\, r'^n P_n(\cos\theta')}_{=0 \text{ UNLESS } n=0}$$

$$= \frac{1}{r}\cdot\frac{4\pi r^3}{3}\rho = Q r^2/a^3 \qquad \left(Q = \frac{4\pi a^3}{3}\rho\right)$$

$$\phi_{r'>r} = \sum_n r^n P_n(\cos\theta)\int_r^a 2\pi r'^2 dr' \underbrace{\int_{-1}^{1} d\cos\theta' \, \rho\frac{P_n(\cos\theta')}{r'^{n+1}}}_{=0 \text{ UNLESS } n=0}$$

$$= 2\pi\rho\left(a^2 - r^2\right) = \frac{3Q}{2a^3}\left(a^2 - r^2\right)$$

SO $\phi(r) = \dfrac{Q}{2a^3}\left(3a^2 - r^2\right)$

$E_r = -\dfrac{\partial\phi}{\partial r} = Q\dfrac{r}{a^3}$ AS EXPECTED.

$\left(= \dfrac{Q}{r^2}\cdot\dfrac{r^3}{a^3}\right)$

The angles in S_n are the coordinates of the terminal point of R_i or R_o. Thus by superposition, the potential due to any surface distribution which can be expanded in the form

$$\sigma = S_0 + S_1 + S_2 + \cdots \tag{6}$$

is given by

$$V_i = \frac{a}{\epsilon}\left[S_0 + \left(\frac{r}{a}\right)\frac{S_1}{3} + \left(\frac{r}{a}\right)^2\frac{S_2}{5} + \cdots\right] \qquad \text{if } r < a$$

$$V_0 = \frac{a}{\epsilon}\left[\frac{a}{r}S_0 + \left(\frac{a}{r}\right)^2\frac{S_1}{3} + \left(\frac{a}{r}\right)^3\frac{S_2}{5} + \cdots\right] \qquad \text{if } r > a \tag{7}$$

5.14. Differential Equations for Surface Harmonics.—The variables θ and ϕ, in the differential equation, 5.12 (4), for the surface harmonics $S = \Theta\Phi$, may be separated by the process already used. Substitute $\Theta\Phi$ for S in 5.12 (4), and divide through by $\Theta\Phi/\sin^2\theta$ giving

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + n(n+1)\sin^2\theta = 0$$

For all values of θ and ϕ, this equation can be satisfied only if

$$\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + n(n+1)\sin^2\theta = K_1 \qquad \text{and} \qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -K_1$$

If we put $K_1 = m^2$, the solution of the second equation is easily seen to be

$$\Phi = C\cos m\phi + D\sin m\phi \tag{1}$$

except when $m = 0$, when it becomes

$$\Phi = M\phi + N \tag{1.1}$$

Putting $K_1 = m^2$ in the first equation and multiplying through by $\Theta/\sin^2\theta$ give

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0 \tag{2}$$

This is the differential equation for Θ.

5.15. Surface Zonal Harmonics. Legendre's Equation.—Before considering a more general solution of 5.14 (2), let us solve the most important special case in which V is independent of ϕ so that Φ is constant and, from 5.14 (1), m is zero. Equation 5.14 (2) then becomes, when μ is written for $\cos\theta$,

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta_n}{d\mu}\right] + n(n+1)\Theta_n = 0 \tag{1}$$

This is Legendre's equation, and its solutions are surface zonal harmonics.

5.151. Series Solution of Legendre's Equation.—To obtain a solution of 5.15 (1) in series, let us assume the solution

$$\Theta_n = \Sigma a_r\mu^r \tag{1}$$

Substituting this in 5.15 (1) gives

$$\Sigma\{r(r-1)a_r\mu^{r-2} + [n(n+1) - r(r+1)]a_r\mu^r\} = 0$$

To be satisfied for all values of μ, the coefficient of each power of μ must equal zero separately so that

$$(r+1)(r+2)a_{r+2} + [n(n+1) - r(r+1)]a_r = 0$$

$$a_r = -\frac{(r+1)(r+2)}{n(n+1) - r(r+1)}a_{r+2} = -\frac{(r+1)(r+2)}{(n-r)(n+r+1)}a_{r+2} \tag{2}$$

To expand in increasing powers of μ, we notice that if $a_r = 0$ then $a_{r-2} = a_{r-4} = \cdots = 0$ and from (2) a_{-1} and a_{-2} are zero if a_0 and a_1 are finite, so that all negative powers of μ disappear. Hence, if we choose $a_0 = 1$ and use the even powers, we have the solution

$$p_n = 1 - \frac{n(n+1)}{2!}\mu^2 + \frac{n(n-2)(n+1)(n+3)}{4!}\mu^4 - \cdots \tag{3}$$

If we choose $a_1 = 1$ and use the odd powers, we have

$$q_n = \mu - \frac{(n-1)(n+2)}{3!}\mu^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}\mu^5 - \cdots \tag{4}$$

A complete solution of 5.15 (1), if $-1 < \mu < +1$, is

$$\Theta_n = A_np_n + B_nq_n$$

regardless of whether n is an integer or a fraction, real or complex, provided the series converges. Recurrence formulas for p_n and q_n may be obtained by subtracting p_{n+1} from p_{n-1}, giving

$$\begin{aligned} p_{n-1} - p_{n+1} &= \left[\frac{(n+1)(n+2) - n(n-1)}{2!}\mu^2 - \frac{(n+1)(n+4) - n(n-3)}{4}\frac{(n-1)(n+2)}{3!}\mu^4 + \cdots\right] \\ &= (2n+1)\mu\left(\mu - \frac{(n-1)(n+2)}{3!}\mu^3 + \cdots\right) \\ &= (2n+1)\mu q_n \end{aligned} \tag{5}$$

By a similar procedure, we obtain

$$(n+1)^2q_{n+1} - n^2q_{n-1} = (2n+1)\mu p_n \tag{6}$$

Differentiating (4) and adding $n\,dq_{n-1}/d\mu$ and $(n+1)\,dq_{n+1}/d\mu$ give

$$\begin{aligned} nq'_{n-1} + (n+1)q'_{n+1} &= \left[2n+1 - \frac{(n-2+n+3)n(n+1)}{2!}\mu^2 + \frac{(n-4+n+5)n(n-2)(n+1)(n+3)}{4!}\mu^4 - \cdots\right] \\ &= (2n+1)\left(1 - \frac{n(n+1)}{2!}\mu^2 + \frac{n(n-2)(n+1)(n+3)}{4!}\mu^4 - \cdots\right) \\ &= (2n+1)p_n \end{aligned} \tag{7}$$

By a similar procedure, we obtain

$$(n+1)p'_{n-1} + np'_{n+1} = -n(n+1)(2n+1)q_n \tag{8}$$

5.152. Legendre Polynomials. Rodrigues's Formula.—If n is a positive even integer, the series in 5.151 (3) evidently terminates and has $\frac{1}{2}(n+2)$ terms and may be written

$$p_n = (-1)^{\frac{n}{2}} 2^n \frac{[(\frac{1}{2}n)!]^2}{n!} \sum_{r=0}^{\frac{1}{2}n} (-1)^{r-\frac{1}{2}n} \frac{(n+2r)!\,\mu^{2r}}{2^n[\frac{1}{2}(n-2r)]!\,[\frac{1}{2}(n+2r)]!\,(2r)!}$$

In this case, we define the polynomials $P_n(\mu)$ to be

$$P_n(\mu) = \frac{(-1)^{\frac{1}{2}n} n!}{2^n[(\frac{1}{2}n)!]^2} p_n \tag{1}$$

If n is a positive odd integer, the series in 5.151 (4) evidently terminates and has $\frac{1}{2}(n+1)$ terms and may be written

$$q_n = (-1)^{\frac{1}{2}(n-1)} 2^{n-1} \frac{\{[\frac{1}{2}(n-1)]!\}^2}{n!} \sum_{r=0}^{\frac{1}{2}(n-1)} (-1)^{r-\frac{1}{2}(n-1)} \frac{(n+2r+1)!\,\mu^{2r+1}}{2^n[\frac{1}{2}(n-2r-1)]!\,[\frac{1}{2}(n+2r+1)]!\,(2r+1)!}$$

In this case, we define the polynomial $P_n(\mu)$ to be

$$P_n(\mu) = \frac{(-1)^{\frac{1}{2}(n-1)} n!}{2^{n-1}\{[\frac{1}{2}(n-1)]!\}^2} q_n \tag{2}$$

Legendre's polynomial, given by $P_n(\mu)$ where n is a positive integer, in ascending powers of μ by (1) and (2) may be written in reverse order by substituting $s = \frac{1}{2}n - r$ in (1) and $s = \frac{1}{2}(n-1) - r$ in (2), both giving the result

$$P_n(\mu) = \sum_{s=0}^{m} (-1)^s \frac{(2n-2s)!}{2^n(s!)(n-s)!(n-2s)!} \mu^{n-2s} \tag{3}$$

where $m = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$, whichever is an integer.

An expression for $P_n(\mu)$, known as Rodrigues's formula, may be obtained by writing (3) in the form

$$P_n(\mu) = \frac{1}{2^n n!} \sum_{s=0}^{m} (-1)^s \frac{n!}{s!(n-s)!} \frac{(2n-2s)!}{(n-2s)!} \mu^{n-2s}$$

$$= \frac{1}{2^n n!} \frac{d^n}{d\mu^n} \sum_{s=0}^{n} (-1)^s \frac{n!}{s!(n-s)!} \mu^{2n-2s}$$

The last summation is the binomial expansion of $(\mu^2-1)^n$ so that

$$P_n(\mu) = \frac{1}{2^n n!} \frac{d^n}{d\mu^n} (\mu^2-1)^n \tag{4}$$

Equations (3) and (4) are valid solutions of Legendre's equation [5.15 (1)] whatever the range of the variable μ. In prolate spheroidal harmonics, we have $0 < \mu < \infty$. For very large values of μ, the highest power outweighs all others; so we have

$$P_n(\mu) \xrightarrow[\mu\to\infty]{} \frac{(2n)!}{2^n(n!)^2} \mu^n \tag{5}$$

5.153. Legendre Coefficients. Inverse Distance.—The polynomials of 5.152 are also known as Legendre's coefficients, the reason being evident from the following considerations. The reciprocal of the distance between two points at distances a and b from the origin, where $b > a$, when the angle between a and b is θ and $\mu = \cos\theta$, can be written:

$$\frac{1}{R} = (a^2+b^2-2ab\mu)^{-\frac{1}{2}} = \frac{1}{b}\left(1 + \frac{a^2-2ab\mu}{b^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{b}\left[1 - \frac{1}{2}\frac{a^2-2ab\mu}{b^2} + \frac{1\cdot 3}{2\cdot 4}\left(\frac{a^2-2ab\mu}{b^2}\right)^2 - \cdots\right]$$

$$= \frac{1}{b}\left[1 + \mu\frac{a}{b} + \frac{3\mu^2-1}{2}\left(\frac{a}{b}\right)^2 + \frac{5\mu^3-3\mu}{2}\left(\frac{a}{b}\right)^3 + \cdots\right]$$

We see that the coefficient of $(a/b)^n$ is exactly the expression for $P_n(\mu)$ in 5.152 (3) so that we may write

$$\frac{1}{R} = \frac{1}{b}\left[P_0(\mu) + \left(\frac{a}{b}\right)P_1(\mu) + \left(\frac{a}{b}\right)^2 P_2(\mu) + \cdots\right] \tag{1}$$

We shall refer to this expansion many times in solving problems.

5.154. Recurrence Formulas for Legendre Polynomials.—If n is an odd integer, we may substitute for p_{n-1}, p_{n+1}, and q_n in 5.151 (5) from 5.152 (1) and (2) and obtain, after dividing out the factor

$$\frac{2^{n-1}\{[\frac{1}{2}(n-1)]!\}^2}{n!(-1)^{\frac{1}{2}(n+1)}}$$

the result

$$nP_{n-1} + (n+1)P_{n+1} = (2n+1)\mu P_n \tag{1}$$

An identical expression is obtained, when n is even, from 5.151 (6). We omit the argument of these polynomials when there is no ambiguity in doing so.

If n is an even integer, we may substitute in 5.151 (7) for p_n, q'_{n-1}, and q'_{n+1} from 5.152 (1) and (2), divide out the factor

$$\frac{2^n(2n+1)[(\frac{1}{2}n)!]^2}{(-1)^{\frac{1}{2}n} n!}$$

and obtain

$$P'_{n+1} - P'_{n-1} = (2n + 1)P_n \tag{2}$$

An identical expression is obtained, when n is odd, from 5.151 (8).

The integral of $P_n(\mu)$ is given by integrating (2), the result being

$$\int P_n(\mu)\, d\mu = \frac{P_{n+1} - P_{n-1}}{2n + 1} \tag{3}$$

The derivative of $P_n(\mu)$ is given, by adding successive equations of the type of (2), to be

$$P'_n(\mu) = (2n - 1)P_{n-1} + (2n - 5)P_{n-3} + \cdots \tag{4}$$

Another useful expression for the derivative may be obtained by differentiating (1) and eliminating P'_{n-1} by (2), giving

$$P'_{n+1} = \mu P'_n + (n + 1)P_n \qquad \text{or} \qquad P'_n = \mu P'_{n-1} + nP_{n-1} \tag{4.1}$$

Eliminating P'_{n-1} and P'_{n+1} between these equations and (2) and combining with (1), (2), or (3) give the following equivalent forms:

$$P'_n = \frac{(n + 1)(\mu P_n - P_{n+1})}{1 - \mu^2} = \frac{-n(\mu P_n - P_{n-1})}{1 - \mu^2}$$

$$= \frac{n(n + 1)}{2n + 1}\frac{P_{n-1} - P_{n+1}}{1 - \mu^2} = \frac{-n(n + 1)}{1 - \mu^2}\int P_n(\mu)\, d\mu \tag{5}$$

5.155. Integral of Product of Legendre Polynomials.—In using Legendre polynomials, the integral of their product over the range $\theta = 0$ to π, or $\mu = -1$ to $+1$, is important. We saw in 5.13 (2) that

$$\int_{-1}^{+1} P_n(\mu)P_m(\mu)\, d\mu = 0 \qquad \text{if } m \neq n \tag{1}$$

If $m = n$, we substitute for one P_n from 5.152 (4)

$$\int_{-1}^{+1} [P_n(\mu)]^2\, d\mu = \frac{1}{2^n n!}\int_{-1}^{+1} P_n(\mu)\frac{d^n}{d\mu^n}(\mu^2 - 1)^n\, d\mu$$

Integrate the right side n times by parts, letting u be the first term and dv the second each time. Since $\dfrac{d^r(\mu^2 - 1)^n}{d\mu^r}$ always contains the factor $(\mu^2 - 1)^{n-r}$, v will always be zero when the limits are inserted and so the product uv drops out and we have finally

$$\int_{-1}^{+1} [P_n(\mu)]^2\, d\mu = \frac{(-1)^n}{2^n n!}\int_{-1}^{+1} \frac{d^n P_n(\mu)}{d\mu^n}(\mu^2 - 1)^n\, d\mu$$

But from 5.152 (3),

$$\frac{d^n P_n(\mu)}{d\mu^n} = \frac{(2n)!n!}{2^n n!n!} = \frac{(2n)!}{2^n n!} \tag{2}$$

so that

$$\int_{-1}^{+1} [P_n(\mu)]^2\, d\mu = \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{+1} (1 - \mu^2)^n\, d\mu$$

$$= \frac{(2n - 1)!!}{(2n)!!}\int_0^{\pi} \sin^{2n+1}\theta\, d\theta \tag{2.1}$$

Integration by Pc 483 or *Dw* 854.1 gives

$$\int_{-1}^{+1} [P_n(\mu)]^2\, d\mu = \frac{2}{2n + 1} \tag{3}$$

5.156. Expansion of Function in Legendre Polynomials.—Any function that can be expanded in Fourier's series in the interval $-1 < \mu < +1$ can also be expanded in a series of Legendre polynomials in the same interval and by a similar method. Assume the expansion

$$f(\mu) = a_0P_0(\mu) + a_1P_1(\mu) + \cdots + a_nP_n(\mu) + \cdots \tag{1}$$

Multiply by $P_m(\mu)$, and integrate from $\mu = -1$ to $+1$. By 5.13 (2), all terms vanish except the mth term; so we have, from 5.155 (3).

$$a_m = \tfrac{1}{2}(2m + 1)\int_{-1}^{+1} f(\mu)P_m(\mu)\, d\mu \tag{2}$$

Note that if $f(\mu) = 0$ when $-1 \leqslant \mu \leqslant +1$ then $a_m = 0$. This means that if we have an expansion in Legendre's polynomials equal to zero, the coefficient of each term must be separately equal to zero. As in the case of a Fourier's series, at a discontinuity this expansion gives half the sum of the values of $f(\mu)$ on each side. A formula for a_m which is frequently more convenient than (2) can be obtained by substituting Rodrigues's formula [5.152 (4)] in (2). Thus

$$a_m = (-1)^m\frac{2m + 1}{2^{m+1}m!}\int_{-1}^{+1} f(\mu)\frac{d^m(1 - \mu^2)^m}{d\mu^m}\, d\mu$$

Integrating this by parts repeatedly, always letting the first member be u and the second dv, we find that $|uv|_{-1}^{+1}$ is zero and $\int_{-1}^{+1} u\, dv$ alternates in sign until finally we are left with

$$a_m = \frac{2m + 1}{2^{m+1}m!}\int_{-1}^{+1} \frac{d^m f(\mu)}{d\mu^m}(1 - \mu^2)^m\, d\mu \tag{3}$$

If the derivatives of $f(\mu)$ are simple, this gives usually an easy integration.

5.157. Table of Legendre Polynomials.—A table of values of $P_n(\mu)$ can be computed from 5.152 (3) or (4). The values of $n < 9$ are

$$P_0(\mu) = 1, \qquad P_1(\mu) = \mu, \qquad P_2(\mu) = \tfrac{1}{2}(3\mu^2 - 1)$$

$$P_3(\mu) = \frac{1}{2}(5\mu^3 - 3\mu), \qquad P_4(\mu) = \frac{(35\mu^4 - 30\mu^2 + 3)}{8}$$

$$P_5(\mu) = \frac{(63\mu^5 - 70\mu^3 + 15\mu)}{8}, \quad P_6(\mu) = \frac{(231\mu^6 - 315\mu^4 + 105\mu^2 - 5)}{16}$$

$$P_7(\mu) = \frac{(429\mu^7 - 693\mu^5 + 315\mu^3 - 35\mu)}{16}$$

$$P_8(\mu) = \frac{(6435\mu^8 - 12{,}012\mu^6 + 6930\mu^4 - 1260\mu^2 + 35)}{128}$$

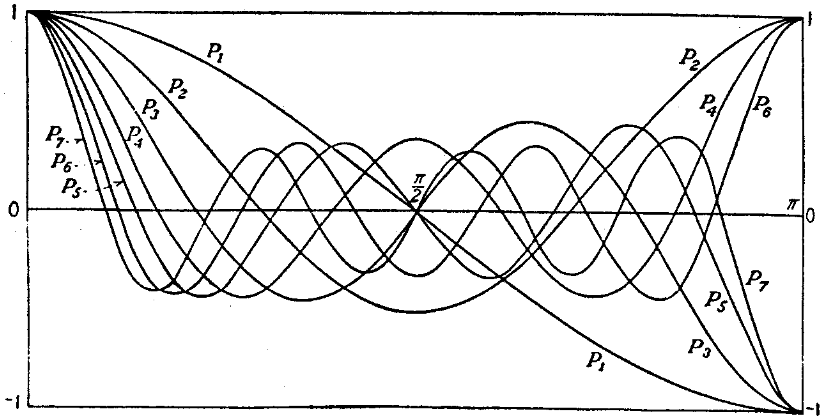


FIG. 5.157.—Legendre polynomials of orders 1 to 7.

Other useful values of $P_n(\mu)$ are

$$\begin{array}{lll}
(n \text{ odd}) & P_n(0) = 0 & \\
(n \text{ even}) & P_n(0) = (-1)^{\frac{1}{2}n}\dfrac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n} & \\
(\text{any } n) & P_n(1) = 1 & \\
(\text{any } n) & P_n(-\mu) = (-1)^n P_n(\mu) & \\
(\text{any } n) & P'_n(0) = -(n+1)P_{n+1}(0) & [\text{from 5.154 (5)}] \\
(\text{any } n) & P'_n(1) = \tfrac{1}{2}n(n+1) & [\text{from 5.15 (1)}]
\end{array}$$

The values of $P_n(\mu)$ for $0 \leqslant n \leqslant 7$ are shown in Fig. 5.157.

5.158. Legendre Polynomial with Imaginary Variable.—We shall have occasion, in oblate spheroidal harmonics, to deal with $P_n(j\zeta)$ where $j = (-1)^{\frac{1}{2}}$ and $0 \leqslant \zeta < \infty$. Substituting $j\zeta$ for μ in 5.152 (3), we have

$$P_n(j\zeta) = (-1)^{\frac{1}{2}n}\sum_{s=0}^{m}\frac{(2n-2s)!}{2^n(s)!(n-s)!(n-2s)!}\zeta^{n-2s} \qquad (1)$$

where $m = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$, whichever is an integer. A similar substitution in 5.152 (5) gives

$$P_n(j\zeta) \xrightarrow[\zeta\to\infty]{} \frac{(2n)!}{2^n(n!)^2}(-1)^{\frac{1}{2}n}\zeta^n \qquad (2)$$

5.16. Potential of Charged Ring.—If V is symmetrical about the x-axis and its value is known at all points on this axis and if this value can be expressed by a finite or convergent infinite series involving only integral powers of x, then the potential at any point can be obtained by multiplying the nth term by $P_n(\cos\theta)$ and writing r for x. The result holds for the same range of values of r as the range of x in the original expansion.

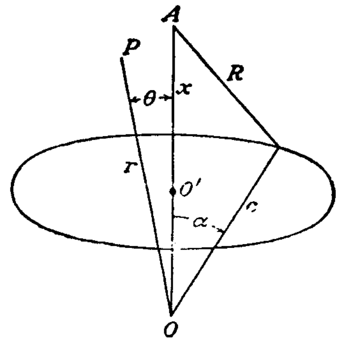


FIG. 5.16.

Let us apply this to a ring carrying a total charge Q (Fig. 5.16) giving

$$4\pi\epsilon V_A = Q(c^2 + x^2 - 2cx\cos\alpha)^{-\frac{1}{2}}$$

Expanding this by 5.153 (1) gives

$$x \geqslant c \qquad V_A = \frac{Q}{4\pi\epsilon c}\sum_{n=0}^{\infty}\left(\frac{c}{x}\right)^{n+1}P_n(\cos\alpha)$$

$$x \leqslant c \qquad V_A = \frac{Q}{4\pi\epsilon c}\sum_{n=0}^{\infty}\left(\frac{x}{c}\right)^{n}P_n(\cos\alpha)$$

The potential at any point P at r, θ is given by

$$r > c, \text{ or } \theta \neq \alpha,\ r = c \qquad V_P = \frac{Q}{4\pi\epsilon c}\sum_{n=0}^{\infty}\left(\frac{c}{r}\right)^{n+1}P_n(\cos\alpha)P_n(\cos\theta)$$

$$r < c, \text{ or } \theta \neq \alpha,\ r = c \qquad V_P = \frac{Q}{4\pi\epsilon c}\sum_{n=0}^{\infty}\left(\frac{r}{c}\right)^{n}P_n(\cos\alpha)P_n(\cos\theta) \qquad (1)$$

Other examples of this method will appear at the end of this and subsequent chapters.

5.17. Charged Ring in Conducting Sphere.—If the value of the potential due to a given fixed charge distribution is known in a certain region, then the values of the potential when an earthed conducting spherical shell is placed there can be found. Expand the original potential in spherical harmonics, and superimpose a second potential, similarly expanded, due to the induced charge such that the sum is zero over the sphere. The latter should vanish at infinity if the original distribution is outside the sphere and be finite at the center if it is inside.

As an example, let us find the potential at any point inside a spherical ionization chamber of radius b, if the collecting electrode is a thin concentric circular ring of radius a. Let us set $\alpha = \frac{1}{2}\pi$ and take $r > a$ in the last problem, inserting the value for $P_n(0)$ from 5.157 and writing $2n$