Princeton University

Ph501
Electrodynamics
Problem Set 11

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1. **Čerenkov Radiation by a Neutron**

A neutron has no electric charge, but it does have a magnetic moment $\mathbf{m}$. Hence, we can expect an accelerated neutron to emit radiation. Here, we ask whether a neutron will emit Čerenkov radiation when traveling inside a dielectric medium with uniform velocity $v > c/n$, where c is the speed of light and n is the index of refraction of the medium?

Towards answering this, consider the spectrum of energy *vs.* angular frequency ω and solid angle Ω of a pulse of radiation due to electric charge e with time-dependent velocity $\mathbf{v}$, p. 250, Lecture 21 of the Notes,
http://kirkmcd.princeton.edu/examples/ph501/ph501lecture21.pdf,

$$\frac{dU_\omega}{d\Omega} = \frac{e^2\omega^2 n}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \boldsymbol{\beta})\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, dt\right|^2 = \frac{\omega^2 n}{4\pi^2 c}\left|\int_{-\infty}^{\infty} e\boldsymbol{\beta} \times \hat{\mathbf{k}}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, dt\right|^2, \tag{1}$$

where $\mathbf{k}$ is the wave vector with $k = n\omega/c$ in case of a medium with index of refraction n,[1] and hence $\hat{\mathbf{k}} = \hat{\mathbf{n}}$ is the unit vector pointing to the observer. Also, $\boldsymbol{\beta} = \mathbf{v}/c$.

A magnetic moment may be thought of as an electric-current loop, so we need a version of eq. (1) for a current rather than an electric charge, For this we note that for a moving charge,[2]

$$e\,\boldsymbol{\beta} \to \frac{\mathbf{J}}{c}\, d\mathrm{Vol}, \tag{2}$$

and hence,[3]

$$\frac{dU_\omega}{d\Omega} = \frac{\omega^2 n}{4\pi^2 c^3}\left|\int\int \mathbf{J} \times \hat{\mathbf{k}}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, dt\, d\mathrm{Vol}\right|^2. \tag{3}$$

For the neutron, relate the current to the magnetic moment by $\mathbf{J} = c\boldsymbol{\nabla} \times \mathbf{m}$ (Lecture 8) and suppose that $\mathbf{m}(\mathbf{r}, t) = m_0\,\hat{\mathbf{z}}\,\delta(x)\,\delta(y)\,\delta(z - vt)$.[4]

Evaluate $dU_\omega/d\Omega$ for a neutron moving in a medium of index of refraction n to show that,

$$\frac{dU_\omega/d\Omega|_{\text{moving neutron}}}{dU_\omega/d\Omega|_{\text{moving charge } e}} = \frac{c^2}{v^2}\frac{k^2 m_0^2}{e^2} = \frac{c^2}{v^2}\frac{m_0^2}{e^2 \lambda\!\!\!^{-2}}. \tag{4}$$

[1]Note that the index n in eq. (1) is the index of the medium in which the radiation is observed.

[2]For a point charge e at the origin is its rest frame we can write its charge density as $\rho^\star = e\,\delta(x^\star)\,\delta(y^\star)\,\delta(z^\star)$, and, of course, the current density is zero, $\mathbf{J}^\star = 0$. In a frame where the charge has velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, the charge and current densities follow from the Lorentz transformations, $\rho = \gamma\rho^\star = \gamma e\,\delta(x^\star)\,\delta(y^\star)\,\delta(z^\star) = \gamma e\,\delta(x)\,\delta(y)\,\delta(\gamma(z - vt)) = e\,\delta(x)\,\delta(y)\,\delta(z - vt)$, and $\mathbf{J} = \gamma\rho^\star v\,\hat{\mathbf{z}} = \gamma e\,\delta(x)\,\delta(y)\,\delta(\gamma(z - vt))v\,\hat{\mathbf{z}} = e\,\delta(x)\,\delta(y)\,\delta(z - vt)v\,\hat{\mathbf{z}} = \rho v\,\hat{\mathbf{z}}$, noting that $\delta(\gamma z) = \delta(z)/\gamma$ since $\int f(z)\,\delta(\gamma z)\, dz = \int (f(z)/\gamma)\,\delta(\gamma z)\, d(\gamma z) = f(0)/\gamma$. That is, while an extended charge density is enhanced by the Lorentz contraction in a frame where the density is in motion, a point charge is not subject to the Lorentz contraction.

[3]Equation (3) also follows from p. 182, Lecture 15 of the Notes, since $|\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{J})| = |\mathbf{J} \times \hat{\mathbf{n}}| = |\mathbf{J} \times \hat{\mathbf{k}}|$.

[4]If the magnetic moment has a component perpendicular to its velocity, then it appears to have an electric-dipole moment as well, which also contributes to Čerenkov radiation. This contribution is different for a magnetic moment due to electric currents and one due to opposite magnetic charges (monopoles).

The earliest computations of Čerenkov radiation by neutrons assumed the latter, while it is now believed that the former assumption is more appropriate. See, for example, G.N. Afanasiev and Y.P. Stepanovsky, Phys. Scripta **61**, 704 (2000), http://kirkmcd.princeton.edu/examples/EM/afanasiev_ps_61_704_00.pdf

Hint: Integrate by parts to absorb the $\boldsymbol{\nabla}$.

Since $m_{\rm neutron} \approx e\hbar/Mc = e\lambdabar_{\rm neutron}$, where $\lambdabar_{\rm neutron}$ is the Compton wavelength of the neutron, the ratio is approximately $\lambdabar^2_{\rm neutron}/\lambdabar$ for the Čerenkov radiation at reduced wavelength $\lambdabar = \lambda/2\pi = 1/k$. That is, Čerenkov radiation by a neutron is an extremely weak effect.

2. **Transition Radiation at a Metal-Vacuum Interface**

   A particle of charge e with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ passes through a metallic beam window at $z = 0$ and emerges into vacuum for $z > 0$. What is the frequency-angle spectrum of the radiation in the region $z > 0$, assuming that the beam window is perfectly conducting and an infinite sheet?

   Hint: Consider an image-charge method.

   Ans: The frequency spectrum, on integrating the frequency-angle spectrum over solid angle, is,

$$U_\omega = \frac{e^2}{\pi c}\left[\frac{1+\beta^2}{\beta}\ln\gamma(1+\beta) - 1\right], \tag{5}$$

   where $\beta = v/c$, *and* $\gamma = 1/\sqrt{1-\beta^2}$.

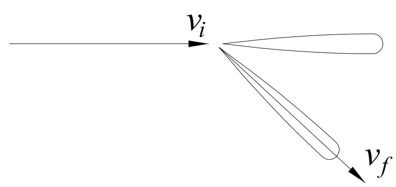
3. a) **Bremsstrahlung Revisited**

A charge e with initial velocity $\mathbf{v}_i$ experiences a brief acceleration, during time interval Δt, which leaves it with final velocity $\mathbf{v}_f$. Show that the frequency-angle spectrum of the radiation is,

$$\frac{dU_\omega}{d\Omega} = \frac{e^2}{4\pi^2 c^3}\left[\frac{\hat{\mathbf{k}} \times \mathbf{v}_i}{1 - \hat{\mathbf{k}} \cdot \mathbf{v}_i/c} - \frac{\hat{\mathbf{k}} \times \mathbf{v}_f}{1 - \hat{\mathbf{k}} \cdot \mathbf{v}_i/c}\right]^2, \tag{6}$$

at least for angular frequencies small compared to $1/\Delta t$.

This behavior is independent of frequency, as discussed in Lecture 20 of the Notes. As $v \to c$, the form (6) indicates that the angular distribution has two peaks, around the directions of the initial and final velocities.

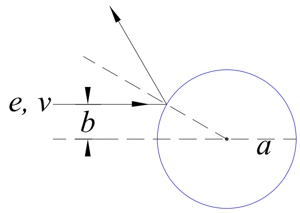


Dividing eq. (6) by $\hbar$, we obtain $dN_\omega/d\Omega$, the frequency-angle distribution of Bremsstrahlung photons, which result is essentially unchanged in quantum electrodynamics.

b) **Neutron Decay**

A free neutron decays into a proton + electron + "something else", with a half life of about 15 minutes. In a classical theory, the "something else" might be a kind of Bremsstrahlung radiation. If so, what is the angle-frequency spectrum $dU_\omega/d\Omega$ of the radiation, and its integral U_ω, approximating the final-state proton as being at rest (for an initial neutron at rest), and the electron as ejected with velocity $\mathbf{v}$?

4. Compute the angle-frequency spectrum $dU_\omega(b,\phi)/d\Omega$ for electromagnetic radiation emitted when an electron of charge e and velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, where $v \ll c$, makes an elastic collision with a hard, transparent sphere of radius a (centered on the origin) at impact parameter b and azimuthal angle ϕ relative to the observer (in the x-z plane, at angle θ to the z axis).



Since U_ω is the energy radiated into unit interval of angular frequency ω, dividing this by the photon energy $\hbar\omega$ gives the number spectrum, N_ω, of photons per unit integral of ω. Dividing this spectrum by $\hbar$ gives the number spectrum, $N_{\hbar\omega}$, of photons per unit interval of photon energy. Then, you can convert the differential spectrum $dN_{\hbar\omega}/d\Omega$ into a kind of differential cross section (with dimensions of area) by integrating over impact parameter b from 0 to a, and over azimuthal angle ϕ (of the incoming electron),

$$\frac{d\sigma}{d\Omega\, d\,\hbar\omega} = \frac{1}{\hbar^2\omega}\int_0^a b\,db\int_0^{2\pi} d\phi\,\frac{dU_\omega(b,\phi)}{d\Omega}\,. \tag{7}$$

Integrate this over solid angle to show that,

$$\frac{d\sigma}{d\,\hbar\omega} = \frac{4\alpha\beta^2 a^2}{3\hbar\omega}\,, \tag{8}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant, and $\beta = v/c$.

5. A particle with electric charge e and rest mass m_0 moves in a plane perpendicular to a uniform magnetic field $\mathbf{B}$, radiating energy and losing velocity, such that its trajectory is an inward spiral. Suppose the spiral is nearly circular at all times, so that it is a good approximation that $\mathbf{a} \perp \mathbf{v}$. Show that the energy loss dU/dt can be integrated to give,

$$\frac{U}{m_0 c^2} = \coth\left[\frac{2e^4 B^2 t}{3m_0^3 c^5} + \text{const}\right]. \tag{9}$$

Thus, it takes forever for the particle's kinetic energy to be radiated away, and $U \to m_0 c^2$.[5]

Note that you must give a relativistic derivation.

[5]This contrasts with the case of a charge in a circular orbit about another, fixed charge, for which the lifetime of the orbit is finite. See, for example, Prob. 8, of Ph501 Set 8,
http://kirkmcd.princeton.edu/examples/ph501set8.pdf, or http://kirkmcd.princeton.edu/examples/orbitdecay.pdf

6. **Nuclear Numerology**

On p. 226, Lecture 19 of the Notes, we mentioned the (attractive) Yukawa potential, $\phi = g\, e^{-\mu r}/r$, as the potential of the force field between nucleons in nuclei.

Consider a single proton, of "nuclear charge" g. Suppose we attribute all of the proton's rest energy, $m_p c^2$, where c is the speed of light in vacuum and m_p is the rest mass of the proton, to the energy of its nuclear force field.[6] What would this mass be if the proton were a spherical shell of radius a_p?

Hint: $U = \int \rho\phi\, d\text{Vol}/2$ *still holds, where* ρ = *density of nuclear charge.*

Relate ρ *to* ϕ *via an appropriate generalization of Poisson's equation,* $\nabla^2\phi = -4\pi\rho$, *where* ρ *is the volume density of charge.*

You should find that $U = \int [(\boldsymbol{\nabla}\phi)^2 + \mu^2\phi^2]\, d\text{Vol}/8\pi$.

In quantum theory, the coupling constants $e^2/\hbar c = \alpha \approx 1/137$ and $g^2/\hbar c$ play important roles. Given that $m_p/m_e = 1836$, estimate the pion-nucleon coupling constant $g^2/\hbar c$ supposing the proton is a spherical shell of nuclear charge of radius $a_p = 0.86 \times 10^{-13}$ cm, and the electron is a spherical shell of electric charge of radius such that all of the electron's rest mass, m_e, is electromagnetic.

This estimate agrees fairly well with experiment. Is this physics or numerology?

[6]The mass of the neutron is about 0.14% higher than the mass of the proton, which suggests that the energy of the proton's electromagnetic field does not contribute significantly to its mass.

7. **Stress and Momentum in a Capacitor That Moves with Constant Velocity**

Consider a parallel-plate capacitor whose plates are held apart by a nonconducting slab of unit (relative) dielectric constant and unit (relative) magnetic permeability.[7] Discuss the energy, momentum and stress in this (isolated) system when at rest and when moving with constant velocity parallel or perpendicular to the electric field.

Does the system contain hidden momentum, $\mathbf{P}_{\rm hidden}$, defined for a subsystem by,

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area}, \qquad (10)$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass," U is its total energy, c is the speed of light in vacuum, $\mathbf{x}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{x}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, and $\mathbf{v}_b$ is the velocity (field) of its boundary?[8]

Fringe-field effects can be ignored. The velocity can be large or small compared to the speed of light.

[7]The use of unit dielectric constant and unit permeability avoids entering into the interesting controversy as to the so-called Abraham and Minkowski forms of the energy-momentum-stress tensor, http://kirkmcd.princeton.edu/examples/ambib.pdf.

[8]The definition (10) was suggested by Daniel Vanzella. See also, http://kirkmcd.princeton.edu/examples/hiddendef.pdf.

8. **Radiation by a Superluminal Source**

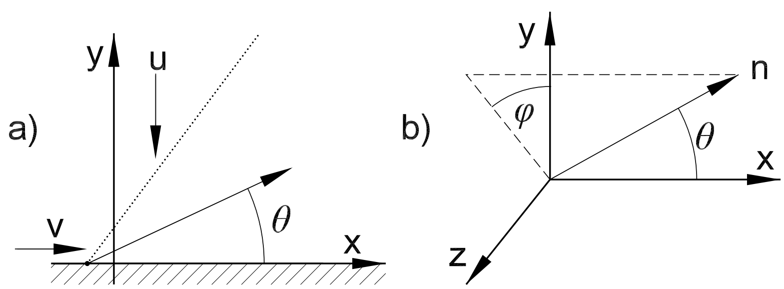
Exotic radiation effects of charges that move at (essentially) constant velocity but cross boundaries between various media can be deduced from the angle-frequency spectrum of radiation, p. 182, Lecture 15 of the Notes,[9]

$$\frac{dU}{d\omega\, d\Omega} = \frac{\omega^2}{4\pi^2 c^3} \left[\int\int dt\, d^3\mathbf{r}\ \hat{\mathbf{n}} \times \mathbf{J}(\mathbf{r}, t)\, e^{i\omega(t-(\hat{\mathbf{n}}\cdot\mathbf{r})/c)} \right]^2, \tag{11}$$

where dU is the radiated energy in angular frequency interval $d\omega$ emitting into solid angle $d\Omega$, $\mathbf{J}$ is the source current density, and $\hat{\mathbf{n}}$ is a unit vector towards the observer.

Consider the example of the sweeping electron beam in an (analog) oscilloscope. In the fastest of such devices (such as the Tektronix model 7104) the speed of the beam spot across the face of an oscilloscope can exceed the velocity of light, although of course the velocity of the electrons does not. Associated with this possibility there should be a kind of Čerenkov radiation, as if the oscilloscope trace were due to a charge moving with superluminal velocity.

As a simple model, suppose a line of charge moves in the $-y$ direction with velocity $u \ll c$, where c is the speed of light, but has a slope such that the intercept with the x axis moves with velocity $v > c$, as shown in the figure below. If the region $y < 0$ is occupied by, say, a metal the charges will emit transition radiation as they disappear into the metal's surface. Interference among the radiation from the various charges then leads to a strong peak in the radiation pattern at angle $\cos\theta = c/v$, which is the Čerenkov effect of the superluminal source – all of which can be deduced from eq. (11).



a) A sloping line of charge moves in the $-y$ direction with velocity $v_y = u \ll c$ such that its intercept with the x axis moves with velocity $v_x = v > c$. As the charge disappears into the conductor at $y < 0$ it emits transition radiation. The radiation appears to emanate from a spot moving at superluminal velocity and is concentrated on a cone of angle $\cos^{-1}(c/v)$.
b) The angular distribution of the radiation is discussed in a spherical coordinates system about the x-axis.

[9] See also eq. (14.70) of http://kirkmcd.princeton.edu/examples/EM/jackson_ce2_75.pdf

Solutions

1. **Čerenkov Radiation by a Neutron**

We compute the frequency-angle spectrum of Čerenkov radiation by a neutron moving with its magnetic moment $\mathbf{m}$ parallel to its uniform velocity $v\,\hat{\mathbf{z}}$ in a medium of index of refraction n using eq. (3), taking the current density to be,

$$\mathbf{J} = c\boldsymbol{\nabla}\times\mathbf{m} = c\boldsymbol{\nabla}\times m_0\,\hat{\mathbf{z}}\,\delta(x)\,\delta(y)\,\delta(z-vt). \tag{12}$$

Hence,

$$\begin{aligned}
\frac{dU_\omega}{d\Omega} &= \frac{\omega^2 n}{4\pi^2 c^3}\left|\int\int \mathbf{J}\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 n}{4\pi^2 c}\left|\int\int (\boldsymbol{\nabla}\times\mathbf{m})\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 n}{4\pi^2 c}\left|-\int\int (-i\mathbf{k}\times\mathbf{m})\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 k^2 n}{4\pi^2 c}\left|\int\int (\hat{\mathbf{n}}\times\mathbf{m})\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 k^2 n}{4\pi^2 c}\left|\int\int m_0\sin\theta\,\delta(x)\,\delta(y)\,\delta(z-vt)\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2 \\
&= \frac{\omega^2 k^2 n m_0^2\sin^2\theta}{4\pi^2 c}\left|\int e^{i\omega t(1-(nv/c)\cos\theta)}\,dt\right|^2,
\end{aligned} \tag{13}$$

where the third line follows from the second via integration by parts with respect to volume, while in the sixth line we note the $k = n\omega/c$ for waves in a medium of index n, and we take θ as the angle between $\hat{\mathbf{n}}$ and the z-axis.

The remaining integral is the same as in an intermediate step of the computation for Čerenkov radiation by electric charge e with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, as on the top of p. 251, Lecture 21 of the Notes, where the prefactor is $e^2\omega^2 n v^2\sin^2\theta/4\pi^2 c^3$. Hence,

$$\frac{dU_\omega/d\Omega|_{\mathrm{neutron}}}{dU_\omega/d\Omega|_{\mathrm{charge}\ e}} = \frac{c^2}{v^2}\frac{k^2 m_0^2}{e^2} = \frac{c^2}{v^2}\frac{m_0^2}{e^2\lambda\!\!\!^{-2}}\,. \tag{14}$$

Since $m_{\mathrm{neutron}} \approx e\hbar/Mc = e\lambda\!\!\!^{-}_{\mathrm{neutron}}$, where $\lambda\!\!\!^{-}_{\mathrm{neutron}}$ is the Compton wavelength of the neutron, the ratio is approximately $\lambda\!\!\!^{-2}_{\mathrm{neutron}}/\lambda\!\!\!^{-2}$ for the Čerenkov radiation at reduced wavelength $\lambda\!\!\!^{-} = \lambda/2\pi = 1/k$. That is, Čerenkov radiation by a neutron is an extremely weak effect.

For another exotic Čerenkov effect, see U. Leonhardt and Y. Rosenberg, *Cherenkov radiation of light bullets*, Phys. Rev. A **100**, 063802 (2019),
http://kirkmcd.princeton.edu/examples/EM/leonhardt_pra_100_063802_19.pdf

The author has made an experimental demonstration of the interference between Čerenkov radiation and synchrotron radiation by a relativistic electron moving on

a circular path in a gas, K.D. Bonin *et al.*, Phys. Rev. Lett. **57**, 2264 (1986),
http://kirkmcd.princeton.edu/examples/EM/bonin_prl_57_2264_86.pdf

Addendum 1: Charge and Current Densities for General $\mathbf{p}_0$ and $\mathbf{m}_0$

As reviewed in http://kirkmcd.princeton.edu/examples/movingdipole.pdf, the Lorentz transformation of electric and magnetic polarization densities, **P** and **M**, from their rest frame (the * frame) to a frame in which they have velocity **v** is,

$$\mathbf{P} = \gamma\left(\mathbf{P}^\star + \frac{\mathbf{v}}{c}\times\mathbf{M}^\star\right) - (\gamma-1)(\hat{\mathbf{v}}\cdot\mathbf{P}^\star)\,\hat{\mathbf{v}}, \tag{15}$$

$$\mathbf{M} = \gamma\left(\mathbf{M}^\star - \frac{\mathbf{v}}{c}\times\mathbf{P}^\star\right) - (\gamma-1)(\hat{\mathbf{v}}\cdot\mathbf{P}^\star)\,\hat{\mathbf{v}}, \tag{16}$$

where c is the speed of light and $\gamma = 1/\sqrt{1-v^2/c^2}$.

For a "point" particle at the origin in its rest frame, with rest-frame electric and magnetic dipole moments $\mathbf{p}_0$ and $\mathbf{m}_0$, we write its electric and magnetic polarization densities as,

$$\mathbf{P}^\star = \mathbf{p}_0\,\delta^3\mathbf{r}^\star, \qquad \text{and} \qquad \mathbf{M}^\star = \mathbf{m}_0\,\delta^3\mathbf{r}^\star, \tag{17}$$

and the associated charge and current densities as,

$$\rho^\star = -\boldsymbol{\nabla}^\star\cdot\mathbf{P}^\star = -(\mathbf{p}_0\cdot\boldsymbol{\nabla}^\star)\,\delta^3\mathbf{r}^\star, \qquad \text{and} \qquad \mathbf{J}^\star = c\boldsymbol{\nabla}^\star\times\mathbf{M}^\star = -c\,\mathbf{m}_0\times\boldsymbol{\nabla}^\star\,\delta^3\mathbf{r}^\star. \tag{18}$$

In a frame where the particle has velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, the charge and current densities are,

$$\rho = \gamma\left(\rho^\star + (\mathbf{J}^\star\cdot\hat{\mathbf{v}})\,\frac{\mathbf{v}}{c^2}\right), \qquad \mathbf{J} = \mathbf{J}^\star + (\gamma-1)(\mathbf{J}^\star\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} + \gamma\rho^\star\mathbf{v}. \tag{19}$$

We note that the Lorentz transformation of the 4-gradient $\partial_\mu = (\partial_t/c, -\boldsymbol{\nabla})$ tells us that $-\partial_x^\star = -\partial_x$, $-\partial_y^\star = -\partial_y$ and $-\partial_z^\star = \gamma(-\partial_z - (v/c)\partial_t/c)$, *i.e.*,

$$\boldsymbol{\nabla}^\star = \boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\,. \tag{20}$$

Hence, eqs. (18) and (19) combine to give,

$$\begin{aligned}\rho = \gamma\Bigg(&-\mathbf{p}_0\cdot\left[\boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(\gamma(z-vt))\\ &-\frac{\mathbf{v}}{c}\hat{\mathbf{v}}\cdot\mathbf{m}_0\times\left[\boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(\gamma(z-vt))\Bigg)\\ &= -\mathbf{p}_0\cdot\left[\boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(z-vt)\\ &-\frac{\mathbf{v}}{c}\hat{\mathbf{v}}\cdot\mathbf{m}_0\times\left[\boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(z-vt),\end{aligned} \tag{21}$$

and,

$$\mathbf{J} = -c\,\mathbf{m}_0 \times \left[\boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(\gamma(z-vt))$$
$$-(\gamma-1)c\,\hat{\mathbf{v}}\left(\hat{\mathbf{v}}\cdot\mathbf{m}_0 \times \left[\boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(\gamma(z-vt))\right)$$
$$-\gamma\mathbf{v}\left(\mathbf{p}_0\cdot\left[\boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(\gamma(z-vt))\right)$$
$$= -c\,\mathbf{m}_0 \times \left[\frac{\boldsymbol{\nabla}}{\gamma} + \frac{\gamma-1}{\gamma}\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(z-vt)$$
$$-\frac{\gamma-1}{\gamma}c\,\hat{\mathbf{v}}\left(\hat{\mathbf{v}}\cdot\mathbf{m}_0 \times \left[\boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(z-vt)\right)$$
$$-\mathbf{v}\left(\mathbf{p}_0\cdot\left[\boldsymbol{\nabla} + (\gamma-1)\,\hat{\mathbf{v}}\,(\hat{\mathbf{v}}\cdot\boldsymbol{\nabla}) + \gamma\beta\,\hat{\mathbf{v}}\frac{\partial}{\partial ct}\right]\,\delta(x)\,\delta(y)\,\delta(z-vt)\right). \tag{22}$$

These expressions are somewhat simpler for the special cases that $\mathbf{p}_0 \perp \mathbf{v}$ and $\mathbf{m}_0 \parallel \mathbf{v}$,

$$\rho = -\left((\mathbf{p}_0\cdot\boldsymbol{\nabla}) + \frac{\mathbf{v}}{c}\,\hat{\mathbf{v}}\cdot\mathbf{m}_0\times\boldsymbol{\nabla}\right)\,\delta(x)\,\delta(y)\,\delta(z-vt), \tag{23}$$
$$\mathbf{J} = -\Big(\mathbf{v}\,(\mathbf{p}_0\cdot\boldsymbol{\nabla}) + c\,\mathbf{m}_0\times\boldsymbol{\nabla}\Big)\,\delta(x)\,\delta(y)\,\delta(z-vt). \tag{24}$$

Equation (12) then follows from eq. (24) when $\mathbf{p}_0 = 0$, and eq. (25) follows when $\mathbf{m}_0 = 0$.

Addendum 2: Čerenkov Radiation by a Moving Point Electric Dipole

We now consider an electrically neutral particle with electric-dipole moment $\mathbf{p}_0$ in its rest frame, and zero magnetic moment there. For simplicity, we also suppose $\mathbf{p}_0$ to be perpendicular to the lab-frame velocity $\mathbf{v}$, where $v > c/n$ in the medium of index of refraction n. Then, from eq. (24), the lab-frame current density is,

$$\mathbf{J} = -\mathbf{v}\,(\mathbf{p}_0\cdot\boldsymbol{\nabla})\,\delta(x)\,\delta(y)\,\delta(z-vt) = \rho\mathbf{v}, \tag{25}$$

and the frequency-angle spectrum of the radiation is given by,

$$\frac{dU_\omega}{d\Omega} = \frac{\omega^2 n}{4\pi c^3}\left|\int\int \mathbf{J}\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2$$
$$= \frac{\omega^2 n}{4\pi^2 c^3}\left|\int\int \mathbf{v}\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}(\mathbf{p}_0\cdot\boldsymbol{\nabla})\,\delta(x)\,\delta(y)\,\delta(z-vt)\,dt\,d\mathrm{Vol}\right|^2$$
$$= \frac{\omega^2 n}{4\pi^2 c^3}\left|-\int\int \mathbf{v}\times\hat{\mathbf{k}}\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,(\mathbf{p}_0\cdot -i\mathbf{k})\,\delta(x)\,\delta(y)\,\delta(z-vt)\,dt\,d\mathrm{Vol}\right|^2$$
$$= \frac{\omega^2 k^2 n p_0^2 v^2\sin^4\theta}{4\pi^2 c^3}\left|\int \delta(x)\,\delta(y)\,\delta(z-vt)\,e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\,dt\,d\mathrm{Vol}\right|^2$$
$$= \frac{\omega^2 k^2 n p_0^2 v^2\sin^4\theta}{4\pi^2 c^3}\left|\int e^{i\omega t(1-(nv/c)\cos\theta)}\,dt\right|^2, \tag{26}$$

where the third line follows from the second via integration by parts with respect to volume, and we take θ as the angle between $\hat{\mathbf{n}}$ and the z-axis.

The remaining integral is the same as in an intermediate step of the computation for Čerenkov radiation by electric charge e with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, as on the top of p. 251, Lecture 21 of the Notes, where the prefactor is $e^2\omega^2 n v^2 \sin^2\theta/4\pi^2 c^3$. Hence,

$$\frac{dU_\omega/d\Omega|_{\text{moving electric dipole}}}{dU_\omega/d\Omega|_{\text{moving charge } e}} = \frac{k^2 p_0^2 \sin^2\theta}{e^2} < \frac{p_0^2}{e^2 \lambdabar^2}\,, \tag{27}$$

where the Čerenkov angle θ is related by $\cos\theta = c/nv$.

The electric dipole $\mathbf{p}_0$ might be that of an atom, in which case $p_0 \approx e R_{\text{Bohr}}$, where the Bohr radius is $R_{\text{Bohr}} \approx 5\times 10^{-11}$ m, and the ratio (27) would be $\approx (R_{\text{Bohr}}/\lambdabar)^2 \approx 2\times 10^{-7}$ for $\lambda = 600$ nm, *i.e.*, $\lambdabar \approx 10^{-7}$ m. While the Čerenkov radiation by such an electric dipole is strong compare to that of a neutron, it is very weak compared to that of an electron.

2. **Transition Radiation at a Metal-Vacuum Interface**

A particle of charge e with velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ passes through a metallic beam window at $z = 0$ and emerges into vacuum at, say, time $t = 0$. Assuming that the beam window is perfectly conducting and an infinite sheet, the electromagnetic fields for $z > 0$ can be thought of as due to the charge e (at $z - vt$), plus an image charge $-e$ at $z = -vt$.[10] We seek to apply eq. (1) for the frequency-angle spectrum of radiation by a moving charge e with velocity $\mathbf{v}$ in vacuum (setting index n to 1). In the present problem, where the fields are due to both the charge e and its image charge $-e$, this formula becomes,[11]

$$\begin{aligned}\frac{dU_\omega}{d\Omega} &= \frac{\omega^2}{4\pi^2 c}\left|\int_0^\infty e\,\boldsymbol{\beta}_e \times \hat{\mathbf{k}}\,e^{i(\omega t - \mathbf{k}\cdot\mathbf{r}_e)}\,dt + \int_0^\infty (-e)\,\boldsymbol{\beta}_{-e} \times \hat{\mathbf{k}}\,e^{i(\omega t - \mathbf{k}\cdot\mathbf{r}_{-e})}\,dt\right|^2 \\ &= \frac{e^2\omega^2\beta^2\sin^2\theta}{4\pi^2 c}\left|\int_0^\infty e^{i\omega t(1-\beta\cos\theta)}\,dt + \int_0^\infty e^{i\omega t(1+\beta\cos\theta)}\,dt\right|^2 \\ = \frac{e^2\omega^2\beta^2\sin^2\theta}{4\pi^2 c}&\left|\frac{e^{i\omega\infty(1-\beta\cos\theta)} - 1}{i\omega(1-\beta\cos\theta)} + \frac{e^{i\omega\infty(1+\beta\cos\theta)} - 1}{i\omega(1+\beta\cos\theta)}\right|^2 = \frac{e^2\beta^2}{\pi^2 c}\frac{\sin^2\theta}{1-\beta^2\cos^2\theta}\,,\end{aligned} \tag{28}$$

where $\mathbf{k} = \omega\,\hat{\mathbf{k}}/c = \omega(\sin\theta, 0, \cos\theta)/c$ is in the direction of the radiation to the observer (located at large $z > 0$), $\hat{\mathbf{k}} = \hat{\mathbf{n}}$, $\boldsymbol{\beta}_{e,-e} = \pm v\,\hat{\mathbf{z}}/c = \pm\beta\,\hat{\mathbf{z}}$, and $\mathbf{r}_{e,-e} = \pm vt\,\hat{\mathbf{z}} = \pm\beta ct\,\hat{\mathbf{z}}$, and we take $e^{i\omega\infty(1\pm\beta\cos\theta)} = 0$, as representing the time-average of the oscillatory behavior at large times.

The frequency spectrum of the transition radiation is, noting that $0 < \theta < \pi/2$ for an observer with $z > 0$,

$$\begin{aligned} U_\omega &= \int \frac{dU_\omega}{d\Omega}\,d\Omega = \frac{2e^2}{\pi c\beta^2}\int_0^1 \frac{1-\cos^2\theta}{(1/\beta^2 - \cos^2\theta)^2}\,d\cos\theta \\ = \frac{2e^2}{\pi c\beta^2}&\left[\frac{\beta^2\cos\theta}{2(1/\beta^2-\cos^2\theta)} + \frac{\beta^3}{4}\ln\frac{1/\beta+\cos\theta}{1/\beta-\cos\theta} - \frac{\cos\theta}{2(1/\beta^2-\cos^2\theta)} + \frac{\beta}{4}\ln\frac{1/\beta+\cos\theta}{1/\beta-\cos\theta}\right]_0^1 \\ &= \frac{2e^2}{\pi c\beta^2}\left[\frac{\beta^4}{2(1-\beta^4)} + \frac{\beta^3}{4}\ln\frac{1+\beta}{1-\beta} - \frac{\beta^2}{2(1-\beta^4)} + \frac{\beta}{4}\ln\frac{1+\beta}{1-\beta}\right] \\ &= \frac{2e^2}{\pi c\beta^2}\left[-\frac{\beta^2}{2} + \frac{\beta(1+\beta^2)}{4}\ln(\gamma^2(1+\beta)^2)\right] = \frac{e^2}{\pi c}\left[\frac{1+\beta^2}{\beta}\ln\gamma(1+\beta) - 1\right], \end{aligned} \tag{29}$$

using Dwight 140.2 and 142.2, http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf, and $\gamma = 1/\sqrt{1-\beta^2}$.

---

[10] The concept of images charges is from electrostatics, where they are a representation of the effect of induced charges on the surfaces of conductors. The induced, static surface charge density on an infinite conducting sheet at $z = 0$ associated with fixed charge at $(0, 0, z)$ extends to infinity, but is significant only for $r = \sqrt{x^2+y^2} \lesssim z$ on the sheet. For a moving charge that emerges from the metal surface at $z = 0$ at time $t = 0$, with subsequent motion $z = vt$, the induced surface charge at time t is restricted to $r < ct$, which is not strictly equivalent to the static surface charge induced by a charge at rest at $z = vt$. However, use of an image charge $-e$ at $z = -vt$ for $t > 0$ also leads to fields at time t within a radius ct of the origin, so this usage seem to be a reasonable approximation, even as $v \to c$.

[11] The result (28) was first obtained by I. Frank and V. Ginsburg, J. Phys. (USSR) **9**, 353 (1945), http://kirkmcd.princeton.edu/examples/EM/frank_jpussr_9_353_45.pdf

In the relativistic limit, $\beta \to 1$,

$$U_\omega \to \frac{2e^2 \ln 2\gamma}{\pi c} \qquad (\beta \to 1). \tag{30}$$

If we had neglected the image charge in eq. (28), the result would have been,

$$\frac{dU_\omega}{d\Omega} = \frac{e^2\beta^2 \sin^2\theta}{4\pi^2 c(1-\beta\cos\theta)^2} \qquad \text{(no image charge)}. \tag{31}$$

$$U_\omega = \int \frac{dU_\omega}{d\Omega}\, d\Omega = \frac{e^2\beta^2}{2\pi c}\int_0^1 \frac{1-\cos^2\theta}{(1-\beta\cos\theta)^2}\, d\cos\theta$$

$$= \frac{e^2\beta^2}{2\pi c}\left[\frac{1}{\beta(1-\beta\cos\theta)} + \frac{1}{\beta^3}\left(1-\beta\cos\theta - 2\ln(1-\beta\cos\theta) - \frac{1}{1-\beta\cos\theta}\right)\right]_0^1$$

$$= \frac{e^2\beta^2}{2\pi c}\left[\frac{1}{1-\beta^2} + \frac{1}{\beta^3}\left(-\beta - \ln\frac{1-\beta}{1+\beta} - \frac{\beta}{1-\beta^2}\right)\right]$$

$$= \frac{e^2\beta^2}{2\pi c}\left[\frac{1}{1-\beta^2}\left(1-\frac{1}{\beta^2}\right) - \frac{1}{\beta^2} + \frac{1}{\beta^3}\ln\frac{1+\beta}{1-\beta}\right] = \frac{e^2}{2\pi c}\left[\frac{2}{\beta}\ln\gamma(1+\beta) - 1\right], \tag{32}$$

using Dwight 90.2 and 92.2. While this is significantly different from eq. (29) for small β, the transition radiation there is so weak that this hardly matters. In the relativistic limit, $\beta \to 1$, where most observations of transition radiation have been made,

$$U_\omega \to \frac{e^2 \ln 2\gamma}{\pi c} \qquad (\text{no image charge},\ \beta \to 1), \tag{33}$$

which is one half of eq. (30), so consideration of the image charge does make a notable difference.

Addendum: Transition Radiation at the Interface between Two Dielectrics

The discussion on pp. 253-256, Lecture 21 of the Notes should be modified to represent better the effects of time-dependent polarization charges near the interface ($z = 0$) between the two semi-infinite dielectric media with (relative) dielectric constants $\epsilon_1(z < 0)$ and $\epsilon_2(z > 0)$. For this, we recall the image method for dielectrics,[12] that when charge e is at $(0, 0, z)$ in medium 2, the electric field for $z > 0$ is that in vacuum due to effective charge e/ϵ_2 at $(0, 0, z)$ and an image charge $-(e/\epsilon_2)(\epsilon_1 - \epsilon_2)/(\epsilon_1 + \epsilon_2)$, while the field for $z < 0$ is that due to effective charge $2e/(\epsilon_1+\epsilon_2)$ at $(0, 0, z)$ in vacuum.

We consider the case of an observer with $z > 0$ (*i.e.*, forward radiation), such that a ray with angle θ_2 to the z-axis at the observer, if it originates with $z < 0$, makes angle θ_1 to the z-axis related by Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$, *i.e.*, $\sqrt{\epsilon_1}\sin\theta_1 = \sqrt{\epsilon_2}\sin\theta_2$. Then, the frequency-angle spectrum of Čerenkov radiation by charge e with position $\mathbf{x} = vt\,\hat{\mathbf{z}}$ and $v > c/n_{1,2}$ follows from eq. (1) as,

[12] See, for example, sec. 2.1.1 of http://kirkmcd.princeton.edu/examples/image.pdf. Conventions differ in the dielectric image method. Sec. 4.4 of http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf supposes that the image charge is not in vacuum, but in a medium with dielectric constant ϵ_2, while sec. 5.05 of http://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf supposes the image charge is in a medium of dielectric constant ϵ_1.

$$
\begin{aligned}
\frac{dU_\omega}{d\Omega} &= \frac{\omega^2 n_2}{4\pi^2 c}\left|\int_0^\infty \frac{e}{\epsilon_2}\boldsymbol{\beta}\times\hat{\mathbf{k}}_2\, e^{i\omega t(1-n_2\beta\cos\theta_2)}\,dt + \int_0^\infty -\frac{e}{\epsilon_2}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}(-\boldsymbol{\beta})\times\hat{\mathbf{k}}_2\, e^{i\omega t(1+n_2\beta\cos\theta_2)}\,dt\right. \\
&\qquad\left. + \int_{-\infty}^0 \frac{2e}{\epsilon_1+\epsilon_2}\boldsymbol{\beta}\times\hat{\mathbf{k}}_1\, e^{i\omega t(1-n_1\beta\cos\theta_1)}\,dt\right|^2 \\
&= \frac{e^2\omega^2 n_2 v^2}{4\pi^2 c^3}\left|\frac{1}{\epsilon_2}\sin\theta_2\frac{-1}{i\omega(1-n_2\beta\cos\theta_2)} + \frac{1}{\epsilon_2}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\sin\theta_2\frac{-1}{i\omega(1+n_2\beta\cos\theta_2)}\right. \\
&\qquad\left. + \frac{2}{\epsilon_1+\epsilon_2}\sin\theta_1\frac{1}{i\omega(1-n_1\beta\cos\theta_1)}\right|^2 \\
&= \frac{e^2\sqrt{\epsilon_2}v^2\sin^2\theta_2}{4\pi^2c^3\epsilon_2^2(\epsilon_1+\epsilon_2)^2}\left|\frac{2\epsilon_1+2\epsilon_2\sqrt{\epsilon_2}\beta\cos\theta_2}{1-\epsilon_2\beta^2\cos^2\theta_2} - \sqrt{\frac{\epsilon_2}{\epsilon_1}}\frac{2\epsilon_2}{1-\sqrt{\epsilon_1}\beta\sqrt{1-(\epsilon_2/\epsilon_1)\sin^2\theta_2}}\right|^2 \\
&= \frac{e^2v^2\sqrt{\epsilon_2}\sin^2\theta_2}{\pi^2c^3}\left|\frac{\sqrt{\epsilon_1}\left(\epsilon_1+\epsilon_2^{3/2}\beta\cos\theta_2\right)\left(1-\beta\sqrt{\epsilon_1-\epsilon_2\sin^2\theta_2}\right) - \epsilon_2^{3/2}\left(1-\epsilon_2\beta^2\cos^2\theta_2\right)}{\sqrt{\epsilon_1}\epsilon_2(\epsilon_1+\epsilon_2)\left(1-\epsilon_2\beta^2\cos^2\theta_2\right)\left(1-\beta\sqrt{\epsilon_1-\epsilon_2\sin^2\theta_2}\right)}\right|^2. \qquad (34)
\end{aligned}
$$

This does vanish if $\epsilon_1 = \epsilon_2$, but is not quite the same as the Ginzburg-Frank result,[13]

$$
\begin{aligned}
\frac{dU_\omega}{d\Omega} = \frac{e^2v^2\sqrt{\epsilon_2}\sin^2\theta_2\cos^2\theta_2}{\pi^2c^3}&\left|\frac{\epsilon_1-\epsilon_2}{\left(1-\epsilon_2\beta^2\cos^2\theta_2\right)\left(1-\beta\sqrt{\epsilon_1-\epsilon_2\sin^2\theta_2}\right)}\right. \\
&\left.\times\frac{\left(1-\beta^2\epsilon_2-\beta\sqrt{\epsilon_1-\epsilon_2\sin^2\theta_2}\right)}{\left(\epsilon_1\cos\theta_2+\sqrt{\epsilon_1\epsilon_2-\epsilon_2^2\sin^2\theta_2}\right)}\right|^2. \qquad (35)
\end{aligned}
$$

Transition radiation is a weak effect, significant only for high frequencies and for relativistic charges, such that the even cruder approximation presented in Lecture 21 of the Notes yields the "correct" results in these limits. See also,
http://kirkmcd.princeton.edu/examples/transition_rad.pdf

[13] See eq. (24.22) of M.I. Ter-Mikaelian, *High-Energy Electromagnetic Processes in Condensed Media* (Interscience, 1972), and eq. (2.41) of V.I. Ginzburg and V.N. Tsytovich, *Transition Radiation and Transition Scattering* (Adam Hilger, 1990). Section 28b of Ter-Mikaelian develops a quasi-classical approximation, and applies it to transition radiation on pp. 283-284. See also,
http://kirkmcd.princeton.edu/examples/EM/ter-mikaelian_np_24_43_61.pdf

3. a) **Bremsstrahlung Revisited**

A charge e travels in vacuum with initial velocity $\mathbf{v}_i$ experiences a brief acceleration, during time interval Δt, which leaves it with final velocity $\mathbf{v}_f$. The frequency-angle spectrum of the radiation can be obtained via eq. (1) (setting index n to 1),

$$\frac{dU_\omega}{d\Omega} = \frac{e^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} \hat{\mathbf{k}} \times \boldsymbol{\beta}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}\, dt\right|^2, \tag{36}$$

We approximate $\boldsymbol{\beta}(t)$ by,

$$\boldsymbol{\beta}(t) = \begin{cases} \frac{\mathbf{v}_i}{c} & (-\infty < t < o \\ \frac{\mathbf{v}_f}{c} & (0 < t < \infty), \end{cases} \tag{37}$$

which omits detailed consideration of the interval $-\Delta t/2 < t < \Delta t/2$ in the computation of eq. (36), which mainly affects frequencies of order $1/\Delta t$ and higher. Then, noting that $\mathbf{r}(t < 0) = \mathbf{v}_i\, t$ and $\mathbf{r}(t > 0) = \mathbf{v}_f\, t$ and $\mathbf{k} = \omega\,\hat{\mathbf{k}}/c$, we find,

$$\frac{dU_\omega}{d\Omega} \approx \frac{e^2\omega^2}{4\pi^2 c^3}\left|\int_{-\infty}^{0} \hat{\mathbf{k}} \times \mathbf{v}_i\, e^{i\omega t(1-\hat{\mathbf{k}}\cdot\mathbf{v}_i/c)}\, dt + \int_{0}^{\infty} \hat{\mathbf{k}} \times \mathbf{v}_f\, e^{i\omega t(1-\hat{\mathbf{k}}\cdot\mathbf{v}_f/c)}\, dt\right|^2$$
$$= \frac{e^2\omega^2}{4\pi^2 c^3}\left|\frac{\hat{\mathbf{k}} \times \mathbf{v}_i}{i\omega(1-\hat{\mathbf{k}}\cdot\mathbf{v}_i/c)}(1 - e^{-i\omega\infty(1-\hat{\mathbf{k}}\cdot\mathbf{v}_i/c)}) + \frac{\hat{\mathbf{k}} \times \mathbf{v}_f}{i\omega(1-\hat{\mathbf{k}}\cdot\mathbf{v}_i/c)}(e^{i\omega\infty(1-\hat{\mathbf{k}}\cdot\mathbf{v}_i/c)} - 1)\right|^2$$
$$= \frac{e^2}{4\pi^2 c^3}\left[\frac{\hat{\mathbf{k}} \times \mathbf{v}_i}{1-\hat{\mathbf{k}}\cdot\mathbf{v}_i/c} - \frac{\hat{\mathbf{k}} \times \mathbf{v}_f}{1-\hat{\mathbf{k}}\cdot\mathbf{v}_i/c}\right]^2, \tag{38}$$

where, as usual, we take e^{iAt} for $t = \pm\infty$ to be zero, *i.e.*, the time-average of the oscillatory exponential function.

b) **Neutron Decay**

In neutron decay from rest, we can approximate the final-state proton as being at rest, since $m_n = 936.965$ MeV$/c^2$, $m_p = 938.272$ MeV$/c^2$, and $m_e = 0.511$ MeV$/c^2$, in which case the angle-frequency spectrum follows from eq. (38), with $\mathbf{v}_i = 0$ and $\mathbf{v}_f = \mathbf{v} = \boldsymbol{\beta}\, c$, as,

$$\frac{dU_\omega}{d\Omega} \approx \frac{e^2\beta^2\sin^2\theta}{4\pi^2 c^3}\left[\frac{1}{1-\beta\cos\theta}\right]^2, \tag{39}$$

where v is the velocity of the final-state electron, and θ is the angle between $\mathbf{v}$ and the direction to the observer.

Note that eq. (39) is the same as eq. (31), the crudest approximation to transition radiation at a metal-vacuum interface. Hence, the frequency spectrum is that found in eq. (32),

$$U_\omega = \int \frac{dU_\omega}{d\Omega}\, d\Omega \approx \frac{e^2}{2\pi c}\left[\frac{2}{\beta}\ln\gamma(1+\beta) - 1\right], \tag{40}$$

Experimentally, the kinetic energy of the ejected electron and the energy of electromagnetic radiation did not equal $(m_n - m_p - m_e)c^2$, so if energy is conserved in neutron decay, there must be "something else" emitted as well. This led Pauli in 1930 to postulate the existence of the neutrino,

http://kirkmcd.princeton.edu/examples/neutrinos/pauli_300430_english.pdf

4. *This is Prob. 7, p. 376 of* W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, 1962),
kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

The angle-frequency spectrum for electromagnetic radiation emitted when an electron of charge e and velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$, where $v \ll c$, makes an elastic collision with a hard, transparent sphere of radius a (centered on the origin) at impact parameter b and azimuthal angle ϕ follows from eq. (6) as,

$$\frac{dU_\omega(b,\phi)}{d\Omega} = \frac{e^2}{4\pi^2 c^3}\left[\frac{\hat{\mathbf{k}}\times\mathbf{v}_i}{1-\hat{\mathbf{k}}\cdot\mathbf{v}_i/c} - \frac{\hat{\mathbf{k}}\times\mathbf{v}_f}{1-\hat{\mathbf{k}}\cdot\mathbf{v}_i/c}\right]^2 \approx \frac{e^2 v^2}{4\pi^2 c^3}\left[\hat{\mathbf{k}}\times(\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_f)\right]^2 \qquad (41)$$

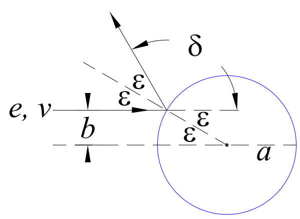
where for an observer in the x-z plane at $\mathbf{r}_o = r(\sin\theta, 0, \cos\theta)$ for large r,

$$\hat{\mathbf{k}} = \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \qquad \hat{\mathbf{v}}_i = (0,0,1), \qquad \hat{\mathbf{v}}_f = (\sin\alpha, 0, \cos\alpha), \qquad (42)$$

and,

$$\delta + 2\epsilon = \pi, \qquad \sin\epsilon = b/a = \cos\left(\frac{\pi}{2} - \epsilon\right) = \cos\frac{\delta}{2}, \qquad (43)$$

with δ being the polar scattering angle.



Then,

$$\hat{\mathbf{k}}\times(\hat{\mathbf{v}}_i - \hat{\mathbf{v}}_f) =$$
$$(-(1-\cos\delta)\sin\theta\sin\phi, -(1-\cos\delta)\sin\theta\cos\phi - \sin\delta\cos\theta, \sin\delta\sin\theta\sin\phi), \qquad (44)$$

and,

$$\frac{dU_\omega(b,\phi)}{d\Omega} = \frac{dU_\omega(\delta,\phi)}{d\Omega} \approx \frac{e^2 v^2}{4\pi^2 c^3}[(1-\cos\delta)^2\sin^2\theta + 2(1-\cos\delta)\sin\delta\sin\theta\cos\theta\cos\phi$$
$$+\sin^2\delta\cos^2\theta + \sin^2\delta\sin^2\theta\sin^2\phi]. \qquad (45)$$

Since U_ω is the energy radiated into unit interval of angular frequency ω, dividing this by the photon energy $\hbar\omega$ gives the number spectrum, N_ω, of photons per unit integral of ω. Dividing this spectrum by $\hbar$ gives the number spectrum, $N_{\hbar\omega}$, of photons per unit interval of photon energy. Then, we can convert the differential spectrum $dN_{\hbar\omega}/d\Omega$ into a kind of differential cross section (with dimensions of area) by integrating over impact parameter b from 0 to a, and over azimuthal angle ϕ (of the incoming electron),

$$\frac{d\sigma}{d\Omega\, d\,\hbar\omega} = \frac{1}{\hbar^2\omega}\int_0^a b\,db\int_0^{2\pi} d\phi\,\frac{dU_\omega(b,\phi)}{d\Omega} = \frac{1}{\hbar^2\omega}\int_0^\pi \frac{a^2}{2}\cos\frac{\delta}{2}\sin\frac{\delta}{2}\,d\delta\int_0^{2\pi} d\phi\,\frac{dU_\omega(\delta,\phi)}{d\Omega}$$

$$
\begin{aligned}
&= \frac{a^2}{4\hbar^2\omega}\int_0^\pi \sin\delta\, d\delta \int_0^{2\pi} d\phi\, \frac{dU_\omega(\delta,\phi)}{d\Omega} \\
\approx \frac{a^2e^2v^2}{16\pi^2\hbar^2\omega c^3}\int_{-1}^1 d\cos\delta \int_0^{2\pi} d\phi\, &[(1-\cos\delta)^2\sin^2\theta + 2(1-\cos\delta)\sin\delta\sin\theta\cos\theta\cos\phi \\
&+ \sin^2\delta\cos^2\theta + \sin^2\delta\sin^2\theta\sin^2\phi] \\
&= \frac{a^2e^2v^2}{16\pi\hbar^2\omega c^3}\int_{-1}^1 d\cos\delta\,[2(1-\cos\delta)^2\sin^2\theta + 2\sin^2\delta\cos^2\theta + \sin^2\delta\sin^2\theta] \\
&= \frac{a^2e^2v^2}{16\pi\hbar^2\omega c^3}\left[2\left(2+\frac{2}{3}\right)\sin^2\theta + 2\left(2-\frac{2}{3}\right)(1-\sin^2\theta) + \left(2-\frac{2}{3}\right)\sin^2\theta\right] \\
&= \frac{a^2e^2v^2}{12\pi\hbar^2\omega c^3}(2+3\sin^2\theta) = \frac{\alpha\beta^2a^2}{12\pi\hbar\omega}(2+3\sin^2\theta), \qquad (46)
\end{aligned}
$$

where $\alpha = e^2/\hbar c = 1/137$ is the fine-structure constant of quantum theory, and $\beta = v/c$.

It follows that,

$$
\frac{d\sigma}{d\,\hbar\omega} = \int \frac{d\sigma}{d\Omega\, d\,\hbar\omega}\, d\Omega = \frac{\alpha\beta^2a^2}{6\hbar\omega}\int_{-1}^1 d\cos\theta\,(5-3\cos^2\theta) = \frac{4\alpha\beta^2a^2}{3\hbar\omega}\,. \qquad (47)
$$

The $1/\hbar\omega$ dependence reminds us that the radiation process is a form of Bremsstrahlung.

5. *This is Prob. 1, §74, p. 203 of* L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Butterworth-Heinemann, 1975),
http://kirkmcd.princeton.edu/examples/EM/landau_ctf_75.pdf

A particle with electric charge e and rest mass m_0 moves in a plane perpendicular to a uniform magnetic field $\mathbf{B}$, radiating energy and losing velocity, such that its trajectory is an inward spiral. Suppose the spiral is nearly circular at all times, so that it is a good approximation that $\mathbf{a} \perp \mathbf{v}$.

Recall that the total radiated power, $dU_{\rm rad}/dt$, is a Lorentz invariant,[14] and hence, we can use the Larmor formula in the instantaneous rest frame of the charge, together with the Lorentz transformation of acceleration between the rest frame and the lab frame, $a^{\star 2} = \gamma^6(a^2 - (\mathbf{v}/c \times \mathbf{a})^2)$,[15] *i.e.*, $a^\star = \gamma^2 a$ for $\mathbf{a} \perp \mathbf{v}$,

$$\frac{dU_{\rm rad}}{dt} = \frac{dU^\star_{\rm rad}}{dt^\star} = \frac{2e^2 a^{\star 2}}{3c^3} = \frac{2\gamma^4 e^2 a^2}{3c^3}\,, \tag{48}$$

where c is the speed of light in vacuum, and $\gamma = 1/\sqrt{1 = v^2/c^2}$. The acceleration of the charge in the uniform magnetic field B is given by the Lorentz force, $\gamma m_0 a = evB/c$ (for $\mathbf{v} \perp \mathbf{B}$), so,

$$\frac{dU_{\rm rad}}{dt} = \frac{2\gamma^2 v^2 e^4 B^2}{3m_0^2 c^5}\,. \tag{49}$$

We now consider the energy $U = \gamma m_0 c^2 = \sqrt{P^2 + m_0^2 c^4}$ of the particle, whose momentum is $\mathbf{P} = \gamma m_0 \mathbf{v}$, such that,

$$\frac{dU}{dt} = -\frac{dU_{\rm rad}}{dt} = -\frac{2P^2 e^4 B^2}{3m_0^4 c^5} = -\frac{2(U^2 - m_0^2 c^4)e^4 B^2}{3m_0^4 c^5}\,, \tag{50}$$

$$\frac{dU}{U^2 - m_0^2 c^4} = -\frac{2e^4 B^2}{3m_0^4 c^5}\,dt, \tag{51}$$

$$\frac{1}{m_0 c^2}\coth^{-1}\frac{U}{m_0 c^2} = -\frac{2e^4 B^2}{3m_0^4 c^5}t + \text{const}\,, \tag{52}$$

$$\frac{U}{m_0 c^2} = \gamma = \coth\left[\frac{2e^4 B^2 t}{3m_0^3 c^5} + \text{const}\right]. \tag{53}$$

using Dwight 140.1. Thus, it takes forever for the particle's kinetic energy to be radiated away, and $U \to m_0 c^2$ ($\gamma \to 1$).

[14] See, for example, p. 243, Lecture 20 of the Notes.
[15] See, for example, p. 221, Lecture 18 of the Notes.

6. **Nuclear Numerology**

Poisson's equation in electrostatics, for the potential ϕ_e due to a static density ρ_e of electric charge, is (pp. 10-10a, Lecture 1 of the Notes),

$$\nabla^2 \phi_e = -4\pi\rho_e. \tag{54}$$

As discussed on p. 226, Lecture 19 of the Notes, Yukawa's equation[16] for the static nuclear potential ϕ_g is $(\nabla^2 - \mu^2)\,\phi_g = 0$, away from a point source of nuclear charge g at the origin, for which,

$$\phi_g = g\frac{e^{-\mu r}}{r}\,, \tag{55}$$

for some constant μ (with dimensions of inverse length). Away from the origin,

$$\nabla^2 \phi_g = \frac{1}{r}\frac{\partial}{\partial r^2}(r\phi_g) = \mu^2 \phi_g, \tag{56}$$

while close to the origin, $\phi_g \approx g/r$, for which,

$$\nabla^2 \phi_g \approx g\nabla^2(1/r) = -4\pi g\,\delta^3(\mathbf{r}) \tag{57}$$

recalling that $\nabla^2(1/r) = -4\pi\,\delta^3(\mathbf{r})$.

This suggests that in case of a (static) volume density ρ_g of nuclear charge, the Yukawa potential is,

$$\phi_g(\mathbf{r}) = \int \frac{\rho_g(\mathbf{r}')\,e^{-\mu|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,d\mathrm{Vol}', \tag{58}$$

and Poisson's equation becomes,[17]

$$(\nabla^2 - \mu^2)\,\phi_g = -4\pi\rho_g. \tag{59}$$

The concept of the potential is that the interaction energy of two (point) charges g_1 and g_2 is,

$$U_{12} = g_1\phi_{g,12} = g_2\phi_{g,21}. \tag{60}$$

For a collection of particles, this leads to the interaction energy,

$$U = \frac{1}{2}\sum_{i,j} g_i\phi_{g,ij} \to \frac{1}{2}\int \rho_g(\mathbf{r})\phi_g(\mathbf{r})\,d\mathrm{Vol}. \tag{61}$$

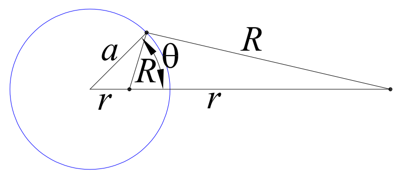
[16]H. Yukawa, *On the Interaction of Elementary Particles. I*, Proc. Phys.-Math. Soc. Japan **17**, 48 (1935), http://kirkmcd.princeton.edu/examples/EP/yukawa_ppmsj_17_48_35.pdf.

[17]Equation (59) is sometimes called the screened Poisson equation.

Then, with ρ_g from eq. (59), we have, recalling that $\phi_g \nabla^2 \phi_g = \boldsymbol{\nabla}(\phi_g \boldsymbol{\nabla}\phi_g) - (\boldsymbol{\nabla}\phi_g)^2$, and using Gauss' theorem,

$$U = -\frac{1}{8\pi}\int \phi_g(\nabla^2 - \mu^2)\phi_g \, d\mathrm{Vol} = \frac{1}{8\pi}\int \left[-\boldsymbol{\nabla}(\phi_g \boldsymbol{\nabla}\phi_g) + (\boldsymbol{\nabla}\phi_g)^2 + \mu^2\phi_g^2\right] d\mathrm{Vol}$$
$$= \frac{1}{8\pi}\int \left[(\boldsymbol{\nabla}\phi_g)^2 + \mu^2\phi_g^2\right] d\mathrm{Vol}, \quad (62)$$

for a charge distribution that is nonzero only within a bounded volume, such that $\phi_g \boldsymbol{\nabla}\phi_g \propto 1/r^3$ for large r.



We now consider a nuclear charge g that is uniformly distributed over a spherical shell (about the origin) of radius a. At a point on the z-axis at distance r from the origin,

$$\phi_g(r) = \int_{-1}^{1} d\cos\theta \, \frac{g}{2}\frac{e^{-\mu R}}{R}\,, \quad (63)$$

where,

$$R^2 = a^2 + r^2 - 2ar\cos\theta, \qquad 2R\,dR = -2ar\,d\cos\theta, \quad (64)$$

and when $\cos\theta = \pm 1$, $R = a + r,\ |a - r|$. Hence,

$$\phi_g(r > a) = \frac{g}{2ar}\int_{r-a}^{r+a} dR\, e^{-\mu R} = \frac{g}{2\mu a r}(e^{-\mu(r-a)} - e^{-\mu(r+a)}) = \mu g \frac{\sinh \mu a}{\mu a}\frac{e^{-\mu r}}{\mu r}\,, \quad (65)$$

$$\phi_g(r < a) = \frac{g}{2ar}\int_{a-r}^{a+r} dR\, e^{-\mu R} = \frac{g}{2\mu a r}(e^{-\mu(a-r)} - e^{-\mu(a+r)}) = \mu g \frac{e^{-\mu a}}{\mu a}\frac{\sinh \mu r}{\mu r}\,. \quad (66)$$

For $\mu = 0$ we recover the form for ordinary electrostatics of a spherical shell of electric charge q, $\phi(r > a) = q/r$, while $\phi(r < a) = q/a$.

The gradient of the potential (65)-(66) is purely radial,

$$\boldsymbol{\nabla}\phi_{g,r}(r > a) = \mu\frac{\partial \phi_g(r > a)}{\partial \mu r} = -\frac{\mu g \sinh \mu a}{a}\left(\frac{e^{-\mu r}}{\mu^2 r^2} + \frac{e^{-\mu r}}{\mu r}\right), \quad (67)$$

$$\boldsymbol{\nabla}\phi_{g,r}(r < a) = \mu\frac{\partial \phi_g(r < a)}{\partial \mu r} = -\frac{\mu g\, e^{-\mu a}}{a}\left(\frac{\sinh \mu r}{\mu^2 r^2} - \frac{\cosh \mu r}{\mu r}\right), \quad (68)$$

and the field energy (62) is, noting that $\sinh^2 x = (\cosh 2x - 1)/2$, $2\sinh x \cosh x = \sinh 2x$, and $\cosh^2 x = (\cosh 2x + 1)/2$,

$$U = \frac{g^2 \sinh^2 \mu a}{2\mu a^2}\int_{\mu a}^{\infty} (\mu r)^2\, d(\mu r)\, e^{-2\mu r}\left(\frac{1}{\mu^4 r^4} + \frac{2}{\mu^3 r^3} + \frac{2}{\mu^2 r^2}\right)$$

$$+\frac{g^2\,e^{-2\mu a}}{2\mu a^2}\int_0^{\mu a}(\mu r)^2\,d(\mu r)\left(\frac{\cosh 2\mu r-1}{2\mu^4 r^4}-\frac{\sinh 2\mu r}{\mu^3 r^3}+\frac{\cosh 2\mu r}{\mu^2 r^2}\right)$$
$$=\frac{g^2}{2\mu a^2}\frac{\cosh 2\mu a-1}{2}\,e^{-2\mu a}\left(1+\frac{1}{\mu a}\right)+\frac{g^2\,e^{-2\mu a}}{2\mu a^2}\left(\frac{1-\cosh 2\mu a}{2\mu a}+\frac{\sinh 2\mu a}{2}\right)$$
$$=\frac{g^2\,e^{-2\mu a}}{2\mu a^2}\left(\frac{\cosh 2\mu a-1+\sinh 2\mu a}{2}\right)=\frac{g^2(1-e^{-2\mu a})}{4\mu a^2}\,, \tag{69}$$

using Dwight 568.2 and 678.12. If $\mu \to 0$, this goes to $g^2/2a$, as expected from electrostatics.

Applying this model to a proton, its rest mass would be,

$$m_p=\frac{U_p}{c^2}=\frac{g^2(1-e^{-2\mu a_p})}{4\mu a_p^2 c^2}\,, \tag{70}$$

where a_p is the radius of the proton (taken to be a spherical shell of nuclear charge).

In nuclear interactions, the range of the Yukawa interaction is about the same as the radius of the proton, $a_p \approx 0.86\times 10^{-13}$ cm,[18] *i.e.*, $\mu a_p \approx 1$, in which case $U_p \approx g^2/4a_p$ for our model of a proton as a spherical shell of nuclear charge.

For an electron modeled as a spherical shell of electric charge e of radius a_e, the electric-field energy $U_e = e^2/2a_e$ equals the electron rest mass (times c^2) for $a_e = r_e/2 = 1.4\times 10^{-13}$ cm, where $r_e = m_e c^2/e^2$ is the so-called classical election radius.[19]

Experimentally, $m_p/m_e = 1836$, so our models imply that,

$$\frac{m_p}{m_e}\approx\frac{g^2/4a_p}{e^2/2a_e}\,, \qquad \frac{g^2}{\hbar c}\approx\frac{e^2}{\hbar c}\frac{m_p}{m_e}\frac{2a_p}{a_e}\approx\frac{1}{137}\cdot 1836\cdot 2\cdot\frac{0.86}{1.4}\approx 16.5. \tag{71}$$

This compares fairly well with the experimental value of ≈ 14.2.[20]

However, the above discussion was for a repulsive Yukawa potential, which doesn't explain why nucleons stick together. If we change the sign in eqs. (55) and (58) to have an attractive potential, then the field energy (62) would be negative, which seems unphysical. Hence, the model presented in this problem cannot be taken too seriously.

[18] https://en.wikipedia.org/wiki/Proton_radius_puzzle

[19] https://en.wikipedia.org/wiki/Classical_electron_radius

[20] See, for example, https://arxiv.org/pdf/hep-ph/0009312.pdf, where our coupling constant $g^2/\hbar c$ is their $g^2/4\pi$.

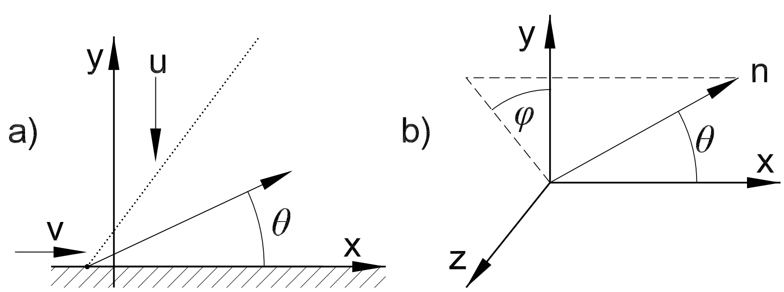
7. The solution to this problem is at http://kirkmcd.princeton.edu/examples/cap_stress.pdf.

8. *The solution to this problem also appears at*
http://kirkmcd.princeton.edu/examples/superluminal.pdf

The possibility of radiation from superluminal sources was first considered by Heaviside in 1888. He considered this topic many times over the next 20 years, deriving most of the formalism of what is now called Čerenkov radiation. However, despite being an early proponent of the concept of a velocity-dependent electromagnetic mass, Heaviside never acknowledged the limitation that massive particles must have velocities less than that of light. Consequently many of his pioneering efforts (and those of his immediate followers, Des Coudres and Sommerfeld), were largely ignored, and the realizable case of radiation from a charge with velocity greater than the speed of light in a dielectric medium was discovered independently in an experiment by Čerenkov in 1934.[21]

An insightful discussion of the theory of Čerenkov radiation by Tamm (J. Phys. U.S.S.R. **1**, 439 (1939), in English!)[22] revealed its close connection with what is now called transition radiation, *i.e.*, radiation emitted by a charge in uniform motion that crosses a boundary between metallic or dielectric media. The present problem was inspired by a work of Bolotovskii and Ginzburg, Sov. Phys. Uspekhi **15**, 184 (1972),[23] on how aggregates of particles can act to produce motion that has superluminal aspects and that there should be corresponding Čerenkov-like radiation in the case of charged particles. The classic example of aggregate superluminal motion is the velocity of the point of intersection of a pair of scissors whose tips approach one another at a velocity close to that of light.

Here we consider the example of a "sweeping" electron beam in a high-speed analog oscilloscope such as the Tektronix 7104. In this device the "writing speed", the velocity of the beam spot across the faceplate of the oscilloscope, can exceed the speed of light. The transition radiation emitted by the beam electrons just before they disappear into the faceplate has the character of Čerenkov radiation from the superluminal beam spot, according to the inverse of the argument of Tamm.



Referring to the figure above, the line of charge has equation,

$$y = \frac{u}{v}x - ut, \qquad z = 0, \tag{72}$$

[21] http://kirkmcd.princeton.edu/examples/EM/cerenkov_pr_52_378_37.pdf
[22] http://kirkmcd.princeton.edu/examples/EM/tamm_jpussr_1_439_39.pdf
[23] http://kirkmcd.princeton.edu/examples/EM/bolotovskii_spu_15_184_72.pdf

so the current density is,

$$\mathbf{J} = -\widehat{\mathbf{y}} Ne\, \delta(z)\, \delta\left(t - \frac{x}{v} + \frac{y}{u}\right), \tag{73}$$

where N is the number of electrons per unit length intercepting the x axis, and $e < 0$ is the electron's charge.

We also consider the effect of the image current,

$$\mathbf{J}_{\rm image} = +\widehat{\mathbf{y}}(-Ne)\, \delta(z)\, \delta\left(t - \frac{x}{v} - \frac{y}{u}\right). \tag{74}$$

We will find that to a good approximation the image current just doubles the amplitude of the radiation. For $u \approx c$ the image current would be related to the retarded fields of the electron beam, but we avoid this complication when $u \ll c$. Note that the true current exists only for $y > 0$, while the image current applies only for $y < 0$.

We insert the current densities (73) and (74) into eq. (11) and integrate using rectangular coordinates, with components of the unit vector $\hat{\mathbf{n}}$ given by,

$$n_x = \cos\theta, \qquad n_y = \sin\theta\cos\phi, \qquad \text{and} \qquad n_z = \sin\theta\sin\phi, \tag{75}$$

as indicated in part b) of the figure. The current impinges only on a length L along the x-axis. The integrals are elementary and we find, noting $\omega/c = 2\pi/\lambda$,

$$\frac{dU}{d\omega\, d\Omega} = \frac{e^2 N^2 L^2}{\pi^2 c}\frac{u^2}{c^2}\frac{\cos^2\theta + \sin^2\theta\sin^2\phi}{(1 - \frac{u^2}{c^2}\sin^2\theta\cos^2\phi)^2}\left(\frac{\sin\left[\frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)\right]}{\frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)}\right)^2. \tag{76}$$

The factor of form $\sin^2\chi/\chi^2$ appears from the x integration, and indicates that this leads to a single-slit interference pattern.

We will only consider the case that $u \ll c$, so from now on we approximate the factor $1 - \frac{u^2}{c^2}\sin^2\theta\cos^2\phi$ by 1.

Upon integration over the azimuthal angle ϕ from $-\pi/2$ to $\pi/2$ the factor $\cos^2\theta + \sin^2\theta\sin^2\phi$ becomes $\frac{\pi}{2}(1 + \cos^2\theta)$.

It is instructive to replace the radiated energy by the number of radiated photons: $dU = \hbar\omega\, dN_\omega$. Thus,

$$\frac{dN_\omega}{d\cos\theta} = \frac{\alpha}{2\pi}\frac{d\omega}{\omega}N^2 L^2 \frac{u^2}{c^2}(1 + \cos^2\theta)\left(\frac{\sin\left[\frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)\right]}{\frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)}\right)^2, \tag{77}$$

where $\alpha = e^2/\hbar c \approx 1/137$. This result applies whether $v < c$ or $v > c$. But for $v < c$, the argument $\chi = \frac{\pi L}{\lambda}(\frac{c}{v} - \cos\theta)$ can never become zero, and the diffraction pattern never achieves a principal maximum. The radiation pattern remains a slightly skewed type of transition radiation. However, for $v > c$ we can have $\chi = 0$, and the radiation pattern has a large spike at angle $\theta_{\check{C}}$ such that

$$\cos\theta_{\check{C}} = \frac{c}{v}, \tag{78}$$

which we identify with Čerenkov radiation. Of course the side lobes are still present, but not very prominent.

Discussion

The present analysis suggests that Čerenkov radiation is not really distinct from transition radiation, but is rather a special feature of the transition radiation pattern which emerges under certain circumstances. This viewpoint actually is relevant to Čerenkov radiation in any real device which has a finite path length for the radiating charge. The walls which define the path length are sources of transition radiation which is always present even when the Čerenkov condition is not satisfied. When the Čerenkov condition is satisfied, the so-called formation length for transition radiation becomes longer than the device, and the Čerenkov radiation can be thought of as an interference effect.

If $L/\lambda \gg 1$, then the radiation pattern is very sharply peaked about the Čerenkov angle, and we may integrate over θ, noting that,

$$d\chi = \frac{\pi, L}{\lambda}\, d\cos\theta \qquad \text{and} \qquad \int_{-\infty}^{\infty} d\chi\, \frac{\sin^2\chi}{\chi^2} = \pi, \tag{79}$$

to find,

$$dN_\omega \approx \frac{\alpha}{2\pi}(N\lambda)^2 \frac{d\omega}{\omega}\frac{L}{\lambda}\frac{u^2}{c^2}\left(1 + \frac{c^2}{v^2}\right). \tag{80}$$

In this we have replaced $\cos^2\theta$ by c^2/v^2 in the vicinity of the Čerenkov angle. We have also extended the limits of integration on χ to $[-\infty, \infty]$. This is not a good approximation for $v < c$, in which case $\chi > 0$ always and dN_ω is much less than stated. For $v = c$ the radiation rate is still about one half of the above estimate.

For comparison, the expression for the number of photons radiated in the ordinary Čerenkov effect is,

$$dN_\omega \approx 2\pi\alpha \frac{d\omega}{\omega}\frac{L}{\lambda}\sin^2\theta_{\check{C}}. \tag{81}$$

The ordinary Čerenkov effect vanishes as $\theta_{\check{C}}^2$ near the threshold, but the superluminal effect does not. This is related to the fact that at threshold ordinary Čerenkov radiation is emitted at small angles to the electron's direction, while in the superluminal case the radiation is at right angles to the electron's motion. In this respect the moving spot on an oscilloscope is not fully equivalent to a single charge as the source of the Čerenkov radiation.

In the discussion thus far we have assumed that the electron beam is well described by a uniform line of charge. In practice the beam is discrete, with fluctuations in the spacing and energy of the electrons. If these fluctuations are too large we cannot expect the transition radiation from the various electrons to superimpose coherently to produce the Čerenkov radiation. Roughly, there will be almost no coherence for wavelengths smaller than the actual spot size of the electron beam at the metal surface. Thus, there will be a cutoff at high frequencies which serves to limit the total radiated energy to a finite amount, whereas the expression derived above is formally divergent. Similarly the effect will be quite weak unless the beam current is large enough that $N\lambda \gg 1$.

We close with a numerical example inspired by possible experiment. A realistic spot size for the beam is 0.3 mm, so we must detect radiation at longer wavelengths. A convenient choice is $\lambda = 3$ mm, for which commercial microwave receivers exist. The bandwidth of a candidate receiver is $d\omega/\omega = 0.02$ centered at 88 GHz. We take $L = 3$ cm, so $L/\lambda = 10$ and the Čerenkov "cone" will actually be about 5° wide, which happens to match the angular resolution of the microwave receiver. Supposing the electron-beam energy to be 2.5 keV, we would have $u^2/c^2 = 0.01$. The velocity of the moving spot is taken as $v = 1.33c = 4 \times 10^{10}$ cm/sec, so the observation angle is 41°. If the electron beam current is 1 μA then the number of electrons deposited per cm along the metal surface is $N \approx 150$, and $N\lambda \approx 45$.

Inserting these parameters into the rate formula (80), we expect about 7×10^{-3} detected photons from a single sweep of the electron beam. This supposes we can collect over all azimuth ϕ which would require some suitable optics. The electron beam will actually be swept at about 1 GHz, so we can collect about 7×10^{6} photons per second. The corresponding signal power is 2.6×10^{-25} Watts/Hz, whose equivalent noise temperature is about 20 mK. This must be distinguished from the background of thermal radiation, the main source of which is in the receiver itself, whose noise temperature is about 100°K. A lock-in amplifier could be used to extract the weak periodic signal; an integration time of a few minutes of the 1-GHz-repetition-rate signal would suffice assuming 100% collection efficiency.

Realization of such an experiment with a Tektronix 7104 oscilloscope would require a custom cathode ray tube that permits collection of microwave radiation through a portion of the wall not coated with the usual metallic shielding layer.

Bremsstrahlung

Early reports of observation of transition radiation were considered by skeptics to be due to Bremsstrahlung instead. The distinction in principle is that transition radiation is due to acceleration of charges in a medium in response to the far field of a uniformly moving charge, while Bremsstrahlung is due to the acceleration of the moving charge in the near field of atomic nuclei. In practice both effects exist and can be separated by careful experiment.

Is Bremsstrahlung stronger than transition radiation in the example considered here? As shown below the answer is no, but even if it were we would then expect a Čerenkov-like effect arising from the coherent bremsstrahlung of the electron beam as it hits the oscilloscope faceplate.

The angular distribution of Bremsstrahlung from a nonrelativistic electron will be $\sin^2\theta$ with θ defined with respect to the direction of motion. The range of a 2.5-keV electron in, say, copper is about 5×10^{-6} (as extrapolated from the table on p. 240 of *Studies in Penetration of Charged Particles in Matter*, National Academy of Sciences – National Research Council, PuB-1133 (1964)),[24] while the skin depth at 88 GHz is

[24] http://kirkmcd.princeton.edu/examples/detectors/nas-1133_64.pdf

about 2.5×10^{-5} cm. Hence, the copper is essentially transparent to the backward hemisphere of bremsstrahlung radiation, which will emerge into the same half space as the transition radiation.

The amount of Bremsstrahlung energy dU_B emitted into energy interval dU is just $Y\, dU$ where Y is the so-called bremsstrahlung yield factor. For 2.5-keV electrons in copper, $Y = 3 \times 10^{-4}$. The number dN of bremsstrahlung photons of energy $\hbar\omega$ in a bandwidth $d\omega/\omega$ is then $dN = dU_B/\hbar\omega = Y\, d\omega/\omega$. For the 2% bandwidth of our example, $dN = 6 \times 10^{-6}$ per beam electron. For a 3-cm-long target region there will be 500 beam electrons per sweep of the oscilloscope, for a total of 3×10^{-4} bremsstrahlung photons into a 2% bandwidth about 88 GHz. Half of these emerge from the faceplate as a background to 7×10^{-3} transition-radiation photons per sweep. Altogether, the Bremsstrahlung contribution would be about 1/50 of the transition-radiation signal in the proposed experiment.