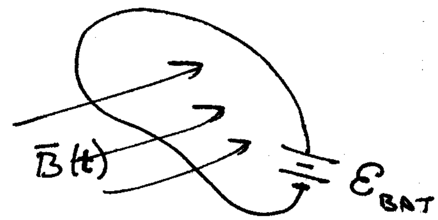
PH 206 LECTURE 9

FARADAY'S LAW (BECKER SEC. 45)

IN 1831 FARADAY MADE THE EXPERIMENTAL DISCOVERY THAT TIME-DEPENDENT MAGNETIC EFFECTS CAUSE ELECTRICAL EFFECTS. IN SOME SENSE, THIS IS THE CONVERSE OF OERSTED'S DISCOVERY THAT MAGNETIC FIELDS ARE DUE TO THE MOTION OF ELECTRICAL CHARGES. BUT FARADAY'S RESULT SHOWS NEW SUBTLETY TO THE RELATION BETWEEN ELECTRICITY AND MAGNETISM, IN THAT HIS ELECTRICAL EFFECTS ARE NOT DUE TO MOTIONS OF MAGNETIC CHARGES (WHICH DON'T APPEAR TO EXIST), BUT MAY BE ASCRIBED TO CHANGES IN THE MAGNETIC FIELD.

A SIMPLE FORM OF FARADAY'S RESULT CAN BE GIVEN IN OUR VECTOR NOTATION. CONSIDER AN ELECTRICAL CIRCUIT IN THE FORM OF A SINGLE CLOSED LOOP, WHICH MAY CONTAIN A SOURCE OF E.M.F. SUCH AS A BATTERY. SUPPOSE ALSO THAT A MAGNETIC FIELD $\bar{B}$ PASSES THRU THE LOOP.



IF THE MAGNETIC FIELD IS CHANGING WITH TIME, THE OHM'S LAW OF THE CIRCUIT MUST BE MODIFIED

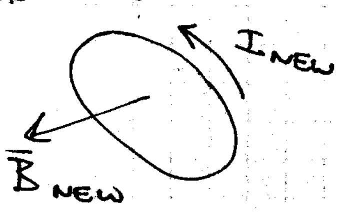
$$I R = \mathcal{E}_{BAT} - \frac{1}{c}\frac{d\Phi_M}{dt},$$

WHERE $\Phi_M = \int_{\text{SURFACE OF LOOP}} \bar{B}\cdot d\bar{S}$ = MAGNETIC FLUX.

Φ_M CAN BE THOUGHT OF AS THE NUMBER OF LINES OF THE MAGNETIC FIELD WHICH PASS THRU THE LOOP.

THUS THE CHANGING MAGNETIC FIELD ACTS LIKE A NEW SOURCE OF E.M.F. $= -\frac{1}{c}\frac{d\Phi_M}{dt}$. THE PRESENCE OF THE MINUS SIGN MAY BE REMEMBERED VIA LENZ' LAW: A CHANGING MAGNETIC FLUX INDUCES ELECTRICAL EFFECTS WHICH TEND TO OPPOSE THE ORIGINAL CHANGE.

EXAMPLE IN THE CIRCUIT SKETCHED ABOVE, SUPPOSE $\bar{B}$ IS INCREASING. THIS CAUSES AN EMF ACCORDING TO FARADAY, WHICH CAUSES ADDITIONAL CURRENT TO FLOW. THIS CURRENT CAUSES A NEW MAGNETIC FIELD WHICH MUST OPPOSE THE INCREASE IN $\bar{B}$. HENCE THE CURRENT MUST FLOW IN THE SENSE SHOWN TO PRODUCE $\bar{B}_{NEW}$. (USE AMPERE'S LAW)



[THIS RESULT ALSO FOLLOWS FROM THE EQUATION ABOVE]

Ph 206 Lecture 9

MAXWELL NOTED THAT A STATEMENT OF FARADAY'S LAW COULD BE GIVEN WHICH PLACED GREATER EMPHASIS ON FIELDS RATHER THAN CIRCUITS.

WE RECALL HOW THE E.M.F. $\mathcal{E}_{BAT}$ COULD BE RELATED TO A NON-ELECTROSTATIC FIELD $\bar{E}'$ INSIDE THE CIRCUIT (P.78),

$$\mathcal{E}_{BAT} = \oint_{\text{CIRCUIT}} \bar{E}' \cdot d\bar{l} \,.$$

LIKEWISE, WE COULD WRITE FOR THE NEW E.M.F.: $-\frac{1}{c}\frac{d\Phi_m}{dt} = \oint \bar{E}'' \cdot d\bar{l}$.

PHYSICALLY WE CANNOT IDENTIFY THIS NEW TERM $\oint \bar{E}'' \cdot d\bar{l}$ AS RESIDING IN ANY PARTICULAR PART OF THE CIRCUIT, SUCH AS THE BATTERY. MAXWELL'S IDEA IS THAT WE SHOULD IDENTIFY THIS PIECE AS DUE TO THE ORDINARY ELECTRIC FIELD! i.e., $\bar{E}'' = \bar{E}$.

WE WOULD EXPECT THE ORDINARY ELECTRIC FIELD TO HAVE ITS TANGENTIAL COMPONENT CONTINUOUS AT THE SURFACE OF THE WIRE. HENCE IF A NEW FIELD $\bar{E}$ IS INDUCED INSIDE THE WIRE WE EXPECT TO FIND $\bar{E}$ OUTSIDE THE WIRE AS WELL. THE PHYSICAL CIRCUIT DOES NOT APPEAR TO BE ESSENTIAL IN THIS ARGUMENT. FOLLOWING MAXWELL, FOR <u>ANY</u> LOOP, WE EXPECT

$$\oint \bar{E} \cdot d\bar{l} = -\frac{1}{c}\frac{d}{dt}\Phi_m = -\frac{1}{c}\frac{d}{dt}\int \bar{B} \cdot d\bar{S} = -\frac{1}{c}\int \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S} \,.$$

BY STOKES THEOREM $\oint \bar{E} \cdot d\bar{l} = \int \bar{\nabla} \times \bar{E} \cdot d\bar{S}$, SO WE CONCLUDE

$$\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t} \,.$$

THIS REPRESENTS A CLEAR DEPARTURE FROM ELECTROSTATICS FOR WHICH $\bar{\nabla} \times \bar{E} = 0$.

REMARK: THE ARGUMENT OF MAXWELL CLAIMS WE SHOULD FIND AN ELECTRIC FIELD OUTSIDE A CURRENT CARRYING WIRE OF FINITE CONDUCTIVITY, SINCE $\bar{J} = \sigma \bar{E}$ REQUIRES A FIELD $\bar{E}$ INSIDE THE WIRE. THIS HAS NOTHING TO DO WITH MAGNETISM. IS THIS REALLY TRUE?

WE ENCOURAGE YOU TO CONSIDER THIS QUESTION ON THE PROBLEM SET.

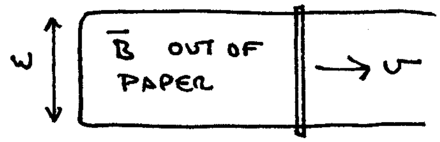
Faraday's Law for Moving Circuits

Faraday also discovered that an additional E.M.F. could be induced in a circuit even if the magnetic field is constant (in the laboratory) but some or all of the circuit is in motion.

We can now recognize this as a relativity of the laws of physics between frames in motion. (So far we would need only 'Galilean' relativity, not the special relativity of Einstein)

However, we do not seem to need a new equation such as $\bar{\nabla} \times \bar{E} = -\frac{1}{c}\frac{\partial \bar{B}}{\partial t}$ to understand this. It is sufficient to consider the Lorentz' force law $\bar{F} = q\left(\bar{E} + \frac{\bar{v}}{c} \times \bar{B}\right)$. (Of course, Lorentz came 60 years after Faraday!)

Example A U-shaped wire lies in a plane $\perp$ to a uniform magnetic field $\bar{B}$ (which is constant in time also).



A cross bar slides along the U with velocity v as shown. Then charges in the moving bar feel a force $q\frac{vB}{c}$ downwards $\downarrow$.

This is as if there is a field $E = \frac{vB}{c}$ pointing downwards inside the cross bar. This leads to an EMF

$$\oint E\, dl = -\frac{vB}{c} w \qquad (w = \text{width}).$$

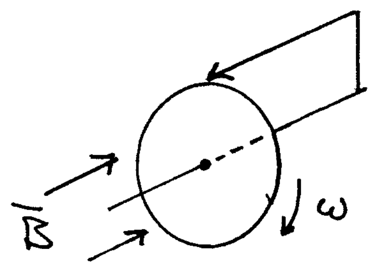
Sense of integration

But note that the rate of change of magnetic flux thru the circuit is $+wvB$ (observing the sign convention as above)

$$\therefore \quad \text{EMF} = -\frac{1}{c}(\text{rate of change of flux})$$

as stated by Faraday.

Example A conducting disc rotates about its axis with angular velocity ω. Constant magnetic field $\bar{B}$ is parallel to the axis.



A wire completes a 'circuit' by connecting the axis of the disc to a point on the rim.

Does current flow in this circuit? (Arago, 1825)

YES, IF THE WIRE WHICH COMPLETES THE CIRCUIT DOES NOT ROTATE, BUT MAKES A SLIDING CONTACT AT THE RIM.

CONSIDER THE LORENTZ FORCE $\bar{F} = q \frac{\bar{v}}{c} \times \bar{B}$.

IF THE WIRE DOES NOT ROTATE, THERE IS NO LORENTZ FORCE ON THE ELECTRONS IN THE WIRE. (OR AT LEAST NONE WITH A COMPONENT ALONG THE WIRE).

BUT ELECTRONS IN THE ROTATING DISC DO FEEL A FORCE, WHICH CAN BE ASCRIBED TO AN EFFECTIVE ELECTRIC FIELD

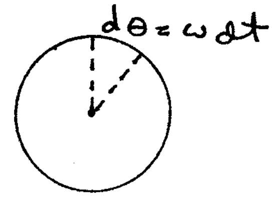
$$E_{eff} = \frac{\omega r B}{c}.$$

HENCE THE EMF INDUCED IN THE CIRCUIT IS $\int E_{eff}\, dr = \frac{\omega R^2 B}{2c}$,

WHERE R = RADIUS OF THE DISC.

WE CAN ALSO RELATE THIS TO EMF $= -\frac{1}{c}\frac{d\Phi_M}{dt} = -\frac{1}{c}\frac{d}{dt}\int \bar{B}\cdot d\bar{S}$.

WE IMAGINE THAT THE 'CIRCUIT' MOVES DUE TO THE ROTATION AS SHOWN IN TIME dt.



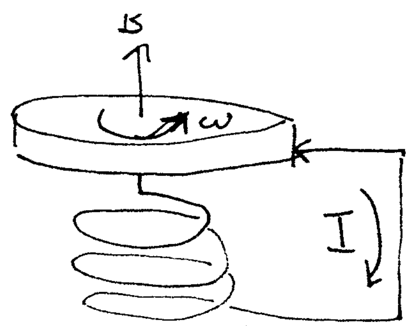
THEN $d\,\text{FLUX} = \int_0^R B\, r\, dr\, d\theta = \frac{BR^2}{2} d\theta = \frac{BR^2 \omega dt}{2}$.

SO EMF $= -\frac{1}{c}\frac{d\,\text{FLUX}}{dt} = -\frac{\omega R^2 B}{2c}$ AS BEFORE.

VARIATION: A LONG WIRE ATTACHED TO A SATELLITE!
[HITE & McCOY, AM. J. PHYS. 56, 222 ('88)]

IN SEC. 45, BECKER SHOWS HOW THE LORENTZ FORCE LAW APPLIED TO AN ARBITRARY MOVING CIRCUIT ALWAYS GIVES RESULTS CONSISTENT WITH FARADAY'S LAW. AS SUGGESTED ABOVE, THE ARGUMENT SUPPOSES THAT IF FIELDS $\bar{E}$ AND $\bar{B}$ ARE OBSERVED IN THE LABORATORY, AN OBSERVER MOVING WITH VELOCITY $\bar{v}$ THINKS THE ELECTRIC FIELD IS REALLY $\bar{E}' = \bar{E} + \frac{\bar{v}}{c}\times\bar{B}$, AS A CONSEQUENCE OF THE LORENTZ FORCE LAW.

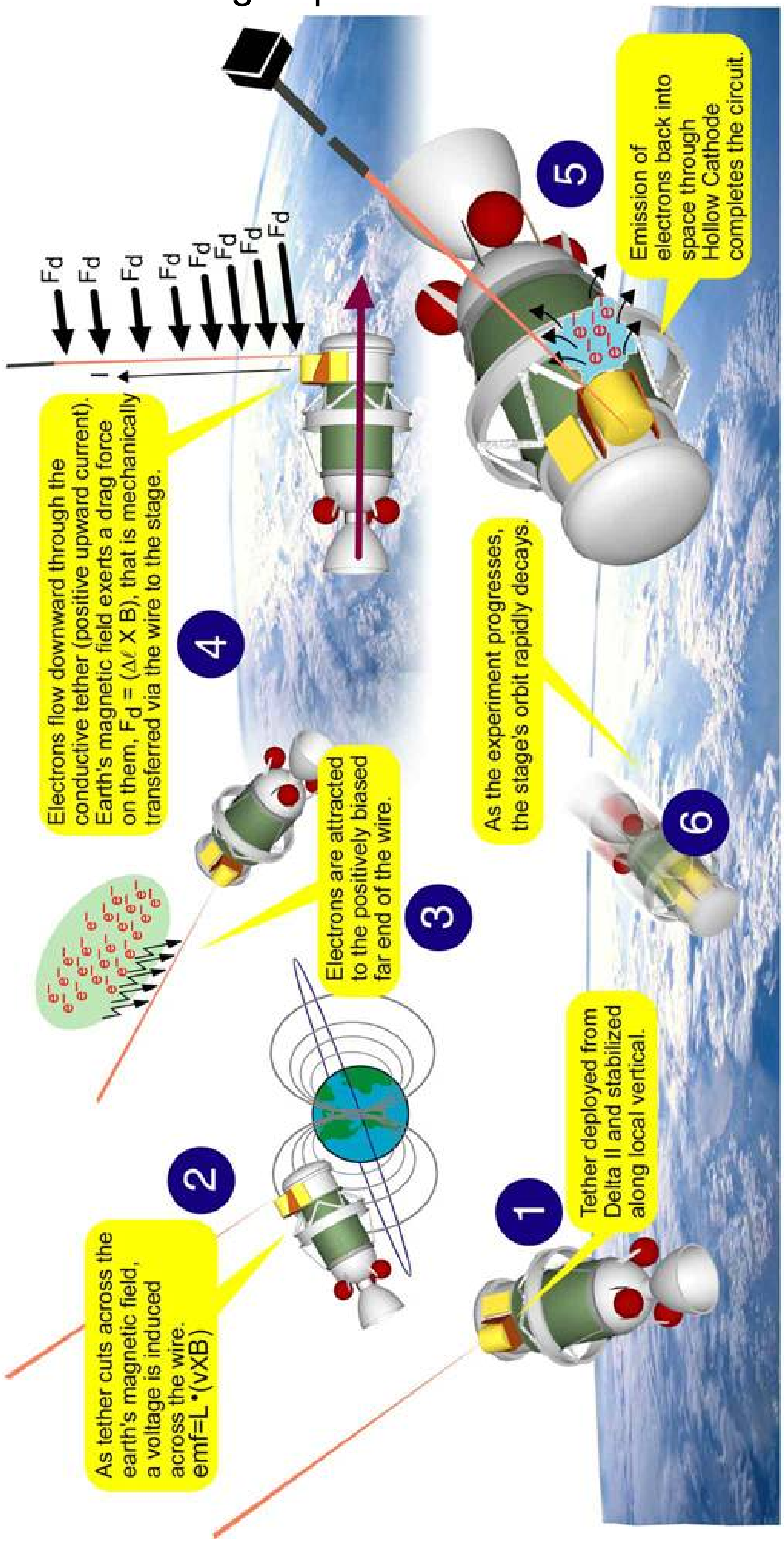
SELF-EXCITED DYNAMO



WIND THE WIRE INTO A COIL SUCH THAT IF CURRENT I FLOWS, THEN MAGNETIC FIELD $\bar{B}$ IS GENERATED WITH THE PROPER SENSE AS TO MAKE MORE CURRENT FLOW. ROTATION $\bar{\omega}$ IS PROVIDED BY EXTERNAL MEANS.

CAN SUCH A CIRCUIT EXCITE ITSELF?

Satellite tether power generation:
http://astp.msfc.nasa.gov/proseds/about.html



COULD THE EARTH'S MAGNETIC FIELD BE DUE TO SUCH A PROCESS?

A GLIMPSE OF MAGNETOHYDRODYNAMICS:

OHM: $\bar{J} = \sigma\left(\bar{E} + \frac{\bar{v}}{c}\times\bar{B}\right) \Rightarrow \bar{E} = \frac{\bar{J}}{\sigma} - \frac{\bar{v}}{c}\times\bar{B} = \frac{c}{4\pi\sigma}\bar{\nabla}\times\bar{B} - \frac{\bar{v}}{c}\times\bar{B}$

AMPÈRE: $\bar{\nabla}\times\bar{B} = \frac{4\pi}{c}\bar{J}$

FARADAY: $\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t} = \frac{c}{4\pi\sigma}\bar{\nabla}\times(\bar{\nabla}\times\bar{B}) - \bar{\nabla}\times\left(\frac{\bar{v}}{c}\times\bar{B}\right)$

$\bar{\nabla}\cdot\bar{B} = 0$, SO $\frac{\partial\bar{B}}{\partial t} = \frac{c^2}{4\pi\sigma}\nabla^2 B + \bar{\nabla}\times(\bar{v}\times\bar{B})$

IF $\bar{v} = 0$, THIS IS A DIFFUSION EQUATION: $\frac{\partial\bar{B}}{\partial t} = \frac{c^2}{4\pi\sigma}\nabla^2\bar{B}$

DIMENSIONAL ANALYSIS: DIFFUSION TIME τ OBEYS $\frac{1}{\tau} \sim \frac{c^2}{4\pi\sigma L^2}$

EARTH: $\sigma \sim \frac{\sigma_{Cu}}{100} \sim 10^{16}$, $L \sim 10^4$ KM $\sim 10^9$ CM

SO $\tau \sim \frac{4\pi\sigma L^2}{c^2} \sim \frac{10\cdot 10^{16}\cdot 10^{18}}{10^{21}} \sim 10^{14}$ S $\sim 3\times 10^6$ YEARS

$\Rightarrow$ ANY INITIAL MAGNETIC FIELD WILL HAVE DIED OUT

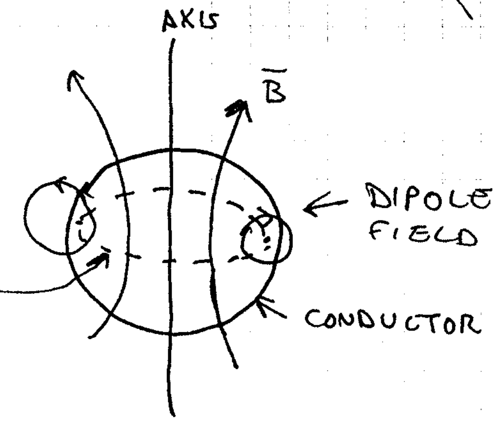
$\Rightarrow$ EARTH'S MAGNETIC FIELD HAS A DYNAMIC ORIGIN!

FOR THE TERM $\nabla\times(\bar{v}\times\bar{B})$ TO OVERCOME THE FIELD DECAY, NEED

$\nabla\times(\bar{v}\times\bar{B}) \sim \frac{v}{L}B = \omega B \gtrsim \frac{c^2}{4\pi\sigma}\nabla^2 B \sim \frac{c^2}{4\pi\sigma L^2}B \Rightarrow \omega > \frac{1}{\tau} \sim 10^{-14}$ Hz

SINCE $\omega_{EARTH} = \frac{2\pi}{1\text{ DAY}} \sim \frac{10}{10^5} \sim 10^{-4}$ Hz, MAYBE THE EARTH IS INDEED A SELF-SUSTAINING DYNAMO (SCHUSTER, 1912).

<u>COWLING'S THEOREM</u> (1934) CLAIMS IT IS IMPOSSIBLE FOR A STATIONARY MAGNETIC FIELD IN AN ORDINARY CONDUCTOR TO BE <u>AXIALLY SYMMETRIC</u>. $\Rightarrow$ ANY SELF-SUSTAINING DYNAMO MUST HAVE A COMPLICATED FIELD TOPOLOGY. SEE http://www.psc.edu/science/glatzmaier.html

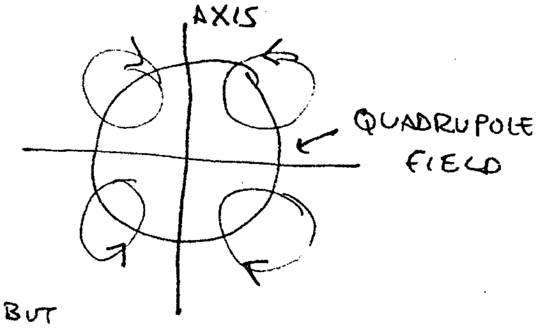


ANY AXIALLY SYMMETRIC MULTIPOLE FIELD HAS AT LEAST ONE RING AROUND WHICH B_r AND B_θ VANISH.

CONSIDER A LOOP THAT LINKS THIS RING.

$$0 \neq \oint \bar{B}\cdot d\bar{l} \qquad \int \bar{\nabla}\times\bar{B}\cdot d\bar{S} = \frac{4\pi}{c}\int J_\phi \, dS$$

$\Rightarrow$ CURRENT $J_\phi \neq 0$ NEAR THIS RING



OHM: $J_\phi = \sigma\left(E_\phi + \underbrace{\left(\frac{\bar{v}}{c}\times\bar{B}\right)_\phi}\right)$

DEPENDS ON B_r & B_θ, BUT NOT ON B_ϕ. HOWEVER $B_r = 0 = B_\theta$ NEAR THE RING

$$\Rightarrow J_\phi = \sigma E_\phi \Rightarrow E_\phi \neq 0$$

SINCE ASSUME NO EXTERNAL EMF, FARADAY $\Rightarrow 0 \neq \int \bar{E}\cdot d\bar{l} = -\frac{1}{c}\frac{d\Phi_M}{dt}$

$\Rightarrow$ MAGNETIC FIELD CANNOT BE STATIONARY!

FACT: THE EARTH'S MAGNETIC FIELD IS TIME DEPENDENT, AND OCCASIONALLY CHANGES SIGN! $\Rightarrow$ FIELD IS <u>DYNAMIC</u> AND/OR <u>NON-AXISYMMETRIC</u>. BOTH EFFECTS OCCUR IN THE EARTH...

TRANSFORMATION OF THE FIELDS

WE PURSUE THIS LINE OF THOUGHT FURTHER, AND SHOW HOW THE ASSUMPTION THAT MAXWELL'S EQUATIONS OBEY GALILEAN RELATIVITY LEADS US TO TRANSFORMATION LAWS WHICH ARE ACTUALLY THOSE OF SPECIAL RELATIVITY.

IN THIS ARGUMENT, C IS ONLY A CONSTANT VELOCITY HAVING TO DO WITH THE DEFINITION OF C.G.S UNITS FOR MAGNETOSTATIC EFFECTS.

THIS ARGUMENT DOES NOT USE THE LORENTZ FORCE LAW.

FOR A CIRCUIT AT REST IN THE LABORATORY FRAME, WE CAST FARADAY'S LAW INTO THE FORM

$$\oint \bar{E}\cdot d\bar{l} = -\frac{1}{c}\frac{d}{dt}\int \bar{B}\cdot d\bar{S}$$

SUPPOSE THE CIRCUIT IS MOVING WITH VELOCITY $\bar{v}$ IN THE LAB. THE IDEA OF GALILEAN RELATIVITY (EMBODIED IN NEWTON'S 1ST LAW) IS THAT IF AN OBSERVER MOVING WITH VELOCITY $\bar{v}$ LABELS THE ELECTRIC AND MAGNETIC FIELDS (S)HE SEES AS $\bar{E}'$ AND $\bar{B}'$.

THEN $$\oint \bar{E}'\cdot d\bar{l} = -\frac{1}{c}\frac{d}{dt}\int \bar{B}'\cdot d\bar{S}.$$ [INTEGRALS EVALUATED BY THE MOVING OBSERVER OVER A LOOP WHICH IS ALSO MOVING]

• NOTE THAT WE ASSUME BOTH OBSERVERS MEASURE DISTANCE AND TIME IN THE SAME WAY. WE ALSO ASSUME THAT THE CONSTANT C ASSOCIATED WITH MAGNETOSTATIC EFFECTS IS MEASURED THE SAME BY BOTH OBSERVERS.

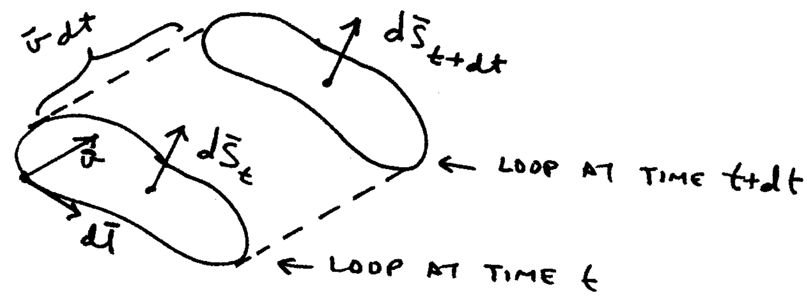
WE START WITH THE ASSUMPTION THAT $\bar{B}' = \bar{B}$, i.e., THE TWO OBSERVERS ALSO SEE THE SAME MAGNETIC FIELD.

WE FIRST SHOW HOW TO CALCULATE $\frac{d}{dt}\int_{\text{MOVING LOOP}} \bar{B}\cdot d\bar{S}$

IN THE LABORATORY FRAME. WE TRY TO REWRITE THIS AS

$$\int_{\text{FIXED LOOP}} \frac{D\bar{B}}{Dt}\cdot dS \qquad \text{WHERE } \frac{D\bar{B}}{Dt} \neq \frac{\partial \bar{B}}{\partial t} \text{ BUT SOMEHOW INCLUDES}$$

THE EFFECTS OF THE MOVING LOOP. (THIS KIND OF QUESTION OFTEN ARISES IN FLUID DYNAMICS)



NOW $$\frac{d}{dt}\int_{\text{MOVING LOOP}} \bar{B}\cdot d\bar{S} = \frac{1}{dt}\left[\int \bar{B}_{t+dt}\cdot d\bar{S}_{t+dt} - \int \bar{B}_t\cdot d\bar{S}_t\right].$$

INTEGRAL EVALUATED IN THE LAB AT INSTANT $t+dt$

OF COURSE WE CAN WRITE $\bar{B}_{t+dt} = \bar{B}_t + \frac{\partial \bar{B}_t}{\partial t}\,dt + \cdots$

SO, $$\frac{d}{dt}\int_{\text{MOVING LOOP}} \bar{B}\cdot d\bar{S} = \int \frac{\partial \bar{B}_t}{\partial t}\cdot dS_{t+dt} + \frac{1}{dt}\left[\int \bar{B}_t\cdot d\bar{S}_{t+dt} - \int \bar{B}_t\cdot d\bar{S}_t\right].$$

WE PLAY A TRICK, INTEGRATING OVER THE ENTIRE SURFACE OF THE VOLUME SWEPT OUT BY THE LOOP IN TIME dt

$$\oint_{\text{WHOLE SURFACE}} \bar{B}\cdot d\bar{S} = \int_{\text{TOP}} \bar{B}\cdot d\bar{S}_{t+dt} - \int_{\text{BOTTOM}} \bar{B}\cdot d\bar{S}_t + \int_{\text{SIDES}} \bar{B}\cdot d\bar{S}$$

↖ NOTE SIGN

$$\int (\bar{\nabla}\cdot\bar{B})\, \underbrace{dvol}_{\bar{v}\,dt\cdot d\bar{S}_t} \qquad\qquad \oint \bar{B}\cdot(d\bar{l}\times\bar{v}\,dt)$$

$$dt\int(\bar{\nabla}\cdot\bar{B})\,\bar{v}\cdot d\bar{S}_t = \int \bar{B}\cdot d\bar{S}_{t+dt} - \int \bar{B}\cdot d\bar{S}_t - dt\underbrace{\oint(\bar{B}\times\bar{v})\cdot d\bar{l}}_{\int\bar{\nabla}\times(\bar{B}\times\bar{v})\cdot d\bar{S}_t}$$

SO ON TAKING THE LIMIT $dt\to 0$,

$$\frac{d}{dt}\int_{\text{MOVING LOOP}} \bar{B}\cdot d\bar{S} = \int \frac{\partial \bar{B}_t}{\partial t}\cdot d\bar{S}_t + \int \bar{\nabla}\times(\bar{B}_t\times\bar{v})\cdot d\bar{S}_t + \int(\bar{\nabla}\cdot\bar{B}_t)\bar{v}\cdot dS_t$$

$$\equiv \int_{\text{FIXED LOOP AT TIME } t} \frac{D\bar{B}}{Dt}\cdot dS$$

WHERE $\frac{D\bar{B}}{Dt} = \frac{\partial\bar{B}}{\partial t} + \bar{\nabla}\times(\bar{B}\times\bar{v}) + (\bar{\nabla}\cdot\bar{B})\bar{v}$ ≡ "CONVECTIVE DERIVATIVE"

THIS IS TRUE FOR ANY VECTOR FIELD.

FOR US, $\bar{\nabla}\cdot\bar{B} = 0$, OF COURSE.

$\bar{v}$ IS NOT A FIELD, SO $\bar{\nabla}$ DOES NOT ACT ON $\bar{v}$
$\Rightarrow \bar{\nabla}\times(\bar{B}\times\bar{v}) = (\bar{v}\cdot\bar{\nabla})B - \bar{v}(\bar{\nabla}\cdot\bar{B})$
$\Rightarrow \frac{D\bar{B}}{Dt} = \frac{\partial\bar{B}}{\partial t} + (\bar{v}\cdot\bar{\nabla})B$, A USEFUL FORM

WE CAN NOW WRITE FARADAY'S LAW FOR THE MOVING CIRCUIT AS

$$\int_{\text{MOVING LOOP}} \bar{E}'\cdot d\bar{l} = -\frac{1}{c}\int_{\text{FIXED LOOP}} \frac{D\bar{B}}{Dt}\cdot dS = \int d\bar{S}\cdot\left[-\frac{1}{c}\frac{\partial\bar{B}}{\partial t} - \frac{1}{c}\bar{\nabla}\times(\bar{B}\times\bar{v})\right]$$

$$\int \bar{\nabla}\times\bar{E}'\cdot d\bar{S}$$ AND THIS CAN BE EVALUATED AT THE INSTANT $t=0$.

HENCE, $\bar{\nabla}\times\left(\bar{E}' - \frac{\bar{v}}{c}\times\bar{B}\right) = \frac{-1}{c}\frac{\partial\bar{B}}{\partial t}$.

BUT THIS ALL BEGAN WITH THE RELATION $\nabla\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t}$.

HENCE, $\bar{E} = \bar{E}' - \frac{\bar{v}}{c}\times\bar{B}$, OR $\underline{\bar{E}' = \bar{E} + \frac{\bar{v}}{c}\times\bar{B}}$.

So far we have merely confirmed our previous use of the Lorentz force on a moving circuit.

But now consider the other Maxwell equation discussed briefly on p.91,

$$\bar{\nabla} \times \bar{B} = \frac{4\pi}{c} \bar{J}_{TOTAL}.$$

In a region free of charges, dielectrics and magnetic materials we are still left with the displacement current

$$\frac{1}{4\pi} \frac{\partial \bar{D}}{\partial t} = \frac{1}{4\pi} \frac{\partial E}{\partial t}, \quad (\text{since } \epsilon = 1).$$

Then, $$\bar{\nabla} \times \bar{B} = \frac{1}{c} \frac{\partial \bar{E}}{\partial t} \quad \text{in vacuum}.$$

We now recognize this as being very similar in form to Faraday's law. It must imply that changing electric fields induce magnetic effects. We are certainly on the threshold of a realm of complex effects relating electricity and magnetism.

For example, we can consider the integral form of this Maxwell equation for moving loops

$$\oint \bar{B} \cdot d\bar{l} = \frac{1}{c} \frac{\partial}{\partial t} \int \bar{E} \cdot d\bar{S}$$

If we follow the preceding argument as applied to this relation, we would conclude that the magnetic field as seen by a moving observer is actually

$$\bar{B}' = \bar{B} - \frac{\bar{v}}{c} \times \bar{E}$$

← since the Maxwell eq for $\bar{\nabla} \times \bar{B}$ has a sign change compared to that for $\bar{\nabla} \times \bar{E}$

If our analysis has been truly consistent, we should expect we can express $\bar{E}$ and $\bar{B}$ in terms of $\bar{E}'$ and B' in the above form, but changing $\bar{v}$ to $-\bar{v}$

$$\bar{E} = \bar{E}' - \frac{\bar{v}}{c} \times \bar{B}'$$

$$\bar{B} = \bar{B}' + \frac{\bar{v}}{c} \times \bar{E}'$$

since relative to the moving frame, the lab frame has velocity $-\bar{v}$.

DOES IT WORK? WE START FROM $\bar{B}' = \bar{B} - \frac{\bar{v}}{c}\times\bar{E}$ AND REARRANGE...

$$\bar{B} = \bar{B}' + \frac{\bar{v}}{c}\times\bar{E} = \bar{B}' + \frac{\bar{v}}{c}\times\bar{E}' \underbrace{- \frac{\bar{v}}{c}\times\left(\frac{\bar{v}}{c}\times\bar{B}\right)}_{+\frac{v^2}{c^2}\bar{B} - \left(\frac{\bar{v}}{c}\cdot\bar{B}\right)\frac{\bar{v}}{c}}$$

ALSO $\quad \bar{E} = \bar{E}' - \frac{\bar{v}}{c}\times\bar{B}$

SO $\left(1-\frac{v^2}{c^2}\right)\bar{B} = \bar{B}' + \frac{\bar{v}}{c}\times\bar{E}' - \left(\frac{\bar{v}}{c}\cdot\bar{B}\right)\frac{\bar{v}}{c}$.

LIKEWISE WE FIND

$$\left(1-\frac{v^2}{c^2}\right)\bar{E} = \bar{E}' - \frac{\bar{v}}{c}\times\bar{B}' - \left(\frac{\bar{v}}{c}\cdot\bar{E}\right)\frac{\bar{v}}{c}.$$

WE ARE STUCK WITH CORRECTIONS OF ORDER $\frac{v^2}{c^2}$ TO OUR DESIRED RELATIONS. WHAT DOES THIS MEAN?

SUPPOSE WE ARE BOLD ENOUGH TO TRY TO FORCE THESE RELATIONS INTO A MORE 'RELATIVISTIC' PATTERN.

I.e., IF $\bar{B}' = f(\bar{E}, \bar{B}, \bar{v})$, THEN $\bar{B} = f(\bar{E}', \bar{B}', -\bar{v})$,

WHERE f IS THE SAME FUNCTION.

CLEARLY WE COULD JUST "SPLIT" THE FACTOR $1-\frac{v^2}{c^2}$

$$\bar{B}' = \frac{1}{\sqrt{1-v^2/c^2}}\left(\bar{B} - \frac{\bar{v}}{c}\times\bar{E}\right) + ?$$

$$\bar{E}' = \frac{1}{\sqrt{1-v^2/c^2}}\left(\bar{E} + \frac{\bar{v}}{c}\times\bar{B}\right) + ?$$

IT IS NOT SO OBVIOUS HOW TO 'SPLIT' THE LEFTOVER TERMS $\left(\frac{\bar{v}}{c}\cdot\bar{B}\right)\frac{\bar{v}}{c}$.

MAYBE JUST KEEP THEM IN THE FORM

$$\bar{B}' = \frac{1}{\sqrt{1-v^2/c^2}}\left(\bar{B} - \frac{\bar{v}}{c}\times\bar{E}\right) + \alpha\left(\frac{\bar{v}}{c}\cdot\bar{B}\right)\frac{\bar{v}}{c},$$

$$\bar{E}' = \frac{1}{\sqrt{1-v^2/c^2}}\left(\bar{E} + \frac{\bar{v}}{c}\times\bar{B}\right) + \beta\left(\frac{\bar{v}}{c}\cdot\bar{E}\right)\frac{\bar{v}}{c}.$$

THEN 'RELATIVITY' OF THE TRANSFORMATION REQUIRES

$$\bar{B} = \frac{1}{\sqrt{1-v^2/c^2}}\left(\bar{B}' + \frac{\bar{v}}{c}\times\bar{E}'\right) + \alpha\left(\frac{\bar{v}}{c}\cdot\bar{B}'\right)\frac{\bar{v}}{c},$$

NO SIGN CHANGE AS THIS TERM DEPENDS ON v^2

ETC.

LET'S TRY IT: REWRITE THE EXPRESSIONS FOR $\bar{B}'$ AND $\bar{E}'$

$$\bar{B} = \sqrt{}\,\bar{B}' + \frac{\bar{v}}{c}\times\bar{E} - \alpha\sqrt{}\left(\frac{\bar{v}}{c}\cdot\bar{B}\right)\frac{\bar{v}}{c}$$

$$\bar{E} = \sqrt{}\,\bar{E}' - \frac{\bar{v}}{c}\times\bar{B} - \beta\sqrt{}\left(\frac{\bar{v}}{c}\cdot\bar{E}\right)\frac{\bar{v}}{c}$$

SO

$$\bar{B} = \sqrt{}\,\bar{B}' + \sqrt{}\,\frac{\bar{v}}{c}\times\bar{E}' + \frac{v^2}{c^2}\bar{B} - \left(\frac{\bar{v}}{c}\cdot\bar{B}\right)\frac{\bar{v}}{c} - \alpha\sqrt{}\left(\frac{\bar{v}}{c}\cdot\bar{B}\right)\frac{\bar{v}}{c}$$

$$\bar{B}\left(1-\frac{v^2}{c^2}\right) = \sqrt{}\left(\bar{B}' + \frac{\bar{v}}{c}\times\bar{E}'\right) - \left(\frac{\bar{v}}{c}\cdot\bar{B}\right)\frac{\bar{v}}{c}\left(\alpha\sqrt{} + 1\right)$$

OR
$$\bar{B} = \frac{1}{\sqrt{}}\left(\bar{B}' + \frac{\bar{v}}{c}\times\bar{E}'\right) \underbrace{- \frac{\left(\frac{\bar{v}}{c}\cdot\bar{B}\right)\frac{\bar{v}}{c}\left(\alpha\sqrt{}+1\right)}{1-v^2/c^2}}_{\text{SHOULD BE } +\alpha\left(\frac{\bar{v}}{c}\cdot\bar{B}'\right)\frac{\bar{v}}{c}}$$

FROM OUR EXPRESSION FOR $\bar{B}'$, WE FIND

$$\left(\frac{\bar{v}}{c}\cdot\bar{B}'\right) = \frac{1}{\sqrt{}}\left(\frac{\bar{v}}{c}\cdot\bar{B}\right) + \alpha\frac{v^2}{c^2}\left(\frac{\bar{v}}{c}\cdot\bar{B}\right)$$

EVERYTHING WOULD BE O.K. IF

$$\alpha\left(\frac{1}{\sqrt{}} + \alpha\frac{v^2}{c^2}\right) = -\frac{\alpha}{\sqrt{}} - \frac{1}{\sqrt{}^{\,2}}$$

WHICH IS SOLVED BY $\alpha = \frac{c^2}{v^2}\left(1 - \frac{1}{\sqrt{}}\right)$

WE DEFINE $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$

THEN
$$\bar{B}' = \gamma\left(\bar{B} - \frac{\bar{v}}{c}\times\bar{E}\right) + (1-\gamma)\left(\bar{B}\cdot\hat{v}\right)\hat{v} \qquad \left(\hat{v} = \frac{\bar{v}}{v}\right)$$

$$\bar{E}' = \gamma\left(\bar{E} + \frac{\bar{v}}{c}\times\bar{B}\right) + (1-\gamma)\left(\bar{E}\cdot\hat{v}\right)\hat{v}$$

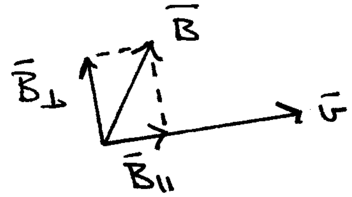
AND
$$\bar{B} = \gamma\left(\bar{B}' + \frac{\bar{v}}{c}\times\bar{E}'\right) + (1-\gamma)\left(\bar{B}'\cdot\hat{v}\right)\hat{v}$$

$$\bar{E}' = \gamma\left(\bar{E}' - \frac{\bar{v}}{c}\times\bar{B}'\right) + (1-\gamma)\left(\bar{E}'\cdot\hat{v}\right)\hat{v}$$

These results are often rewritten by decomposing $\bar{B} = \bar{B}_\parallel + \bar{B}_\perp$

where $\bar{B}_\parallel = (B \cdot \hat{v})\hat{v}$

$\bar{B}_\perp = \bar{B} - \bar{B}_\parallel$



Then

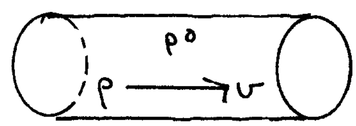
$$\bar{B}_\parallel' + B_\perp' = \gamma\left(\bar{B}_\parallel + \bar{B}_\perp - \frac{\bar{v}}{c} \times \bar{E}\right) + (1-\gamma)\bar{B}_\parallel$$

or

$$\boxed{\begin{aligned} \bar{B}_\parallel' &= \bar{B}_\parallel & \bar{E}_\parallel' &= \bar{E}_\parallel \\ \bar{B}_\perp' &= \gamma\left(\bar{B}_\perp - \frac{\bar{v}}{c} \times \bar{E}\right) & \bar{E}_\perp' &= \gamma\left(\bar{E}_\perp + \frac{\bar{v}}{c} \times \bar{B}\right) \end{aligned}}$$

This is the complete transformation as given by Einstein (1905). Work along this line had been done by Lorentz in 1904. But Einstein added the major insights of the relativity of simultaneity, time dilation, etc. Einstein himself was led to special relativity by considerations of moving circuits .....

Example Problem ③ a, Set 4.



Electron charge density ρ flows with velocity v. ρ_0 is the positive ion density at rest. How are ρ and ρ_0 related so that there is no radial force on the electrons?

We go to the rest frame of the electrons. Then $\bar{F}' = q\bar{E}'$ only and we can ignore magnetic effects.

In the lab frame, the electric field of the positive ions is

$2\pi r E_\perp = 4\pi \rho_0 \cdot \pi r^2$ or $E_\perp = 2\pi r \rho_0$ by Gauss' law.

In the electrons frame this becomes $E_\perp' = 2\pi \gamma r \rho_0$.

This must be balanced by the field due to the electrons themselves, which is $E_\perp'' = -2\pi r \rho' = -2\pi r \frac{\rho}{\gamma}$ where $\rho' = \frac{\rho}{\gamma}$ is the electron density in the moving frame — due to the Lorentz contraction.

Thus $\rho = -\gamma^2 \rho_0 = -\frac{\rho_0}{1 - v^2/c^2}$.

MAXWELL'S EQUATIONS

WE NOW HAVE RECONSTRUCTED ALL OF THE DIFFERENTIAL EQUATIONS FOR THE ELECTRIC AND MAGNETIC FIELDS, AS PROPOSED BY MAXWELL

MICROSCOPIC VIEW	MACROSCOPIC VIEW
$\bar{\nabla}\cdot\bar{E} = 4\pi\rho_{TOTAL}$	$\bar{\nabla}\cdot\bar{D} = 4\pi\rho_{FREE}$
$\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t}$	$\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial\bar{B}}{\partial t}$
$\bar{\nabla}\cdot\bar{B} = 0$	$\bar{\nabla}\cdot\bar{B} = 0$
$\bar{\nabla}\times\bar{B} = \frac{4\pi}{c}\bar{J}_{CHARGED} + \frac{1}{c}\frac{\partial\bar{E}}{\partial t}$	$\bar{\nabla}\times\bar{H} = \frac{4\pi}{c}\bar{J}_{FREE} + \frac{1}{c}\frac{\partial\bar{D}}{\partial t}$

THE MICRO- AND MACRO- VIEWS ARE RELATED BY

$$\rho_{TOTAL} = \rho_{FREE} - \bar{\nabla}\cdot\bar{P} \qquad \bar{D} = \bar{E} + 4\pi\bar{P} = \epsilon\bar{E}$$

$$\bar{J}_{CHARGED} = \bar{J}_{FREE} + \frac{\partial\bar{P}}{\partial t} + c\,\bar{\nabla}\times\bar{M} \qquad \bar{H} = \bar{B} - 4\pi\bar{M} = \frac{1}{\mu}\bar{B}$$

DESPITE THE EXTRA COMPLEXITY OF THESE EQUATIONS COMPARED TO THOSE OF ELECTRO- AND MAGNETOSTATICS, WE CAN STILL USE POTENTIAL METHODS FOR THEIR SOLUTION.

$$\bar{\nabla}\cdot\bar{B} = 0 \Rightarrow \underline{\bar{B} = \bar{\nabla}\times\bar{A}} \quad \text{STILL HOLDS}$$

THEN $\bar{\nabla}\times\bar{E} = -\frac{1}{c}\frac{\partial B}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{A}) \Rightarrow \nabla\times\left(\bar{E} + \frac{1}{c}\frac{\partial\bar{A}}{\partial t}\right) = 0$

HENCE WE WILL BE ABLE TO WRITE $\bar{E} + \frac{1}{c}\frac{\partial\bar{A}}{\partial t} = -\bar{\nabla}\phi$

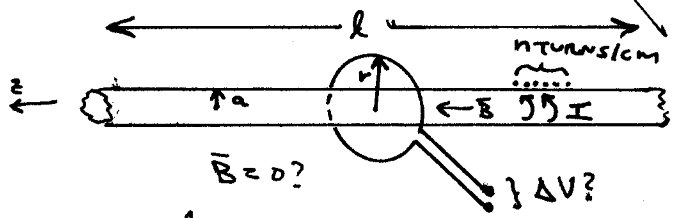
OR $$\underline{\bar{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\bar{A}}{\partial t}}$$

WE DEFER DISCUSSION OF THE SOLUTION FOR THE POTENTIALS ϕ AND $\bar{A}$ IN TIME DEPENDENT SITUATIONS UNTIL LECTURE 15.

THE PART OF $\bar{E}$ INDUCED BY CHANGING MAGNETIC FIELDS CAN BE WRITTEN $\bar{E}_{IND} = -\frac{1}{c}\frac{\partial\bar{A}}{\partial t}$. HENCE $\mathcal{E}_{MF,\,INDUCED} = \oint\bar{E}\cdot d\bar{l} = -\frac{1}{c}\frac{\partial}{\partial t}\oint\bar{A}\cdot d\bar{l}$.

NOTE THAT $\oint\bar{A}\cdot d\bar{l} = \int\bar{\nabla}\times\bar{A}\cdot d\bar{S} = \int\bar{B}\cdot d\bar{S} = \Phi_{MAG}$ = "NUMBER OF LINES" LINKED BY THE LOOP.

A PARADOXICAL TRANSFORMER



A LONG SOLENOID OF RADIUS a HAS n TURNS/cm CARRYING CURRENT I.

A SINGLE-TURN PICKUP LOOP OF RADIUS r, $a < r \ll \ell$ SURROUNDS THE SOLENOID. IF THE CURRENT I VARIES, WHAT IS THE VOLTAGE INDUCED AROUND THE PICKUP LOOP?

FARADAY: $\Delta V = \oint_{\text{LOOP}} \bar{E} \cdot d\bar{\ell} = -\frac{1}{c}\frac{d}{dt}\int \bar{B} \cdot d\bar{S}$

AS A FIRST APPROXIMATION WE CALCULATE $\bar{B}$ SUPPOSING CURRENT I IS CONSTANT.

THEN AMPERE'S LAW QUICKLY TELLS US THAT $B_z = \frac{4\pi n I}{c}$ $r < a$ & $B = 0$ $r > a$.

(SOME PEOPLE FEEL THE NEED TO CONFIRM THIS BY EXPLICIT EVALUATION OF $\bar{A} = \frac{1}{c}\int \frac{\bar{J}}{r} d\text{vol}$.)

THEN $\int \bar{B} \cdot d\bar{S} = 4\pi^2 n a^2 I / c$ AND $\Delta V = -4\pi^2 n a^2 \dot{I} / c^2$,

APPROXIMATING $\bar{B}(t)$ AS THE MAGNETOSTATIC FIELD DUE TO CURRENT $I(t)$.

WE ARE LED TO INTERPRET THIS AS DUE TO AN INDUCED ELECTRIC FIELD AT RADIUS r THAT CIRCULATES AROUND THE LOOP:

$$\Delta V = 2\pi r E_\phi \Rightarrow E_\phi = -\frac{2\pi n a^2 \dot{I}}{c^2 r}$$

THIS IS A BIT MYSTERIOUS: IF $\bar{B}(t) = 0$ OUTSIDE THE SOLENOID, IT APPEARS THAT $\bar{E}$ IS INDUCED ONLY BE THE DISTANT MAGNETIC FIELD AT $r < a$. BUT A MAIN POINT OF THE FIELD CONCEPT WAS TO AVOID ACTION AT A DISTANCE??

WE OFFER VARIOUS POSSIBLE RESOLUTIONS:

A. WE HAVE SEEN THAT $\bar{E}$ CAN BE RELATED TO THE POTENTIALS AS $\bar{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \bar{A}}{\partial t}$.

THE VECTOR POTENTIAL FOR THE SOLENOID WITH CURRENT I IS

$$A_\phi = \frac{2\pi n I}{c}\begin{cases} r & r < a \\ a^2/r & r > a \end{cases}$$

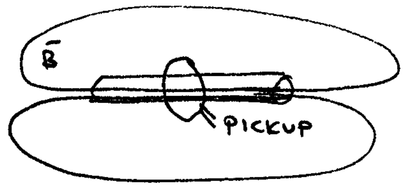
BOTH B_z AND E_ϕ AS FOUND ABOVE CAN BE DEDUCED FROM $A_\phi(t)$ ASSUMING IT TO BE VALID FOR VARYING CURRENT $I(t)$.

SO PERHAPS WE HAVE FOUND A CLASSICAL SITUATION IN WHICH THE VECTOR POTENTIAL MUST BE REGARDED AS 'REAL'.

HOWEVER, THE PARADOX CAN BE RESOLVED, I BELIEVE, BY MORE DETAILED CONSIDERATION OF THE $\bar{E}$ & $\bar{B}$ FIELDS.

B. Faraday would have said that $\bar{E}$ is induced around the loop by magnetic field lines that cut across the loop.

Now as I varies, the number of field lines inside the solenoid varies also. These lines all form loops beyond the ends of the long solenoid:



As these loops of $\bar{B}$ are created or destroyed they must cut across the pickup loop, and can create $\hat{E}$ there.

So we shouldn't have to depend on the vector potential to understand things.

This argument suggests that $\bar{B} \neq 0$ for $r > a$ when I is varying.

Indeed, the 4th Maxwell equation tells us $\bar{\nabla} \times \bar{B} = \frac{1}{c}\frac{\partial \bar{E}}{\partial t}$ for $r > a$

Assuming $\bar{B} = B_z \hat{z}$ by symmetry we have $\frac{1}{c}\frac{\partial E_\phi}{\partial t} = -\frac{\partial B_z}{\partial r}$

so $B_z(t) = \frac{2\pi n a^2 \ddot{I}}{c^3} \ln r \qquad r > a$

(This $B_z(t)$ in turn induces a little more E_ϕ .... $\Rightarrow$ use more powerful analysis in time-varying situations!)

C. The paradox is largely under control, but the special case that $I(t) = Kt$ remains somewhat mysterious.

It appears that $\{\bar{B} = 0,\ \bar{E} = -2\pi n a^2 K/cr\}$ is a formal solution to all of Maxwell's equations for $r > a$ for an infinite solenoid and assuming $I(t) = Kt$ has held for all time.

If, however, the current $I(t) = 0$ for $t < 0$ and varies as Kt for $t > 0$ then one finds nonzero $E_\phi(r)$ & $B_z(r)$ for $t > r/c$ and these converge on the paradoxical forms only as $t \to \infty$.

[See J.D. Templin, Am. J. Phys. 64, 1330 (1996)]

This interesting problem seems to have been first discussed by O. Lodge, Phil. Mag. 27, 469 (1889).