Princeton University

Ph501
Electrodynamics
Problem Set 1

Kirk T. McDonald

(1998)

kirkmcd@princeton.edu

http://kirkmcd.princeton.edu/examples/

References:

R. Becker, *Electromagnetic Fields and Interactions* (Dover, 1982).

D.J. Griffiths, *Introductions to Electrodynamics*, 3rd ed. (Prentice Hall, 1999), http://kirkmcd.princeton.edu/examples/EM/griffiths_em3.pdf

J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

The classic is, of course:

J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 1 (Clarendon Press, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_73.pdf
3rd ed. (Clarendon Press, 1892, 1904), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_04.pdf
Vol. 2 (Clarendon Press, 1873), kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf
Vol. 2, 3rd ed. (Clarendon Press, 1892), kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_92.pdf

For greater detail:

L.D. Landau and E.M. Lifshitz, *Classical Theory of Fields*, 4th ed. (Butterworth-Heinemann, 1975), http://kirkmcd.princeton.edu/examples/EM/landau_ctf_71.pdf
Electrodynamics of Continuous Media, 2nd ed. (Butterworth-Heinemann, 1984), http://kirkmcd.princeton.edu/examples/EM/landau_ecm2.pdf

N.N. Lebedev, I.P. Skalskaya and Y.S. Ulfand, *Problems of Mathematical Physics* (Prentice-Hall, 1965), http://kirkmcd.princeton.edu/examples/EM/lebedev_65.pdf

W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill), 1968), http://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf

J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, 1941), http://kirkmcd.princeton.edu/examples/EM/stratton_electromagnetic_theory.pdf

Excellent introductions:

R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. 2 (Addison-Wesely, 1964), https://www.feynmanlectures.caltech.edu/II_toc.html

E.M. Purcell and D.J. Morin, *Electricity and Magnetism*, 3rd ed. (Cambrighe U. Press, 2013), http://kirkmcd.princeton.edu/examples/EM/purcell_em_13.pdf

History:

O. Darrigol, *Electrodynamics from Ampère to Einstein* (Oxford U. Press, 2000), http://kirkmcd.princeton.edu/examples/EM/darrigol_em_00.pdf

B.J. Hunt, *The Maxwellians* (Cornell U. Press, 1991).

E. Whittaker, *A History of the Theories of Aether and Electricity* (Longmans, Green, 1910), kirkmcd.princeton.edu/examples/EM/whittaker_history.pdf

Online E&M Courses:

http://www.ece.rutgers.edu/~orfanidi/ewa/

http://farside.ph.utexas.edu/teaching/jk1/jk1.html

1. (a) Show that the mean value of the potential over a spherical surface is equal to the potential at the center, provided that no charge is contained within the sphere.

   (A related result is that the mean value of the electric field over the volume of a charge-free sphere is equal to the value of the field at its center.)

   (b) Demonstrate Earnshaw's theorem: A charge cannot be held at equilibrium solely by an electrostatic field.[1]

   (c) Demonstrate that an electrostatic field $\mathbf{E}$ cannot have a local maximum of E^2, using the mean value theorem mentioned in part (a) – or any other technique.

   Remark: An interesting example of nonelectrostatic equilibrium is laser trapping of atoms. Briefly, an atom of polarizability α takes on an induced dipole moment $\mathbf{p} = \alpha\mathbf{E}$ in an electric field. The force on this dipole is then (Notes, p. 26), $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{p}\cdot\mathbf{E}) = \alpha\boldsymbol{\nabla}E^2$. Since an electrostatic field cannot have a local maximum of E^2, it cannot trap a polarizable atom. But consider an oscillatory field, in particular a focused light wave. The time-average force, $\langle\mathbf{F}\rangle = \alpha\boldsymbol{\nabla}\langle E^2\rangle$ draws the atom into the laser focus where the electric field is a maximum. See, http://kirkmcd.princeton.edu/examples/tweezers.pdf

[1] http://kirkmcd.princeton.edu/examples/EM/earnshaw_tcps_7_97_39.pdf

2. Calculate the potential $\phi(z)$ along the axis of a disk of radius R in two cases:

   (a) The disk is a uniform layer of charge density σ, and

   (b) The disk is a uniform dipole layer of dipole moment density $\mathbf{p} = p\hat{\mathbf{z}}$ per unit area.

3. Suppose the electric field of point charge q were $\mathbf{E} = q\hat{\mathbf{r}}/r^{2+\delta}$ where $\delta \ll 1$, rather then $\mathbf{E} = q\hat{\mathbf{r}}/r^2$.

   (a) Calculate $\boldsymbol{\nabla} \cdot \mathbf{E}$ and $\boldsymbol{\nabla} \times \mathbf{E}$ for $r \neq 0$. Find the electric potential for such a point charge.

   (b) Two concentric spherical conducting shells of radii a and b are joined by a thin conducting wire. Show that if charge Q_a resides on the outer shell, then the charge on the inner shell is

$$Q_b \simeq -\frac{Q_a \delta}{2(a-b)} \left[2b \ln 2a - (a+b)\ln(a+b) + (a-b)\ln(a-b)\right] \qquad (1)$$

4. (a) Starting from the dipole potential $\phi = \mathbf{p} \cdot \mathbf{r}/r^3$ explicitly show that

$$\mathbf{E} = \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} - \frac{4\pi \mathbf{p}}{3} \delta^3(\mathbf{r}). \tag{2}$$

Hint: to show the need for the $\delta^3(\mathbf{r})$ term, consider the volume integral of $\mathbf{E}$ over a small sphere about the dipole. You may need a variation of Gauss' theorem:

$$\int_V \boldsymbol{\nabla}\phi \; d\text{Vol} = \oint_S \phi \hat{\mathbf{n}} \; dS, \tag{3}$$

where $\hat{\mathbf{n}}$ is the outward normal to the surface.

(b) The geometric definition of the "lines of force" is that this family of curves obeys the differential equation:

$$\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}. \tag{4}$$

For a dipole $\mathbf{p} = p\hat{\mathbf{z}}$, find the equation of the lines of force in the x-z plane. It is easiest to work in spherical coordinates. Compare with the figure on the cover of the book by Becker.

5. Find the two lowest-order nonvanishing terms in the multipole expansion of the potential due to a uniformly charged ring of radius a carrying total charge Q. Take the origin at the center of the ring, and neglect the thickness of the ring.

6. (a) A long, very thin rod of dielectric constant ϵ is oriented parallel to a uniform electric field $\mathbf{E}_{\text{ext}}$. What are $\mathbf{E}$ and $\mathbf{D}$ inside the rod?

   (b) What are $\mathbf{E}$ and $\mathbf{D}$ inside a very thin disc of dielectric constant ϵ if the disc is perpendicular to $\mathbf{E}_{ext}$?

   (c) Find $\mathbf{E}$ and $\mathbf{D}$ everywhere due to a sphere of fixed uniform polarization density $\mathbf{P}$. Then calculate $\int \mathbf{E} \cdot \mathbf{D}\, d\text{Vol}$ for the two volumes inside and outside the sphere's surface.

   Hint: this problem is equivalent to two oppositely charged spheres slightly displaced.

   (d) Show that for any finite electret, a material with fixed polarization $\mathbf{P}$,

$$\int_{\text{all space}} \mathbf{E} \cdot \mathbf{D}\, d\text{Vol} = 0. \tag{5}$$

7. A spherical capacitor consists of two concentric conducting shells of radii a and b. The gap is half filled with a (non-conducting) dielectric liquid of constant ϵ. You may assume the fields are radial. The inner shell carries charge $+Q$, the outer shell $-Q$.

   Calculate $\mathbf{E}$ and $\mathbf{D}$ in the gap, and the charge distribution in the inner shell. Also calculate the capacitance, defined as $C = Q/V$, where V is the potential difference between the inner and outer shells.

8. (a) As a classical model for atomic polarization, consider an atom consisting of a fixed nucleus of charge $+e$ with an electron of charge $-e$ in a circular orbit of radius a about the nucleus. An electric field is applied at right angles to the plane of the orbit. Show that the polarizability α is approximately a^3. (This happens to be the result of Becker's (26-6), but the model is quite different!)

   Assuming that radius a is the Bohr radius, $\sim 5.3 \times 10^{-9}$ cm, use the model to estimate the dielectric constant ϵ of hydrogen gas at S.T.P. Empirically, $\epsilon \sim 1 + 2.5 \times 10^{-4}$.

   (b) Another popular classical model of an atom is that the electron is bound to a neutral nucleus by a spring whose natural frequency of vibration is that of some characteristic spectral line. For hydrogen, a plausible choice is the Lyman line at 1225 Angstroms. In this model, show that $\alpha = e^2/m\omega^2$, and estimate ϵ. Recall that $e = 4.8 \times 10^{-10}$ esu, and $m = 9.1 \times 10^{-28}$ g.

Solutions

1. a) We offer two solutions: the first begins by showing the result holds for small spheres, and then shows the result is independent of the size of the (charge-free) sphere; the second applies immediately for spheres of any size, but is more abstract.

We consider a charge-free sphere of radius R centered on the origin.

In a charge-free region, the potential $\phi(\mathbf{r})$ satisfies Laplace's equation:

$$\nabla^2\phi = 0. \tag{6}$$

First, we simply expand the potential in a Taylor series about the origin:

$$\phi(\mathbf{r}) = \phi(0) + \sum_i \frac{\partial\phi(0)}{\partial x_i}x_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2\phi(0)}{\partial x_i \partial x_j}x_i x_j + ... \tag{7}$$

We integrate (7) over the surface of the sphere:

$$\oint_S \phi(\mathbf{r})dS = 4\pi R^2\phi(0) + \sum_i \frac{\partial\phi(0)}{\partial x_i}\oint_S x_i dS + \frac{1}{2}\sum_{i,j}\frac{\partial^2\phi(0)}{\partial x_i \partial x_j}\oint_S x_i x_j dS + ... \tag{8}$$

For a very small sphere, we can ignore all terms except the first, In this case, eq. (8) becomes

$$\frac{1}{4\pi R^2}\oint_S \phi(\mathbf{r})dS = \phi(0), \qquad [R \text{ "small"}], \tag{9}$$

which was to be shown.

Does the result still hold at larger radii? One might expect that since "small" is not well defined, (9) holds for arbitrary R, so long as the sphere is charge free.

Progress can be made staying with the Taylor expansion. By spherical symmetry, the integral of the product of an odd number of x_i vanishes. Hence, only the terms with even derivatives of the potential survive. And of these, only some terms survive. In particular, for the 2nd derivative, only the integrals of x_1^2, x_2^2, and x_3^2 survive, and these 3 are all equal. Thus, the 2nd derivative term consists of

$$\frac{1}{2}\left(\frac{\partial^2\phi(0)}{\partial x_1^2} + \frac{\partial^2\phi(0)}{\partial x_2^2} + \frac{\partial^2\phi(0)}{\partial x_3^2}\right)\oint_S x_1^2 dS, \tag{10}$$

which vanishes, since $\nabla^2\phi(0) = 0$.

It is less evident that the terms with 4rth and higher even derivatives vanish, although this can be shown via a systematic multipole expansion in spherical coordinates, which emphasizes the spherical harmonics Y_l^m.

But by a different approach, we can show that the mean value of the potential over a charge-free sphere is independent of the radius of the sphere. That is, consider

$$M(r) = \frac{1}{4\pi r^2}\oint_S \phi dS = \frac{1}{4\pi}\int d\cos\theta \int d\varphi\phi(r,\theta,\varphi), \tag{11}$$

in spherical coordinates (r, θ, φ). Then

$$\begin{aligned}\frac{dM(r)}{dr} &= \frac{1}{4\pi}\int d\cos\theta \int d\varphi \frac{\partial\phi}{\partial r} = \frac{1}{4\pi}\int d\cos\theta \int d\varphi \boldsymbol{\nabla}\phi \cdot \hat{\mathbf{r}} \\ &= \frac{1}{4\pi r^2}\oint_S \boldsymbol{\nabla}\phi \cdot d\mathbf{S} = \frac{1}{4\pi r^2}\int_V \nabla^2\phi d\text{Vol} = 0, \end{aligned} \tag{12}$$

for a charge-free volume. Hence, the mean value of the potential over a charge-free sphere of finite radius is the same as that over a tiny sphere about the center of the larger sphere. But, as shown in the argument leading up to (9), this is just the value of the potential at the center of the sphere.

A second solution is based on one of Green's theorems (sec. 1.8 Of Jackson). Namely, for two reasonable functions $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$,

$$\int_V \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) d\text{Vol} = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) dS, \tag{13}$$

where n is coordinate normally outward from the closed surface S surrounding a volume V.

With ϕ as the potential satisfying Laplace's equation (6), the second term on the l.h.s. of (13) vanishes. We seek an auxiliary function ψ such that $\nabla^2\psi = \delta^3(0)$, so the l.h.s. is just $\phi(0)$. Further, it will be helpful if ψ vanishes on the surface of the sphere of radius R, so the second term on the r.h.s. vanishes also.

These conditions are arranged with the choice

$$\psi = \frac{1}{4\pi}\left(\frac{1}{R} - \frac{1}{r}\right), \tag{14}$$

recalling pp. 8-9 of the Notes. On the surface of the sphere, coordinate n is just the radial coordinate r, so

$$\frac{\partial\psi}{\partial n} = \frac{1}{4\pi r^2}. \tag{15}$$

Thus, we can evaluate the expression (13), and get:

$$\psi(0) = \frac{1}{4\pi R^2}\oint_S \phi dS, \tag{16}$$

which means that the value of the potential ϕ at the center of a charge-free sphere of any size is the average of the potential on the surface of the sphere.

b) The potential energy of a charge q at point $\mathbf{r}$, due to interaction with an electrostatic field derivable from a potential $\phi(\mathbf{r})$, is:

$$U = q\phi(\mathbf{r}). \tag{17}$$

For the point $\mathbf{r}_0$ to be the equilibrium point for a particle, the potential ϕ should have a minimum there. But we can infer from part (a) that:

Harmonic functions do not have minima,

harmonic functions being a name for solutions of Laplace's equation (6).

Indeed, a minimum of U at point $\mathbf{r}_0$ would imply that there is a small sphere centered on $\mathbf{r}_0$ such that $\phi(\mathbf{r}_0)$ is less than ϕ at any point on that sphere. This would contradict what we have shown in part (a): $\phi(\mathbf{r}_0)$ is the average of ϕ over the sphere.

We continue with an example of Earnshaw's theorem. Consider 8 unit charges located at the corners of a cube of edge length 2, *i.e.*, the charges are at the locations (x_i, y_i, z_i) = (1,1,1), (1,1,-1), (1,-1,1), (1,-1, -1), (-1,1,1), (-1,1,-1), (-1,-1,1), (-1,-1,1). It is suggestive, but not true, that the electric field near the origin points inwards and could trap a positive charge.

The symmetry of the problem is such that a series expansion of the electric potential near the origin will have terms with only even powers, and we must go to 4rth order to see that the potential does not have a maximum at the origin. To simplify the series expansion, we consider the electric field, for which we need expand only to third order.

The electric potential is given by

$$\phi = \sum_{i=1}^{8} \frac{1}{\sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2}}. \tag{18}$$

The x component of the electric field is

$$\begin{aligned}
E_x &= -\frac{\partial \phi}{\partial x} = -\sum_{i=1}^{8} \frac{x_i - x}{[(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2]^{3/2}} \\
&= -\frac{1}{3^{3/2}} \sum_{i=1}^{8} \frac{x_i - x}{[1 + (-2x_i x - 2y_i y - 2z_i z + x^2 + y^2 + z^2)/3]^{3/2}} \\
&\approx -\frac{1}{3^{3/2}} \sum_{i=1}^{8} x_i \left[1 + y_i y + z_i z + \frac{-2x^2 + y^2 + z^2 + 5y_i y z_i z}{3}\right] \\
&\quad -\frac{1}{3^{3/2}} \sum_{i=1}^{8} x_i \left[\frac{11 y_i y^3 + 11 z_i z^3}{54} - \frac{x^2 y_i y + x^2 z_i z}{6} + \frac{3y^2 z_i z + 3z^2 y_i y}{2}\right] \\
&\quad -\frac{1}{3^{3/2}} \sum_{i=1}^{8} \left[\frac{2x y_i y + 2x z_i z}{3} - \frac{7x^3}{54} + \frac{7xy^2 + 7xz^2}{6} + \frac{20 x y_i y z_i z}{9}\right] \\
&= \frac{28}{81\sqrt{3}}(x^3 - 9xy^2 - 9xz^2),
\end{aligned} \tag{19}$$

noting that $x_i^2 = y_i^2 = z_i^2 = 1$ and that $\sum x_i = 0 = \sum x_i y_i$, *etc.* Similarly,

$$E_y \approx \frac{28}{81\sqrt{3}}(y^3 - 9yx^2 - 9yz^2) \qquad \text{and} \qquad E_z \approx \frac{28}{81\sqrt{3}}(z^3 - 9zx^2 - 9zy^2). \tag{20}$$

The radial component of the electric field is therefore

$$E_r = \frac{\mathbf{E} \cdot \mathbf{r}}{r} = \frac{xE_x + yE_y + zE_z}{r} \approx \frac{28}{81\sqrt{3}r}[x^4 + y^4 + z^4 - 18(x^2y^2 + y^2z^2 + z^2x^2)]. \tag{21}$$

Along the x axis, $r = x$ and the radial field varies like

$$E_r \approx \frac{28}{81\sqrt{3}} r^3 > 0, \tag{22}$$

but along the diagonal $x = y = z = r/\sqrt{3}$ it varies like

$$E_r \approx -\frac{476}{243\sqrt{3}} r^3 < 0. \tag{23}$$

It is perhaps not intuitive that the electric field is positive along the positive x axis, although Earnshaw assures us that the radial electric field must be positive in some direction. A clue is to consider the point (1,0,0) on the face of the cube whose corners hold the charges. At this point the electric fields due to the 4 charges with $x = 1$ sum to zero, so the field here is due only to the 4 charges with $x = -1$, and now "obviously" the x component of the electric field is positive. The charges at the corners of the cube force a positive charge toward the origin along the diagonals, but cannot prevent that charge from escaping near the centers of the faces of the cube.

c) If E^2 has a local maximum at some point P in a charge-free region, then there is a nonzero r such that $E^2 < E^2(P)$ for all points (other than P) within a sphere of radius r about P. Consequently, $E < E(P)$ in that sphere.

Let $\hat{\mathbf{z}}$ point along $\mathbf{E}(P)$. Then the mean-value theorem can be written

$$\int E_z d\text{Vol} = \frac{4\pi r^3}{3} E(P), \tag{24}$$

for the sphere about P. In general, $E_z \le E$, and by assumption $E < E(P)$ for all points other than P within the sphere, so

$$\int E_z d\text{Vol} \le \int E d\text{Vol} < \int E(P) d\text{Vol} = \frac{4\pi r^3}{3} E(P), \tag{25}$$

which contradicts eq. (24). Hence, E^2 cannot be locally maximal at P.

However, E^2 can take on a local minimum....

2. a) The potential $\phi(z)$ along the axis of a disk of radius R of charge density σ (per unit area) is given by the integral:

$$\phi(z) = \int_{r=0}^{R} 2\pi r dr \frac{\sigma}{\sqrt{r^2+z^2}} = 2\pi\sigma\left(\sqrt{R^2+z^2} - |z|\right). \tag{26}$$

Notice that $\phi(z)$ behaves as $\pi\sigma R^2/|z|$ for $|z| \gg R$, which is consistent with the observation that in this limit the disk may be considered as a point charge $q = \pi\sigma R^2$. Notice also, that at $z = 0$ the potential is continuous, but it's first derivative $(-\mathbf{E})$ jumps from $-2\pi\sigma$ at $z = 0_+$ to $2\pi\sigma$ at $z = 0_-$. This reflects the fact that at small $|z|$ ($|z| \ll R$) the potential may be calculated, in first approximation, as the potential for the infinite plane with charge density σ.

b) A disk of dipole-moment density $\mathbf{p} = p\hat{\mathbf{z}}$ can be thought of as composed of a layer of charge density $+\sigma$ separated in z from a layer of charge density $-\sigma$ by a distance $d = p/\sigma$. Say, the + layer is at $z = d/2$, and the − layer is as $z = -d/2$. Then, the potential ϕ_b at distance z along the axis could be written in terms of $\phi_a(z)$ found in (26) as

$$\phi_b(z) = \phi_a(z - d/2) - \phi_a(z + d/2) \to -d\frac{\partial \phi_a(z)}{\partial z} = 2\pi p\left(\text{sign}(z) - \frac{z}{\sqrt{R^2+z^2}}\right). \tag{27}$$

In this, we have taken the limit as $d \to 0$ while $\sigma \to \infty$, but $p = \sigma d$ is held constant.

At large z we get the potential of a dipole $P = \pi p R^2$ on its axis. But near the plate ($|z| \ll R$), the potential has a discontinuity:

$$\phi_b(0_+) - \phi_b(0_-) = 4\pi p. \tag{28}$$

We may explain this in our model of the dipole layer as a system of two close plates with charge density σ and distance d between them, where $p = \sigma d$. The field between the plates is $E_z = -4\pi\sigma$, and the potential difference potential between two plates is

$$\Delta\phi = -E_z d = 4\pi p, \tag{29}$$

as found in (28).

3. a) For a charge q at the origin, the proposed electric field is

$$\mathbf{E} = q\frac{\hat{\mathbf{r}}}{r^{2+\delta}} = q\frac{\mathbf{r}}{r^{3+\delta}}. \tag{30}$$

This still has spherical symmetry, so we can easily evaluate the divergence and curl in spherical coordinates:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = q\frac{\boldsymbol{\nabla}\cdot\mathbf{r}}{r^{3+\delta}} + q\mathbf{r}\cdot\boldsymbol{\nabla}\frac{1}{r^{3+\delta}} = \frac{3q}{r^{3+\delta}} - q\frac{3+\delta}{r^{3+\delta}} = -\frac{q\delta}{r^{3+\delta}}; \tag{31}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = q\frac{\boldsymbol{\nabla}\times\mathbf{r}}{r^{3+\delta}} + \boldsymbol{\nabla}\frac{1}{r^{3+\delta}}\times q\mathbf{r} = 0 - (3+\delta)\frac{\mathbf{r}}{r^{5+\delta}}\times q\mathbf{r} = 0. \tag{32}$$

Since $\boldsymbol{\nabla}\times\mathbf{E} = 0$, the field can be derived from a potential. Indeed,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi, \qquad \text{where} \qquad \phi = -\int_{\infty}^{r} E_r dr = -q\int_{\infty}^{r}\frac{dr}{r^{2+\delta}} = \frac{1}{\delta+1}\frac{q}{r^{\delta+1}}. \tag{33}$$

b) Let us first compute the potential due to a the spherical shell of radius a that carries charge Q, as observed at a distance r from the center of the sphere. We use (33) and integrate in spherical coordinates (r,θ,ϕ) to find

$$\phi(r) = \frac{Q}{4\pi a^2}\frac{1}{1+\delta}\int_{-1}^{1}\frac{2\pi a r^2\, d\cos\theta}{(a^2+r^2-2ar\cos\theta)^{\frac{1+\delta}{2}}} = \frac{Q}{2ar}\frac{(a+r)^{1-\delta} - |a-r|^{1-\delta}}{1-\delta^2}. \tag{34}$$

Now consider the addition of a sphere of radius $b < a$ that carries charge Q_b. The total potential at $r = a$ is

$$\phi_a = \frac{Q_a}{2a^2}\frac{(2a)^{1-\delta}}{1-\delta^2} + \frac{Q_b}{2ab}\frac{(a+b)^{1-\delta} - (a-b)^{1-\delta}}{1-\delta^2}, \tag{35}$$

while that at $r = b$ is

$$\phi_b = \frac{Q_a}{2ab}\frac{(a+b)^{1-\delta} - (a-b)^{1-\delta}}{1-\delta^2} + \frac{Q_b}{2b^2}\frac{(2b)^{1-\delta}}{1-\delta^2}. \tag{36}$$

We require Q_b such that $\phi_a = \phi_b$, (as guaranteed by the wire connecting the two spheres). However, we neglect terms of $\mathcal{O}(\delta^2)$, assuming δ to be small. Then, (35-36) lead to the relation

$$Q_b = -Q_a\frac{(b/a)(2a)^{1-\delta} - (a+b)^{1-\delta} + (a-b)^{1-\delta}}{-(a/b)(2b)^{1-\delta} + (a+b)^{1-\delta} - (a-b)^{1-\delta}} \tag{37}$$

What about the factors of form x^δ, which are approximately 1 for small δ? Let $x^\delta \approx 1 + \epsilon$. Taking logarithms, $\delta\ln x \approx \ln(1+\epsilon) \approx \epsilon$. Thus,

$$x^\delta \approx 1 + \delta\ln x, \qquad \text{so} \qquad x^{1-\delta} \approx \frac{x}{1+\delta\ln x} \approx x(1-\delta\ln x). \tag{38}$$

Using (38) in (37), we find

$$Q_b = -\frac{Q_a\delta}{2(a-b)}\left[2b\ln 2a - (a+b)\ln(a+b) - (a-b)\ln(a-b)\right]. \tag{39}$$

Measurement of the ratio Q_b/Q_a provides a stringent test of the accuracy of the $1/r^2$ law for electrostatics.

4. a) The first terms in $\mathbf{E}$ can be obtained by explicit differentiation, perhaps best done using vector components. For $r > 0$,

$$E_i = -\frac{\partial}{\partial x_i}\frac{p_j x_j}{r^3} = \frac{3(p_j x_j)x_i}{r^5} - \frac{p_i}{r^3}. \tag{40}$$

To justify the δ-term, consider the integral over a small sphere surrounding the (point) dipole:

$$\int_V \mathbf{E} d\text{Vol} = -\int_V \boldsymbol{\nabla}\phi d\text{Vol} = \oint_S \phi\hat{\mathbf{n}} dS = \oint_S \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\hat{\mathbf{r}} dS, \tag{41}$$

according to the form of Gauss' law (3) given in the hint. Evaluating the last integral in a spherical coordinate system with z axis along $\mathbf{p}$, we find that only the $\hat{\mathbf{p}}$ component is nonzero:

$$\int \mathbf{E} d\text{Vol} = \mathbf{p}\int_{-1}^{1} 2\pi d\cos\theta\, \cos^2\theta = \frac{4\pi\mathbf{p}}{3}. \tag{42}$$

No matter how small the sphere, the integral (42) remains the same.

On the other hand, if we insert the field $\mathbf{E}$ from (40) in the volume integral, only the z component (along $\mathbf{p}$) does not immediately vanish, but then

$$\int_V E_z d\text{Vol} = \int_0^r 2\pi r dr \int_{-1}^{1} d\cos\theta\, \frac{p(3\cos^2\theta - 1)}{r^3} = 0, \tag{43}$$

if we adopt the convention that the angular integral is performed first.

To reconcile the results (42) and (43), we write that the dipole field has a spike near the origin symbolized by $-(4\pi\mathbf{p}/3)\delta^3(\mathbf{r})$.

Another qualitative reason for the δ-term is as follows. Notice, that without this term we would conclude that the electric field on the axis of the dipole (z-axis) would always along $+z$. This would imply that if we moved some distribution of charge from large negative z to large positive z, then that charge would gain energy from the dipole field. But this cannot be true: after all, the dipole may be thought as a system of two charges, separated by a small distance, and for such a configuration the potential at large distances is certainly extremely small.

We can also say that the δ-term, which points in the $-\mathbf{p}$ direction, represents the large field in the small region between two charges that make up the dipole.

b) In spherical coordinates with $\hat{\mathbf{z}}$ along $\mathbf{p}$, the dipole potential in the x-z plane is

$$\phi(r,\theta) = \frac{p\cos\theta}{r^2} \tag{44}$$

Then,

$$E_r = -\frac{\partial\phi}{\partial r} = \frac{2p\cos\theta}{r^3}, \qquad \text{and} \qquad E_\theta = -\frac{\partial\phi}{r\partial\theta} = \frac{p\sin\theta}{r^3}. \tag{45}$$

Thus, the differential equation of the field lines

$$\frac{dr}{E_r} = \frac{r d\theta}{E_\theta}, \qquad \text{implies} \qquad \frac{dr}{r} = 2\frac{d\sin\theta}{\sin\theta}, \tag{46}$$

which integrates to

$$r = C\sin^2\theta = C\frac{x^2}{r^2}, \tag{47}$$

etc.

5. The multipole expansion of the potential ϕ about the origin due to a localized charge distribution $\rho(\mathbf{r})$ is

$$\phi(\mathbf{r}) = \frac{Q}{r} + \frac{\mathbf{P}\cdot\hat{\mathbf{r}}}{r^2} + \frac{1}{2}\frac{Q_{ij}\hat{r}_i\hat{r}_j}{r^3} + \ldots \tag{48}$$

where

$$Q = \int \rho(\mathbf{r})d\mathrm{Vol}, \qquad P_i = \int \rho r_i d\mathrm{Vol}, \qquad Q_{ij} = \int \rho(3r_ir_j - \delta_{ij}r^2)d\mathrm{Vol}, \qquad \ldots \tag{49}$$

For the ring, Q is just the total charge, while the dipole moment $\mathbf{P}$ is zero because of the symmetry.

Let us find the quadrupole moment, Q_{ij}. Take the z axis to be along that of the ring. Then, the quadrupole tensor is diagonal, and due to the rotational invariance,

$$Q_{xx} = Q_{yy} = -\frac{1}{2}Q_{zz}. \tag{50}$$

We compute Q_{xx} in spherical coordinates (r,θ,φ):

$$Q_{xx} = \int \rho(3x^2 - r^2)d\mathrm{Vol} = \int_0^{2\pi} \frac{Q}{2\pi a}a^2(3\cos^2\varphi - 1)ad\varphi = \frac{Qa^2}{2}. \tag{51}$$

Thus, the third term in the expansion (48) is:

$$\frac{1}{2r^3}\left(Q_{xx}\sin^2\theta\cos^2\varphi + Q_{yy}\sin^2\theta\sin^2\varphi + Q_{zz}\cos^2\theta\right) = -\frac{Qa^2}{4r^3}(3\cos^2\theta - 1). \tag{52}$$

6. a) Remember that $\mathbf{E}$ is defined as a mean electric field, due to both the external field and the microscopic charges.

   In the presence of external field $\mathbf{E}_{\text{ext}}$, polarization $\mathbf{P}$ is induced in the rod. Since $\boldsymbol{\nabla} \times \mathbf{E} = 0$, the tangential electric field is continuous across the surface of the rod. This suggests that inside the rod, which is parallel to $\mathbf{E}_{\text{ext}}$, we have $\mathbf{E} = \mathbf{E}_{\text{ext}}$.

   To check for consistency, note that in this case, $\mathbf{P}$ is constant apart from the ends. There is no net polarization charge in the bulk of the rod, and no change in the electric field from $\mathbf{E}_{\text{ext}}$. But at the ends there are charges $\pm Q$, where $Q = PA$, and A is the cross-sectional area of the rod. Since A is very small for the thin rod, and the ends of a long rod are far away from most of the rod, these charges do not contribute significantly to the electric field. Hence our hypothesis is satisfactory.

   The electric displacement can then be deduced as $\mathbf{D} = \epsilon\mathbf{E} = \epsilon\mathbf{E}_{\text{ext}}$ inside the rod.

   b) For a dielectric disk perpendicular to $\mathbf{E}_{\text{ext}}$, the normal component of the fields is naturally emphasized. Recall that $\boldsymbol{\nabla} \cdot \mathbf{D}$ implies that the normal component of the displacement $\mathbf{D}$ is continuous across the dielectric boundary.

   Outside the disk, where $\epsilon = 1$, $\mathbf{D} = \mathbf{E} = \mathbf{E}_{\text{ext}}$ is normal to the surface. (That $\mathbf{E} = \mathbf{E}_{\text{ext}}$ may be justified by noting that the induced charges on two surfaces of the thin disk have opposite signs and do not contribute to the field outside of the disk.) Thus, inside the disk, $\mathbf{D} = \mathbf{E}_{\text{ext}}$. Lastly, inside the disk $\mathbf{E} = \mathbf{D}/\epsilon = \mathbf{E}_{\text{ext}}/\epsilon$.

   c) The problem of a dielectric sphere of uniform polarization density $\mathbf{P}$ is equivalent to two homogeneous spheres with charge densities ρ and $-\rho$, displaced by distance $\mathbf{d}$, such that in the limit $d \to 0$, $\rho \to \infty$ but $\rho\mathbf{d} = \mathbf{P}$.

   Recall that for a sphere of uniform charge density ρ, the interior electric field is

$$\mathbf{E} = \frac{4\pi}{3}\rho\mathbf{r}. \tag{53}$$

   Thus, **inside** the polarized sphere we have

$$\mathbf{E} = \lim\left\{\frac{4\pi}{3}\rho(\mathbf{r} - \mathbf{d}) - \frac{4\pi}{3}\rho\mathbf{r}\right\} = -\frac{4\pi}{3}\lim\rho\mathbf{d} = -\frac{4\pi\mathbf{P}}{3}. \tag{54}$$

   The displacement is

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \frac{8\pi\mathbf{P}}{3}. \tag{55}$$

   Then,

$$\int_{\text{inside}} \mathbf{E} \cdot \mathbf{D} d\text{Vol} = -\frac{4\pi P}{3}\frac{8\pi P}{3}\frac{4\pi r^3}{3} = -\frac{128\pi^2 r^3 P^2}{27}. \tag{56}$$

   Outside the sphere, the electric field is effectively due to two point charges $q = 4\pi\rho r^3/3$ separated by small distance $\mathbf{d}$, where $\rho\mathbf{d} = \mathbf{P}$. The external field is simply that of a point dipole at the origin of strength

$$\mathbf{p} = \frac{4\pi r^3\mathbf{P}}{3}. \tag{57}$$

Then,

$$\mathbf{E} = \frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}}{r^3}, \tag{58}$$

and $\mathbf{D} = \mathbf{E}$. Thus,

$$\int_{\text{outside}} \mathbf{E}\cdot\mathbf{D}d\text{Vol} = \int \frac{p^2 + 3(\mathbf{p}\cdot\hat{\mathbf{r}})^2)}{r^6} d\text{Vol} \tag{59}$$

$$= \quad p^2 \int_r^\infty \frac{4\pi r^2 dr}{r^6} + 3p^2 \int_r^\infty \frac{2\pi r^2 dr}{r^6} \int_{-1}^{1} \cos^2\theta \; d\cos\theta = \frac{8\pi p^2}{3r^3} = \frac{128\pi^3 r^3 P^2}{27},$$

and the sum of the inside and outside integrals vanishes.

d) Let us show, on general grounds, that for an electret the integral of $\mathbf{E}\cdot\mathbf{D}$ over the whole space is zero. First, note that $\boldsymbol{\nabla}\cdot\mathbf{D} = 0$ everywhere for an electret, and that the electric field can be derived from a potential, ϕ. Then

$$\int_{\text{all space}} \mathbf{E}\cdot\mathbf{D}d\text{Vol} = -\int_V \boldsymbol{\nabla}\phi\cdot\mathbf{D}d\text{Vol} = -\int_V \boldsymbol{\nabla}\cdot(\phi\mathbf{D})d\text{Vol} = -\oint_S \phi\mathbf{D}\cdot d\mathbf{S}. \tag{60}$$

But if electret occupies finite volume, then $\phi \simeq 1/r$ at large r, as seen from multipole expansion; in the same limit, $D \simeq 1/r^2$, while $dS \simeq r^2$. So the integral over the surface at infinity is zero.

What would be different in this argument if the material were not an electret? In general, we would then have $\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho_{\text{free}}$, and the 3rd step in (60) would have the additional term $\int_V \phi\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\int_V \rho_{\text{free}}\phi = 8\pi U$. Thus, the usual electrostatic energy is contained in $\int_V \mathbf{E}\cdot\mathbf{D}/8\pi$, as expected.

This argument emphasizes that the work done in putting a field on an ordinary dielectric can be accounted for using only the free charges (which establish $\mathbf{D}$). One need not explicitly calculate the energy stored in the polarization charge distribution, which energy is accounted for via the modification to the potential ϕ in the presence of the dielectric. But the whole argument fails for an electret, which remains polarized (energized) even in the absence of a free charge distribution.

7. In the upper half of the capacitor (where there is no dielectric) we have via Gauss' law:

$$E_{\rm up}(r) = D_{\rm up}(r) = \frac{4\pi Q_{\rm up}}{2\pi r^2} = \frac{2Q_{\rm up}}{r^2}, \tag{61}$$

where $Q_{\rm up}$ is the charge on the upper half of the inner sphere, assuming the fields are radial.

In the lower part we have:

$$E_{\rm down}(r) = E_{\rm up}(r), \tag{62}$$

which follows from the continuity of the tangential component of $\mathbf{E}$ across the boundary between dielectric and vacuum. Also,

$$D_{\rm down}(r) = \epsilon E_{\rm down}(r), \tag{63}$$

and

$$D_{\rm down}(r) = \frac{2Q_{\rm down}}{r^2}, \tag{64}$$

as follows from $\boldsymbol{\nabla}\cdot\mathbf{D} = 4\pi\rho_{\rm free}$, where $Q_{\rm down}$ is the charge on the lower part of the inner sphere.

Combining (61-64),

$$Q_{\rm down} = \epsilon Q_{\rm up}, \tag{65}$$

holds for the "free" charge on the inner shell.

What about the total charge distribution, which include polarization charges in the dielectric? The polarization vector $\mathbf{P}$ obeys $4\pi\mathbf{P} = (\epsilon - 1)\mathbf{E}$. Thus, the charge density, $\sigma = 4\pi\mathbf{P}\cdot\hat{\mathbf{n}}$, which appears microscopically on the boundary of the dielectric adjacent to the inner shell, equals $1-\epsilon$ times the charge density on the upper shell (since $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$). In other words, the total *microscopic* charge densities on the lower and upper parts of the shell are equal. This ensures that the electric field is the same in the upper and lower part of the capacitor.

Of course,

$$Q_{\rm down} = Q - Q_{\rm up}. \tag{66}$$

From (65) and (67),

$$Q - Q_{\rm up} = \epsilon Q_{up}, \tag{67}$$

and hence,

$$Q_{\rm up} = \frac{Q}{1+\epsilon}. \tag{68}$$

This implies that the potential difference V between the shells is

$$V = -\int_a^b E dr = -\frac{2Q}{1+\epsilon}\int_a^b \frac{dr}{r^2} = \frac{2Q}{1+\epsilon}\left[\frac{1}{b} - \frac{1}{a}\right]. \tag{69}$$

So, the capacitance is

$$C = \frac{Q}{V} = \frac{1+\epsilon}{2}\frac{ab}{a-b}. \tag{70}$$

8. a) In the presence of an electric field along the z direction, which is perpendicular to the plane of the orbit of our model atom, the plane is displaced by a distance z. We calculate this displacement from the equilibrium condition: the axial component of the Coulomb force between the electron and the proton should be equal to the force eE on the electron of charge e due to the electric field. Namely,

$$F_z = \frac{e^2}{r^2}\frac{z}{r} = eE, \tag{71}$$

where $r = \sqrt{a^2 + z^2}$, and a is the radius of the orbit of the electron. The induced atomic dipole moment p is

$$p = ez = r^3 E \approx a^3 E, \tag{72}$$

where the approximation holds for small displacements, *i.e.*, small electric fields. Since $p = \alpha E$ in terms of the atomic polarizability α, we estimate that

$$\alpha \approx a^3, \tag{73}$$

where a is the radius of the atom.

The dielectric constant ϵ is related to the atomic polarizability via

$$\epsilon - 1 = 4\pi N\alpha, \tag{74}$$

where N is the number of atoms per cm^3. For hydrogen, there are 2 atoms per molecule, and 6×10^{23} molecules in 22.4 liters, at S.T.P. Hence $N = 2(6 \times 10^{23})/(22.4 \times 10^3) = 5.4 \times 10^{19}$ atoms/cm^3. Estimating the radius a as the Bohr radius, 5.3×10^{-9} cm, we find

$$\epsilon - 1 \approx 4\pi(5.4 \times 10^{19})(5.3 \times 10^{-9})^3 \approx 1.0 \times 10^{-4}. \tag{75}$$

b) If an electric field E is applied to the springlike atom, then the displacement d of the electron relative to the fixed (neutral) nucleus is related by $F = kd = eE$, where $k = m\omega^2$ is the spring constant in terms of characteristic frequency ω. The induced dipole moment p is given by

$$p = ed = \frac{e^2 E}{k} = \frac{e^2 E}{m\omega^2}. \tag{76}$$

Thus the polarizability α is given by

$$\alpha = \frac{e^2}{m\omega^2}. \tag{77}$$

The frequency ω that corresponds to the Lyman line at 1225
A is

$$\omega = \frac{2\pi c}{\lambda} = \frac{2\pi(3 \times 10^{10})}{1225 \times 10^{-8}} = 1.54 \times 10^{16} \text{ Hz}. \tag{78}$$

From (77),

$$\alpha = \frac{(4.8 \times 10^{-10})^2}{(9.1 \times 10^{-28})(1.54 \times 10^{16})^2} = 1.07 \times 10^{-24} \text{ cm}^3. \tag{79}$$

Finally,

$$\epsilon - 1 = 4\pi N\alpha = 4\pi(5.4 \times 10^{19})(1.07 \times 10^{-24}) = 7.3 \times 10^{-4} \tag{80}$$

Thus, our two models span the low and high side of the empirical result. Of course, we have neglected the fact that the hydrogen atoms are actually paired in molecules.