Princeton University

Ph501
Electrodynamics
Problem Set 10

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1. **Gravity Waves**

On pp. 229-233 of Lecture 19 of the Notes,
http://kirkmcd.princeton.edu/examples/ph501/ph501lecture19.pdf, we showed that (plane) waves of the gravitational tensor potential $\phi_{\mu\nu} = \epsilon_{\mu\nu}\, e^{i(kz-\omega t)}$ are transverse, and have only two polarizations. Furthermore, $\phi_{\mu\nu}$ is symmetric and traceless.

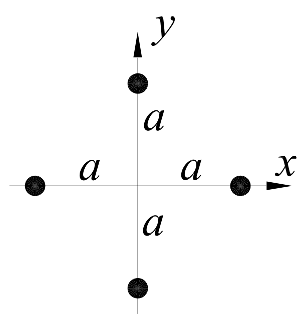
But, what is the physical significance of $\phi_{\mu\nu}$?

Einstein tells us to write the square of the invariant length as $ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$ instead of $ds^2 = dx_\mu\, dx^\nu$. Then,

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} + \epsilon_{\mu\nu}\, e^{i(kz-\omega t)}, \tag{1}$$

for gravity waves in "empty space", far from any star (or planet). Then, we may say that $\phi_{\mu\nu}$ is a wave in the structure of space and time.

Consider a plane wave $\epsilon_{\mu\nu}\, e^{i(kz-\omega t)}$ that is incident on four equal masses m in he x-y plane. We have argued that conservation of energy and momentum restrict the simplest gravitational wave to be a kind of quadrupole radiation.



Use you "physical intuition" to the only two distinct quadrupole oscillations that the gravity wave could induce on the 4-mass system. Identify the forms of the polarization tensor $\phi_{\mu\nu}$ corresponding to these two oscillations.

Show that $\sum_{\text{masses}} \Delta s^2$ remains invariant under the oscillations induced by the (weak) gravity wave.

A "practical" gravity-wave detector might consist of a massive sphere, so as to be sensitive to waves from all directions. Sketch the oscillations of a sphere by waves of the two polarizations that you found above.

Measure Δx_μ from the origin at some fixed time, say $t = 0$. Let $\epsilon \ll 1$ be the strength of a component of $\epsilon_{\mu\nu}$ and $\delta \ll a$ be the amplitude of the oscillation of the masses.

It suffices to show this for only 1 of the 2 possible oscillations.

2. Show that the angular distribution,

$$\frac{dP^\star}{d\Omega^\star} = \frac{d^2U^\star}{dt^\star\, d\Omega^\star} = f(\cos\theta^\star, \phi^\star), \tag{2}$$

of the power of electromagnetic radiation in the far zone of a system whose center of mass/energy is instantaneously at rest in the (inertial) $\star$ frame has the form,

$$\frac{dP_{\rm source}}{d\Omega} = \frac{dU}{dt\, d\Omega} = \frac{1}{\gamma^4(1-\beta\cos\theta)^3}\, f\left(\frac{\cos\theta - \beta}{1-\beta\cos\theta}, \phi\right), \tag{3}$$

for radiation by the source in the (inertial) lab frame where the system has velocity $\mathbf{v}$ along the polar $(z,\, z^\star)$ axes of the spherical coordinate systems (r, θ, ϕ) and $(r^\star, \theta^\star, \phi^\star)$, $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, with c being the speed of light in vacuum. Comment on the angular distribution of radiation as detected by distant, fixed observers in the lab frame.

Since $dP_\mu = (dU, c\, d\mathbf{P})$ *is a 4-vector, we know that* $dU^\star = \gamma(dU - \mathbf{v}\cdot d\mathbf{P})$.

How are dU *and* $d\mathbf{P}$ *related (in the far zone)?*

Note that θ *is an angle of a light ray, so you can transform the 4-vector* $(k, k\sin\theta, 0, k\cos\theta)$ *to find the relation between* $\cos\theta$ *and* $\cos\theta^\star$*, etc.*

3. Use the result of Prob. 2 above to transform the Larmor formula,

$$f = \frac{e^2\, a^{\star 2} \sin^2 \theta^\star}{4\pi c^3}\,, \tag{4}$$

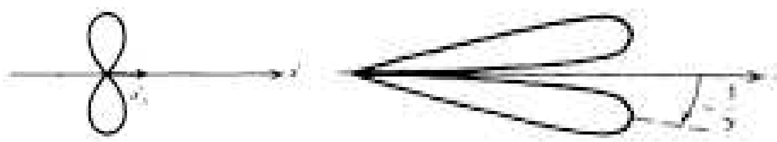
to the lab frame in the two cases; a) $\mathbf{a}^\star \parallel \mathbf{v}$, and b) $\mathbf{a}^\star \perp \mathbf{v}$.

Use $a^{\star 2} = -c^2 a_\mu a^\mu$ to eliminate $a^\star$ in favor of a in the lab frame.

Ans: (with $\beta = v/c$)

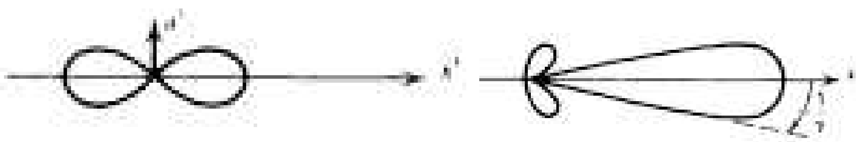
a) $\mathbf{a}^\star \parallel \mathbf{v}$

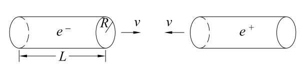
$$\frac{dU}{d\Omega\, dt} = \frac{e^2\, a^2}{4\pi c^3} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}\,. \tag{5}$$



b) $\mathbf{a}^\star \perp \mathbf{v}$

$$\frac{dU}{d\Omega\, dt} = \frac{e^2\, a^2}{4\pi c^3} \frac{(1 - \beta \cos \theta)^2 - (1 - \beta^2) \sin^2 \theta \cos^2 \phi}{(1 - \beta \cos \theta)^5}\,. \tag{6}$$





4. Suppose the acceleration of a charge e is entirely due to external electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$.

   (a) As discussed on pp. 236-236, Lecture 20 of the Notes, the charge radiates energy and momentum with the 4-vector,

$$dP_\mu = d(U, c\,\mathbf{P}) = -\frac{2e^2 c}{3} a_\nu a^\nu \, dx_\mu, \tag{7}$$

   The particle obeys $\mathbf{F} = m\mathbf{a}$ in relativistic form (p. 222, Lecture 18 of the Notes),

$$\frac{dp_\mu}{ds} = f_\mu = F_{\mu\nu} u^\nu, \tag{8}$$

   where $p_\mu = (E, c\,\mathbf{p}) = m_0 c^2 u_\mu$ is the particle's 4-momentum, and m_0 its rest mass. Use these facts to show that in an arbitrary inertial frame,

$$\frac{dU}{dt} = \frac{2e^4\gamma^2}{3m_0^2 c^3}\left[\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)^2 - \left(\mathbf{E} \cdot \frac{\mathbf{v}}{c}\right)^2\right]. \tag{9}$$

   (b) Reconsider part (a) from another point of view, noting that the radiated power dU/dt is a Lorentz invariant,

$$\frac{dU}{dt} = \frac{dU^\star}{dt^\star} = \frac{2e^2 a^{\star 2}}{3c^3}. \tag{10}$$

   It then suffices to relate $a^\star$ to the lab-frame Lorentz force $\mathbf{F}$ on charge e due to $\mathbf{E}$ and $\mathbf{B}$. Show that,

$$\frac{dU}{dt} = \frac{2e^2}{3m_0^2 c^3}\begin{cases} \mathbf{F}^2 & \text{if } \mathbf{a} \parallel \mathbf{v}, \\ \gamma^2 \mathbf{F}^2 & \text{if } \mathbf{a} \perp \mathbf{v}. \end{cases} \tag{11}$$

   (c) The maximum laboratory electric field that can be applied to a relativistic charged particle is about 10^8 V/m, while the maximum (static) magnetic field is about $10\text{ T} = 100\text{ kG}$. For an electron with $\gamma = 10^5$, as at the Stanford Linear Accelerator Center, what is the energy radiated per cm for $\mathbf{E} \parallel \mathbf{v}$, $\mathbf{E} \perp \mathbf{v}$, and $\mathbf{B} \perp \mathbf{v}$?

5. A relativistic particle of charge e passes a fixed charge Ze such that b is the distance of closet approach (impact parameter). What is the total energy radiated, $U = \int (dU/dt)\, dt$, assuming that the deflections of the charges are negligible?

   Ans:

$$U = \frac{\pi Z^2 e^6}{12 c^3 m_0\, b^3 v} \frac{4 - (v/c)^2}{1 - (v/c)^2}\,. \tag{12}$$

   As a check, make a quick estimate for $v \approx c$ in the spirit of the "short-cut" method used for Rutherford scattering on p. 134, Ph205 Lecture 12,
   http://kirkmcd.princeton.edu/examples/Ph205/ph205l12.pdf

6. Charge e_1 with mass m_1 passes by charge e_2 with mass m_2 on parallel trajectories, initially separated by distance b in their center-of-mass frame, such that the relative velocity obeys $v \ll c$. However, v is large enough that we can approximate the motion of the charges as along straight lines at all times. Supposing that the motion lies in the x-y plane, with $\mathbf{v} = v\,\hat{\mathbf{x}}$, show that the angular distribution of the emitted radiation in the center of mass frame, far from the charges, is,

$$\frac{dU}{d\Omega} = \int \frac{d^2U}{d\Omega\,dt} = \frac{e_1^2 e_2^2}{32c^3 b^3 v}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 (4 - n_x^2 - 3n_y^2), \tag{13}$$

where unit vector $\hat{\mathbf{n}}$ is in the direction of the radiation at the distant observer.

Recall that for $v \ll c$, the radiation is well approximated as that associated with the 2^{nd} *time derivative of the electric-dipole moment* $\mathbf{p}$ *of the system, described by the Larmor formula (p. 186, Lecture 16 of the Notes),*

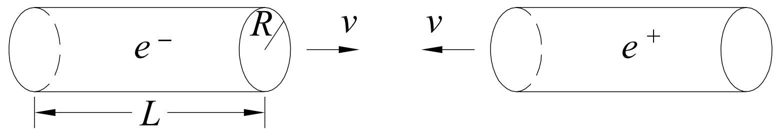
$$\frac{d^2U}{d\Omega\,dt} = \frac{(\hat{\mathbf{n}} \times \ddot{\mathbf{p}})^2}{4\pi c^3}\,. \tag{14}$$

As discussed on p. 187 of the notes, electric-dipole radiation vanishes for a system in which all particles have the same charge-to-mass ratio.

Integrate over $d\Omega$ supposing charge 1 is an electron, $e_2 = Ze$, and $m_2 \to \infty$ to find the result of Prob. 5 above for $v \ll c$.

7. **Colliding Bunches**

At the Stanford Linear Collider, two "bunches" of electrons and positrons of 46-GeV energy collide head on. Each bunch has N particles, of charge e in one bunch and charge $-e$ in the other. Approximate the bunches as (coaxial) cylinders of radius R and length $L \gg R$, with uniform charge density.



Estimate the transverse-momentum "kick" given to an electron as it passes through the positron bunch. In a "thin-lens" approximation, the "kick" is applied at the (circular) midplane of the bunch, and the trajectories are straight before and after this. Note that all electrons are deflected so as to cross the axis of the bunch at the same place. Each bunch acts as a lens for the particles in the other bunch!

Show that the focal length of this lens is,

$$f \approx \frac{\gamma m_0 c^2 R^2}{2Ne^2} = \frac{\gamma R^2}{2N r_0}, \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \qquad r_0 = \frac{e^2}{m_0 c^2} \tag{15}$$

Due to the deflection, the particles emit radiation (sometimes called beamstrahlung. Estimate the total energy dU radiated by a particle at the outer radius R of a bunch.

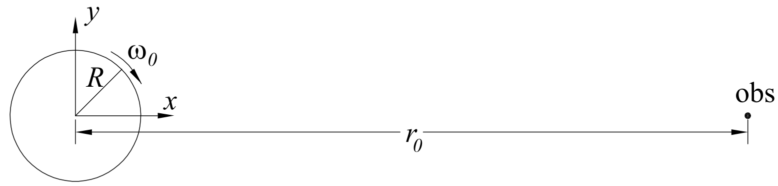
Ans:

$$\frac{dU}{U} \approx \frac{16\gamma N^2 r_0^3}{3LR^3}. \tag{16}$$

What is dU/U for the typical operating parameters of the SLC, $N = 10^{10}$, $L = 1$ mm, $R = 1\,mu$m, $\gamma = 10^5$?

8. **Synchrotron Radiation**

A particle of charge e moves in a circle of radius R about the origin in the x-y plane, with angular velocity ω_0: $x = R\cos\omega_0 t$, $y = R\sin\omega_0 t$. An observer is at $(x, y) = (r_0, 0$ where $r_0 \gg R$.



The electric field seen by the observer is, in the dipole approximation,

$$\mathbf{E} \approx \left[\frac{(\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}}{c^2 r}\right] \approx -\frac{e}{c^2 r_0}\frac{d^2}{dt^2} y(t' = t - r(t')/c)\hat{,} \tag{17}$$

where t' is the retarded time.

Evaluate $\ddot{y}(t')$ to show this is,

$$\omega_0^2 R \frac{cps\omega_0 t' - \beta}{(1 - \beta\cos\omega_0 t')^3}\,, \tag{18}$$

where $\beta = v/c$. This is big only for $\cos\omega_0 t' \approx 2n\pi$. For the pulse of radiation around $t' = 0$, eliminate t' in favor of $T = t - r_0/c$ to show that,

$$E(T) \propto \frac{1 - 4\gamma^6\omega_0^2 T^2}{1 + 12\gamma^6\cos\omega_0^2 T^2}\,, \tag{19}$$

for $\beta \to 1$, $T \approx 0$, and $\gamma = 1/\sqrt{1 - \beta^2}$.

Consider the frequency spectrum E_ω of this pulse, *i.e.*,

$$E_y(T) = \int_{-\infty}^{\infty} E_\omega\, e^{-i\omega t}\, dt. \tag{20}$$

Show that $E_\omega \propto e^{-\omega/2\omega_C}$, where the "critical frequency" is $\omega_C = \sqrt{3}\gamma^3\,\omega_0$.

Note that the pulse energy has Fourier analysis $U = \int_{-\infty}^{\infty} U_\omega\, d\omega$ with $U_\omega \propto E_\omega^2 \propto e^{-\omega/\omega_C}$.

This approximation breaks down at large T, and so we misestimate the low-frequency part of the spectrum. But, this analysis provides a good understanding of the high-frequency tail of the synchrotron radiation spectrum.

"Exact" calculation indicates that $U_\omega \propto \sqrt{\omega}\, e^{-\omega/\omega_C}$ *with* $\omega_C = 3\gamma^3\,\omega_0/2$. J. Schwinger, Phys. Rev. **75**, 1912 (1949), http://kirkmcd.princeton.edu/examples/EM/schwinger_pr_75_1912_49.pdf

A more sophisticated, but still fairly simple, analysis (following a suggestion by Fermi) is reviewed at http://kirkmcd.princeton.edu/examples/weizsacker.pdf

9. Charge e_1 of mass m makes a head-on collision with charge e_2, of the same sign as e_1, where charge 2 is fixed at the origin. Show that the total energy radiated in the collision is,

$$\Delta U = \frac{16}{45}\frac{e_1}{e_2}\left(\frac{v_0}{c}\right)^3 \frac{m v_0^2}{2}, \tag{21}$$

if the initial velocity v_0 of charge 1 is small compared to c.

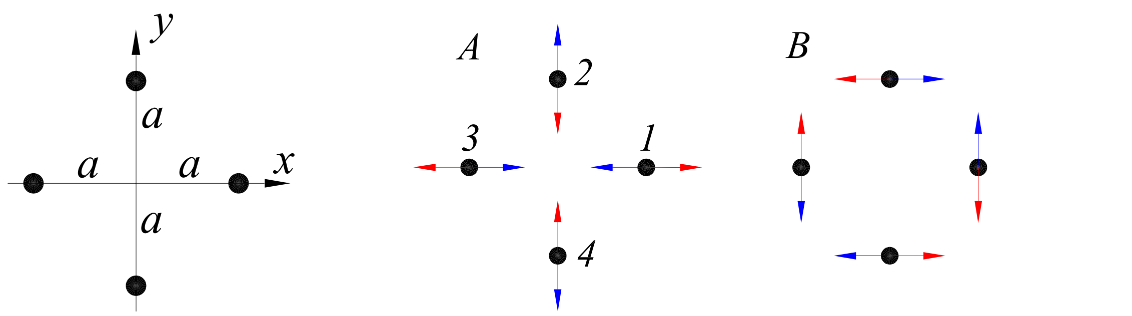
Note that the expression for dU/dt contains the force $F = ma$ on charge 1, which permits integration over time to be replaced by integration over velocity.

That $\Delta U/U_0 \propto (v_0/c)^3$ is characteristic of electric dipole radiation for $v_0 \ll c$, as seen in Prob. 8b, Set 8, http://kirkmcd.princeton.edu/examples/ph501set8.pdf

Solutions

1. **Gravity Waves**

For (tranverse) gravity waves of potential $\phi_{\mu\nu} = \epsilon_{\mu\nu}\, e^{i(kz-\omega t)}$ incident on 4 masses in the x-y plane as shown on the left below, one polarization leads to the motion sketched in the center figure, and the other polarization leads to that shown in the right figure.



For these two modes, A and B, which lead to quadrupole deformations, the polarization tensors are,

$$\epsilon^{(A)}_{\mu\nu} = \epsilon \begin{pmatrix} 0 & & & 0 \\ & 1 & 0 & \\ & 0 & -1 & \\ 0 & & & 0 \end{pmatrix}, \qquad \epsilon^{(B)}_{\mu\nu} = \epsilon \begin{pmatrix} 0 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & 0 \end{pmatrix}, \tag{22}$$

For mode A at time $t = 0$ the displacements of the four masses relative to the origin are,

$$\begin{aligned} \Delta x_{(1)} &= (0, a+\delta, 0, 0), \qquad & \Delta x_{(2)} &= (0, 0, a-\delta, 0), \\ \Delta x_{(3)} &= (0, -a-\delta, 0, 0), \qquad & \Delta x_{(4)} &= (0, 0, -a+\delta, 0), \end{aligned} \tag{23}$$

The $\Delta s^2 = g_{\mu\nu}\, \Delta x^\mu\, \Delta x^\nu = (\eta_{\mu\nu} + \epsilon_{\mu\nu})\, \Delta x^\mu\, \Delta x^\nu$ associated with these four displacements at time $t = 0$ are, recalling that $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$,

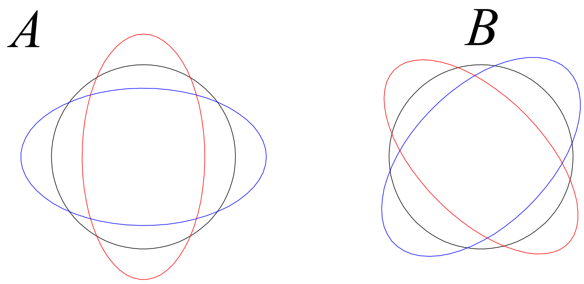
$$\begin{aligned} \Delta s^2_{(1)} &= (-1+\epsilon)(a+\delta)^2, \qquad & \Delta s^2_{(2)} &= (-1-\epsilon)(a-\delta)^2, \\ \Delta s^2_{(3)} &= (-1+\epsilon)(-a-\delta)^2, \qquad & \Delta s^2_{(4)} &= (-1-\epsilon)(-a+\delta)^2, \end{aligned} \tag{24}$$

The sum of these four terms is,

$$\sum \Delta s^2_{(i)} = -4a^2 - 4\delta^2 + 4\epsilon\delta \;\approx -4a^2, \tag{25}$$

ignoring the terms of second order of smallness. The sum is invariant under the deformations induced by the gravity wave to first order.

For a spherical gravity-wave detector, the deformations induced by gravity-wave modes A and B would have the forms sketched below, which differ by a rotation of 45° about the z-axis (in contrast to the response of a single electric charge to plane electromagnetic waves of the two independent linear polarizations, which differ by a rotation of 90°).



2. The solution to this problem is in sec. 2.2.1 of
http://kirkmcd.princeton.edu/examples/moving_far.pdf

3. The solution to this problem is in secs. 2.2.3-4 of
   http://kirkmcd.princeton.edu/examples/moving_far.pdf

4. (a) *Part (a) of this problem is the topic of §73 of* http://kirkmcd.princeton.edu/examples/EM/landau_ctf_71.pdf.

As discussed on pp. 236-236, Lecture 20 of the Notes, the charge radiates energy and momentum with the 4-vector,

$$dP_\mu = d(U, c\,\mathbf{P}) = -\frac{2e^2c}{3} a_\nu a^\nu \, dx_\mu, \tag{26}$$

The particle obeys $\mathbf{F} = m\mathbf{a}$ in relativistic form (p. 222, Lecture 18 of the Notes),

$$a_\mu = \frac{1}{m_0c^2}\frac{dp_\mu}{ds} = \frac{1}{m_0c^2} F_{\mu\nu} u^\nu = \frac{f_\mu}{m_0c^2} = \frac{e\gamma}{m_0c^2}\left(\mathbf{E}\cdot\frac{\mathbf{v}}{c}, \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right), \tag{27}$$

where $p_\mu = (E, c\,\mathbf{p}) = m_0c^2 u_\mu$ is the particle's 4-momentum, m_0 its rest mass and c is the speed of light in vacuum. Then,

$$\begin{aligned} \frac{dU}{dt} &= c\frac{dP_0}{dx_0} = c\frac{2e^2c}{3}\frac{{}^2\gamma^2}{m_0^2c^4}\left[\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)^2 - \left(\mathbf{E}\cdot\frac{\mathbf{v}}{c}\right)^2\right] \\ &= \frac{2e^4\gamma^2}{3m_0^2c^3}\left[\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)^2 - \left(\mathbf{E}\cdot\frac{\mathbf{v}}{c}\right)^2\right]. \end{aligned} \tag{28}$$

(b) We also recall that,

$$\frac{dU}{dt} = \frac{dU^\star}{dt^\star} = \frac{2e^2 a^{\star 2}}{3c^3}, \tag{29}$$

using the Larmor formula in the instantaneous rest frame of the charge, where (p. 221, Lecture 18) $a_\mu = (0, \mathbf{a}^\star/c^2)$.

We also have that (p. 222, Lecture 18),

$$a_\mu = \frac{1}{m_0c^2} f_\mu = \frac{1}{m_0c^2}\left(\gamma\mathbf{F}\cdot\frac{\mathbf{v}}{c}, \gamma\mathbf{F}\right), \tag{30}$$

where $\mathbf{F}$ is the lab-frame Lorentz force on charge e due to $\mathbf{E}$ and $\mathbf{B}$. The Lorentz transformation of the 4-acceleration from the rest frame to the lab frame then gives,

$$\mathbf{a} = \gamma\mathbf{a}^\star, \qquad \mathbf{a}^\star = \frac{\mathbf{a}}{\gamma} = \frac{\mathbf{F}}{m_0c^2} \qquad (\mathbf{a} \parallel \mathbf{v}), \tag{31}$$

$$\mathbf{a} = \mathbf{a}^\star, \qquad \mathbf{a}^\star = \mathbf{a} = \frac{\gamma\mathbf{F}}{m_0c^2} \qquad (\mathbf{a} \perp \mathbf{v}), \tag{32}$$

Then, CEQ. (29) gives,

$$\frac{dU}{dt} = \frac{2e^2}{3m_0^2c^3}\begin{cases} \mathbf{F}^2 & \text{if } \mathbf{a} \parallel \mathbf{v}, \\ \gamma^2\mathbf{F}^2 & \text{if } \mathbf{a} \perp \mathbf{v}. \end{cases} \tag{33}$$

(c) The maximum laboratory electric field that can be applied to a relativistic charged particle is about 10^8 V/m = $10^4/3$ statvolt/cm, while the maximum (static) magnetic field is about 10 T = 10^5 G. For an electron with $\gamma = 10^5$, $v \approx c$, as at the Stanford Linear Accelerator Center, the energy radiated per cm for $\mathbf{E} \parallel \mathbf{v}$ is, from eq. (33),

$$\begin{aligned}\frac{dU(\mathbf{E} \parallel \mathbf{v})}{vdt} &= \frac{2e^2}{3m_0^2 vc^3} e^2 E^2 \approx \frac{2r_e^2 E^2}{3} \\ &\approx \frac{2}{3}(3 \times 10^{-13} \text{ cm})^2 (3.3 \times 10^3 \text{ statvolt/cm})^2 \\ &\approx 7 \times 10^{-19} \text{ erg/cm} \approx 4 \times 10^{-7} \text{ eV/cm},\end{aligned} \tag{34}$$

where $r_e = e^2/m_0c^2 = 2.8 \times 10^{-13}$ cm is the classical electron radius, and 1 eV = 1.6×10^{-12} erg. The radiation here is negligible compared to the final energy of the electrons of 50 GeV ($\gamma \approx 10^5$), even over the total length (2 miles) of the accelerator.

The energy radiated per cm for $\mathbf{E} \perp \mathbf{v}$ is,

$$\begin{aligned}\frac{dU(\mathbf{E} \perp \mathbf{v})}{vdt} &= \frac{2e^2}{3m_0^2 vc^3} e^2 \gamma^2 E^2 \approx \frac{2r_e^2 \gamma^2 E^2}{3} \\ &\approx \frac{2}{3}(3 \times 10^{-13} \text{ cm})^2 (10^{10})(3.3 \times 10^3 \text{ statvolt/cm})^2 \\ &\approx 7 \times 10^{-9} \text{ erg/cm} \approx 4 \times 10^3 \text{ eV/cm} = 4 \text{ keV/cm}.\end{aligned} \tag{35}$$

The energy radiated per cm for 50-GeV electrons deflected by a 10-T magnetic field is,

$$\begin{aligned}\frac{dU(\mathbf{B} \perp \mathbf{v})}{vdt} &= \frac{2e^2}{3m_0^2 vc^3} e^2 \gamma^2 B^2 \approx \frac{2r_e^2 \gamma^2 B^2}{3} \\ &\approx \frac{2}{3}(3 \times 10^{-13} \text{ cm})^2 (10^{10})(10^{10}) \\ &\approx 6 \times 10^{-5} \text{ erg/cm} \approx 4 \times 10^7 \text{ eV/cm} = 40 \text{ MeV/cm}.\end{aligned} \tag{36}$$

If the electrons were in a circular ring of magnets with 10-T field (parallel to the axis of the ring), the radius r of the ring would be related by,

$$F = \frac{evB}{c} = \frac{\gamma m v^2}{r}, \qquad r \approx \frac{\gamma m c^2}{eB} \approx \frac{10^5 \cdot 10^{-27} \text{ g} \cdot (3 \times 10^{10} \text{ cm/s})^2}{5 \times 10^{-10} \text{ statcoulomb} \cdot 10^5 \text{ G}} \approx 2 \times 10^3 \text{ cm}, \tag{37}$$

so the radiated energy per turn would be about 500 GeV. In a practical storage ring, the radiated energy must be replenished (by azimuthally accelerating electric fields). For 50-GeV electrons such a ring could have a magnetic field of only a fraction of 1 Tesla, to avoid excessive power bills.

5. *This problem is Prob. 1, §73, p. 196 of*
http://kirkmcd.princeton.edu/examples/EM/landau_ctf_71.pdf

A relativistic particle of charge e passes a fixed charge Ze such that b is the distance of closet approach (impact parameter).

We take charge ze to be at the origin, and charge e at $(vt, b, 0)$, assuming that the deflection of the charge is negligible. At time t it experiences electric field,

$$\mathbf{E} = \frac{Ze}{v^2t^2 + b^2} \left(\frac{vt\,\hat{\mathbf{x}} + b\,\hat{\mathbf{y}}}{\sqrt{v^2t^2 + b^2}} \right), \tag{38}$$

and zero magnetic field. Using eq. (28) of Prob. 4 above, we have,

$$\frac{dU}{dt} = \frac{2Z^2e^6\gamma^2}{3m_0^2c^3} \left(\frac{1}{(v^2t^2 + b^2)^2} - \frac{v^2t^2(v^2/c^2)}{(v^2t^2 + b^2)^3} \right), \tag{39}$$

$$dU = \frac{2Z^2e^6\gamma^2}{3m_0^2c^3v} \int_{-\infty}^{\infty} v\,dt \left(\frac{1}{(v^2t^2 + b^2)^2} - \frac{v^2t^2(v^2/c^2)}{(v^2t^2 + b^2)^3} \right)$$

$$U = \frac{2Z^2e^6\gamma^2}{3m_0^2c^3v} \int_{-\infty}^{\infty} v\,dt \left[\frac{1}{2b^3} \tan^{-1}\frac{vt}{b} - \frac{v^2}{c^2}\frac{1}{8b^3} \tan^{-1}\frac{vt}{b} \right]_{-\infty}^{\infty}$$

$$= \frac{\pi Z^2e^6\gamma^2}{3m_0^2c^3vb^3} \left(1 - \frac{v^2}{4c^2} \right) = \frac{\pi Z^2e^6}{12c^3m_0^2\,b^3v}\, \frac{4 - (v/c)^2}{1 - (v/c)^2} \approx \frac{\pi Z^2e^6\gamma^2}{4c^4m_0^2\,b^3}, \tag{40}$$

using Dwight 120.1 and 122.3, http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf, and the approximation follows for $v \approx c$. As usual, $\gamma = 1/\sqrt{1 - v^2/c^2}$.

As a check, we make a quick estimate in the spirit of the "short-cut" method used for Rutherford scattering on p. 134, Ph205 Lecture 12,
http://kirkmcd.princeton.edu/examples/Ph205/ph205l12.pdf.
That is, we estimate that, noting that $v \approx c$, and $dU/dt|_{\max}$ occurs at $t = 0$, with $\Delta t \approx 2b/c$,

$$U \approx \left.\frac{dU}{dt}\right|_{\max} \Delta t = \frac{2Z^2e^6\gamma^2}{3m_0^2c^3b^4}\, \frac{2b}{c} = \frac{4Z^2e^6\gamma^2}{3m_0^2c^4b^3}, \tag{41}$$

which agrees with eq. (40) to within a factor of $16/3\pi = 1.7$. *The agreement would be even better taking* $\Delta t = b/c$.

6. *This problem is Prob. 4, p. 376 of* W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, $2^{\rm nd}$ ed. (Addison-Wesley, 1962),
kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

Charge e_1 with mass m_1 passes by charge e_2 with mass m_2 on parallel trajectories, initially separated by distance b, such that the relative velocity obeys $v \ll c$. However, v is large enough that we can approximate the motion of the charges as along straight lines at all times. We suppose that the motion lies in the x-y plane, with $\mathbf{v} = v\,\hat{\mathbf{x}}$. Then, the radiation is well approximated as that associated with the $2^{\rm nd}$ time derivative of the electric-dipole moment $\mathbf{p}$ of the system, described by the Larmor formula (for $v \ll c$),

$$\frac{d^2U}{d\Omega\,dt} = \frac{(\hat{\mathbf{n}} \times \ddot{\mathbf{p}})^2}{4\pi c^3}\,, \tag{42}$$

where unit vector $\hat{\mathbf{n}}$ is the direction of the radiation at the distant observer.

Denoting the locations in their center-of-mass frame of the two charges as $\mathbf{x}_1$ and $\mathbf{x}_2$ at any time t, their electric-dipole moment at that time is,

$$\mathbf{p} = e_1\mathbf{x}_1 + e_2\mathbf{x}_2, \qquad \ddot{\mathbf{p}} = e_1\ddot{\mathbf{x}}_1 + e_2\ddot{\mathbf{x}}_2 = \frac{e_1}{m_1}m_1\ddot{\mathbf{x}}_1 + \frac{e_2}{m_2}m_2\ddot{\mathbf{x}}_2, \tag{43}$$

For $v \ll c$, we can approximate the forces on the charges as the instantaneous Coulomb forces,

$$= m_1\ddot{\mathbf{x}}_1 = \frac{e_1e_2(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \equiv \frac{e_1e_2\,\mathbf{d}}{d^3} = -m_2\ddot{\mathbf{x}}_\mathbf{2}. \tag{44}$$

Taking the origin to be at the center of mass, and $t = 0$ to be the time of their closest approach, at separation b, we have that,

$$\mathbf{d} = (vt, b, 0), \qquad d^2 = (vt)^2 + b^2. \tag{45}$$

Then,

$$\ddot{\mathbf{p}} = \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)\frac{e_1e_2\,\mathbf{d}}{d^3} = \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)\frac{e_1e_2\,(vt, b, 0)}{((vt)^2 + b^2)^{3/2}}\,, \tag{46}$$

$$\hat{\mathbf{n}} \times \ddot{\mathbf{p}} = \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)\frac{e_1e_2\,(-\hat{n}_zb, \hat{n}_zvt, \hat{n}_xb - \hat{n}_yvt)}{(v^2t^2 + b^2)^{3/2}}\,, \tag{47}$$

$$(\hat{\mathbf{n}} \times \ddot{\mathbf{p}})^2 = \left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\frac{e_1^2e_2^2\,[\hat{n}_z(b^2 + v^2t^2) + \hat{n}_x^2b^2 - 2\hat{n}_x\,\hat{n}_y\,bvt + \hat{n}_y^2v^2t^2]}{(v^2t^2 + b^2)^3}\,, \tag{48}$$

$$\frac{dU}{d\Omega} = \int_{-\infty}^{\infty}\frac{d^2U}{d\Omega\,dt}\,dt = \int_{-\infty}^{\infty}\frac{(\hat{\mathbf{n}} \times \ddot{\mathbf{p}})^2}{4\pi c^3v}\,d(vt)$$

$$= \frac{e_1^2e_2^2}{4\pi c^3v}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\int_{-\infty}^{\infty}d(vt)\,\frac{\hat{n}_z(b^2 + v^2t^2) + \hat{n}_x^2b^2 - 2\hat{n}_x\,\hat{n}_y\,bvt + \hat{n}_y^2v^2t^2}{(v^2t^2 + b^2)^3}$$

$$= \frac{e_1^2e_2^2}{4\pi c^3v}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\left(\frac{\pi n_z^2}{2b^3} + \frac{3\pi n_x^2}{8b^3} + \frac{\pi n_y^2}{8b^3}\right)$$

$$= \frac{e_1^2e_2^2}{32c^3b^3v}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2(4 - n_x^2 - 3n_y^2), \tag{49}$$

using Dwight 120.1, 120.3 and 122.3.

For the case that charge 1 is an electron, $e_1 = e$, $m_1 = m_0$, and charge 2 is Ze with $m_2 = \infty$ (*i.e.*, fixed in place), eq. (49) becomes, in spherical coordinates (r, θ, ϕ),

$$\frac{dU}{d\Omega} = \frac{Z^2 e^6}{32 c^3 m_0^2 b^3 v}(4 - \sin^2\theta\cos^2\phi - 3\sin^2\theta\sin^2\phi), \quad (50)$$

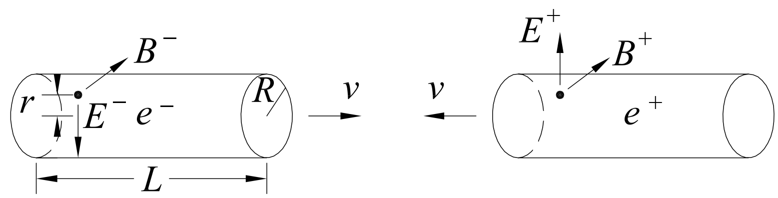
$$U = \int \frac{dU}{d\Omega}\, d\Omega = \frac{Z^2 e^6}{32 c^3 m_0^2 b^3 v}\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi\, (4 - \sin^2\theta\cos^2\phi - 3\sin^2\theta\sin^2\phi)$$

$$= \frac{Z^2 e^6}{32 c^3 m_0^2 b^3 v}\left(16\pi - \frac{4\pi}{3} - \frac{12\pi}{3}\right) = \frac{\pi Z^2 e^6}{3 c^3 m_0^2 b^3 v}, \quad (51)$$

which is the same as eq. (40) when $v \ll c$.

7. **Colliding Bunches**

At the Stanford Linear Collider, two "bunches" of electrons and positrons of 46-GeV energy collide head on. Each bunch has N particles, of charge e in one bunch and charge $-e$ in the other. The particles have $v \approx c$, Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ and momentum $P \approx \gamma m_0 c$.



We approximate the bunches as (coaxial) cylinders of radius R and length $L \gg R$ in the lab frame, with uniform charge density, such theat the (radial) electric field inside a bunch at radius $r < R$ is independent of the axial position, and has magnitude,

$$E^\star = 4\pi \frac{Ner^2}{R^2} \frac{1}{2\pi r L^\star} = \frac{2Ner}{L^\star R^2}\,, \tag{52}$$

in the rest frame of a bunch, where $L^\star = \gamma L$. The lab-frame fields are, since $\mathbf{E}^\star$ is transverse to $\mathbf{v}$,

$$E = \gamma E^\star = \frac{2Ner}{LR^2}\,, \qquad B \approx \gamma E^\star = E. \tag{53}$$

The directions of $\mathbf{E}$ and $\mathbf{B}$ for the fields inside the two bunches are sketched above.

Assuming that the deflection of a particle is negligible during the time $\Delta t = L/2c$ that it is inside the oncoming bunch, the transverse momentum kick on an electron is,

$$\begin{aligned}\Delta \mathbf{P} = \mathbf{F}\,\Delta t &\approx e^- \left(\mathbf{E}^- + \frac{\mathbf{v}}{c} \times \mathbf{B}^- + \mathbf{E}^+ + \frac{\mathbf{v}}{c} \times \mathbf{B}^+\right) \frac{L}{2c} \approx e^- \left(\mathbf{E}^- - \mathbf{E}^- + \mathbf{E}^+ + \mathbf{E}^+\right) \frac{L}{2c} \\ &= -\frac{4Ne^2 r}{LR^2}\,\hat{\mathbf{r}} \frac{L}{2c} = -\frac{2Ne^2 r}{cR^2}\,\hat{\mathbf{r}},\end{aligned} \tag{54}$$

noting that the force on an electron due the electric and magnetic fields of the electron bunch cancel, while they add for the positron bunch. The resulting deflection angle is,

$$\Delta\theta = \frac{\Delta P}{P} = \frac{2Ne^2 r}{\gamma m_0 c^2 R^2} = \frac{2N r_0 r}{\gamma R^2}\,, \tag{55}$$

where $r_0 = e^2/m_0 c^2 = 2.8 \times 10^{-15}$ m is the classical electron radius (and is the same for both charges e and $-e$).

Approximating the deflection as occuring at the (circular) midplane of the oncoming bunch, the particle crosses the bunch axis at distance,

$$f \approx \frac{r}{\Delta\theta} = \frac{\gamma R^2}{2N r_0}\,, \tag{56}$$

from the centers of the two bunches at the moment they coincide (often call the (bunch) crossing point or intersection point). The distance f can be called the focal length of the effective lens formed by the oncoming bunch.

Due to the deflection, the particles emit radiation (sometimes called beamstrahlung. The radiated power is given by eq. (9) of Prob. 4 above,

$$\frac{dU}{dt} = \frac{2e^4\gamma^2}{3m_0^2c^3}\left[\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)^2 - \left(\mathbf{E} \cdot \frac{\mathbf{v}}{c}\right)^2\right] \approx \frac{2e^4\gamma^2}{3m_0^2c^3}(2E^{+2})^2 \approx \frac{32\gamma^2N^2e^6r^2}{3m_0^2c^3L^2R^2} \qquad (57)$$

The total radiated energy is,

$$dU \approx \frac{dU}{dt}\Delta t \approx \frac{32\gamma^2N^2e^6r^2}{3m_0^2c^3L^2R^2}\frac{L}{2c} = \frac{16\gamma^2N^2e^6r^2}{3m_0^2c^4LR^2}\,, \qquad (58)$$

and the fraction of the particle's energy, $U = \gamma m_0\, c^2$, that is radiated away during the bunch crossing is, for a particle at the outer radius $r = R$ of the bunch,

$$\frac{dU}{U} \approx \frac{16\gamma N^2e^6r^2}{3m_0^3c^6LR^4} = \frac{16\gamma N^2r_0^3}{3LR^2}\,. \qquad (59)$$

For the typical operating parameters of the SLC, $N = 10^{10}$, $L = 1$ mm, $R = 1\,\mu$m, $\gamma = 10^5$, then $dU/U \approx 1.5 \times 10^{-3}$ for particles at $r = R$.

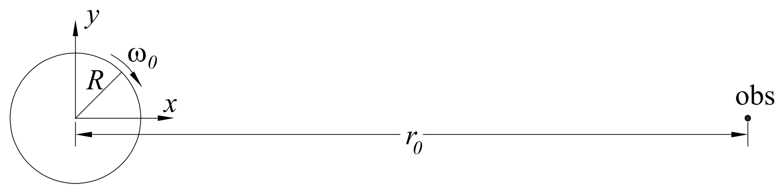
8. **Synchrotron Radiation**

This problem elaborates on discussion by Feynman in
https://www.feynmanlectures.caltech.edu/I_34.html
See also sec. 5.1 of http://kirkmcd.princeton.edu/examples/synchrad.pdf

A charge e moves in a circle of radius R about the origin in the x-y plane,

$$\begin{aligned} x &= R \sin \omega_0 t, \\ y &= R \cos \omega_0 t. \end{aligned} \tag{60}$$

We observe the radiation at $(x, y) = (r_0, 0)$ where $r_0 \gg R$.



Feynman tells us that the radiation field has y-component,

$$E_y = -\frac{e}{c^2 r_0} \frac{d^2 y(t')}{dt^2}, \tag{61}$$

where $t' = t - r(t')/c \approx t - r_0/c + (R/c) \sin \omega_0 t'$ is the retarded time. Then, noting that,

$$t \approx t' - \frac{R}{c} \sin \omega_0 t' + \frac{r_0}{c}, \qquad \frac{dt}{dt'} \approx 1 - \beta \cos \omega_0 t', \tag{62}$$

where $\beta = R\,\omega_0/c$ is the particle's velocity, we find,

$$\frac{dy(t')}{dt} = \frac{dy(t')}{dt'} \frac{dt'}{dt} = -\omega_0 R \sin \omega_0 t' \frac{dt'}{dt} = -\frac{\omega_0 R \sin \omega_0 t'}{1 - \beta \cos \omega_0 t'}, \tag{63}$$

$$\begin{aligned} \frac{d^2 y(t')}{dt^2} &= \frac{d}{dt'} \left(\frac{dy(t')}{dt} \right) \frac{dt'}{dt} = -\frac{\omega_0^2 R \cos \omega_0 t'}{1 - \beta \cos \omega_0 t'} \frac{dt'}{dt} - + \frac{\omega_0 R \sin \omega_0 t'}{(1 - \beta \cos \omega_0 t')^2} \beta\, \omega_0 \sin \omega_0 t' \frac{dt'}{dt} \\ &= \frac{\omega_0^2 R}{(1 - \beta \cos \omega_0 t')^3} \left[-\cos \omega_0 t' (1 - \beta \cos \omega_0 t') + \beta (1 - \cos^2 \omega_0 t') \right] \\ &= \omega_0^2 R \frac{\beta - \cos \omega_0 t'}{(1 - \beta \cos \omega_0 t')^3}. \end{aligned} \tag{64}$$

The radiation is big only for $\omega_0 t' \approx 2n\pi$. We will make a Fourier analysis of only the pulse near $t' = 0$. For this, we eliminate t' in favor of $T = t - r_0/c$. For $\beta \approx 1$ we find,

$$t' \approx T + \frac{R}{c} \sin \omega_0 t' \approx T + \frac{R}{c} \omega_0 t' = T + \beta t', \qquad t' \approx \frac{T}{1 - \beta} = T \frac{1 + \beta}{1 - \beta^2} \approx 2\gamma^2 T, \tag{65}$$

$$\cos \omega_0 t' - \beta \approx 1 - \frac{\omega_0^2 (2\gamma^2 T)^2}{2} - \beta = \frac{1 - \beta^2}{1 + \beta} - \frac{4\omega^2 \gamma^4 T^2}{2} \approx \frac{1 - 4\gamma^6 \omega_0^2 T^2}{2\gamma^2}, \tag{66}$$

$$1 - \beta \cos \omega_0 t' \approx 1 - \beta \left(1 - \frac{\omega_0^2 (2\gamma^2 T)^2}{2} \right) = \frac{1 - \beta^2}{1 + \beta} + \frac{4\omega^2 \gamma^4 T^2}{2} \approx \frac{1 + 4\gamma^6 \omega_0^2 T^2}{2\gamma^2}, \tag{67}$$

and hence,

$$E_y(T) \propto \frac{\beta - \cos\omega_0 t'}{(1-\beta\cos\omega_0 t')^3} \propto \frac{1 - 4\gamma^6\omega_0^2 T^2}{1 + 12\gamma^6\omega_0^2 T^2}. \tag{68}$$

Using 3.767.1-2, p. 423 of http://kirkmcd.princeton.edu/examples/EM/gradshteyn_80.pdf, the Fourier transform of this varies as,

$$E_y(\omega) \propto e^{-\omega/2\sqrt{3}\gamma^3\omega_0}, \tag{69}$$

noting that Gradshteyn's a is our ω, and his γ is our $1/\sqrt{12}\,\gamma^3\,\omega_0$. The power spectrum of the pulse, U_ω, goes as,

$$U_\omega \propto E_y^2(\omega) \propto e^{-\omega/\sqrt{3}\gamma^3\omega_0} \equiv e^{-\omega/\omega_C}, \tag{70}$$

where the critical frequency is,

$$\omega_C = \sqrt{3}\gamma^3\omega_0. \tag{71}$$

The critical frequency (71) can be anticipated by a simpler argument.

First, we note that the acceleration is perpendicular to the velocity, so the angular distribution of the radiation follows from Prob. 2 above as

$$\frac{dU}{dt\,d\Omega} = \frac{e^2\gamma^4\beta^2 c}{4\pi\rho^2}\frac{(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}, \tag{72}$$

where $\beta = v/c$ and angles (θ, ϕ) are measured with respect to the direction of the electron's motion and with the x-axis towards the center of the electron's orbit.

For highly relativistic motion, $\gamma \gg 1$, the radiation is peaked forward with characteristic angle $\theta \approx 1/\gamma$. Then, $1 - \beta\cos\theta \approx (\theta^2 + 1/\gamma^2)/2$. In the plane of the orbit we have,

$$\frac{dU}{dt\,d\Omega} \approx \frac{2e^2\gamma^{10}c}{\pi\rho^2}\frac{[1-(\gamma\theta)^2]^2}{[1+(\gamma\theta)^2]^5}. \tag{73}$$

The "cone" of synchrotron radiation passes over a fixed angle θ in (retarded) time,

$$\Delta t' \approx \frac{\Delta\theta}{\omega_0} = \frac{1}{\gamma\omega_0}, \tag{74}$$

where for $\gamma \gg 1$ the width of the angular distribution is $\Delta\theta \approx 1/\gamma$.

For a distant observer, the corresponding time interval Δt is related by the usual transformation for retarded time,

$$\Delta t = \Delta t'(1 - \boldsymbol{\beta}\cdot\hat{\mathbf{r}}) \approx \Delta t'(1-\beta) \approx \frac{\Delta t'}{2\gamma^2} = \frac{1}{2\gamma^3\omega_0}. \tag{75}$$

Then, a Fourier analysis of the pulse of width Δt will have spectral width $\Delta\omega \approx 1/\Delta t = 2\gamma^3\omega_0$. The spectrum peaks near $\omega = \gamma^3\omega_0$, which provides an estimate of the critical frequency.

The radiation is strong at high harmonics of the orbital frequency and can be regarded as a continuum spectrum.

Detailed analysis of the frequency spectrum leads one to define the "critical frequency" as (eq. II.17 of http://kirkmcd.princeton.edu/examples/EM/schwinger_pr_75_1912_49.pdf),

$$\omega_C \equiv \frac{3}{2}\gamma^3\omega_0 = \frac{3}{2}\frac{\gamma^3 c}{R}\,, \tag{76}$$

although the spectrum peaks very close to $\omega = \gamma^3 c/R$.

Fourier analysis of the pulse train, rather than only a single pulse as above, show that at low frequencies $U_\omega \propto \omega^{1/3}$ while at high frequencies $U_\omega \propto \sqrt{\omega}\, e^{-\omega/\sqrt{3}\gamma^3\omega_0}$. These limiting features of the frequency spectrum can also be deduced via the so-called method of virtual quanta (pioneered by Fermi, Weizsäcker and Williams), as reviewed in sec. 2 of http://kirkmcd.princeton.edu/examples/weizsacker.pdf.

9. Charge e_1 of mass m and initial velocity $v_0 \ll c$ makes a head-on collision with charge e_2, of the same sign as e_1, where charge 2 is fixed at the origin. According to eq. (9), the power radiated by charge 1 due to its interaction with charge 2 is,

$$\frac{dU}{dt} = \frac{2e_1^4\gamma^2}{3m^2c^3}\left[\left(\mathbf{E}_2 + \frac{\mathbf{v}}{c}\times\mathbf{B}_2\right)^2 - \left(\mathbf{E}_2\cdot\frac{\mathbf{v}}{c}\right)^2\right] \approx \frac{2e_1^4e_2^2}{3m^2c^3r^4}, \tag{77}$$

since $v < v_0 \ll c$, where r is the distance between the two charges.

Conservation of energy tells us that,

$$\frac{mv_0^2}{2} = \frac{mv(r)^2}{2} + \frac{e_1e_2}{r}. \tag{78}$$

Then, the total energy radiated is, noting that the force on charge 1 is $F = e_1e_2/r^2 = ma$,

$$\Delta U = \int \frac{dU}{dt}\,dt = \int \frac{2e_1^4e_2^2}{3m^2c^3r^4}\,dt = \int \frac{2e_1^3e_2}{3m^2c^3r^2}F\,dt = \int \frac{2e_1^3e_2}{3m^2c^3r^2}ma\,dt$$

$$= 2\int_0^{v_0} \frac{2e_1^3e_2}{3mc^3}\left(\frac{m(v_0^2 - v^2)}{2e_1e_2}\right)^2 dv = \frac{m}{3c^3}\frac{e_1}{e_2}\left(v_0^5 - \frac{2v_0^5}{3} + \frac{v_0^5}{5}\right) = \frac{16}{45}\frac{e_1}{e_2}\left(\frac{v_0}{c}\right)^3\frac{mv_0^2}{2}. \tag{79}$$

$$\frac{\Delta U}{U_0} = \frac{16}{45}\frac{e_1}{e_2}\left(\frac{v_0}{c}\right)^3. \tag{80}$$

That $\Delta U/U_0 \propto (v_0/c)^3$ is characteristic of electric dipole radiation for $v_0 \ll c$, as seen in Prob. 8b, Set 8, http://kirkmcd.princeton.edu/examples/ph501set8.pdf