Princeton University

Ph501
Electrodynamics
Problem Set 12

Kirk T. McDonald

(2001)

kirkmcd@princeton.edu

http://kirkmcd.princeton.edu/examples/

1. A linearly polarized, plane electromagnetic wave of angular frequency ω is incident on a free electron, leading to motion of the latter transverse to the direction of the wave. Where does the transverse momentum of the electron come from (such that total momentum is conserved)?

   It must be that there is electromagnetic-field momentum equal and opposite to the mechanical momentum of the electron, but the momentum of the electromagnetic wave is in the direction of the wave.

   Show that the electromagnetic momentum associated with the interaction of the wave with the time-average static field of the electron is equal and opposite to the transverse momentum of the electron, in the frame where the electron is at rest on average.[1]

   You may suppose the incident wave is weak enough that the velocity of the electron (in its average rest frame) is always small compared to the speed of light.

[1]If the plane wave overtakes an electron initially at rest, the electron takes on a drift velocity in the direction of the wave, as first noted in
http://kirkmcd.princeton.edu/examples/accel/mcmillan_pr_79_498_50.pdf

2. A circularly polarized electromagnetic wave of angular frequency ω is incident on a free electron of charge e and rest mass m. Find the resulting motion, in the average rest frame of the electron, even for strong fields, in which $v \to c$.

   (a) Show that the total radiated intensity, measured at the electron, is,

$$\frac{dU}{dt} = \frac{2\gamma^2 e^4 E^2}{3m^2 c^3} = \frac{2e^4 E^2}{3m^2 c^3}(1 + \eta^2), \qquad \text{where} \qquad \eta = \frac{eE}{m\omega c}\,. \tag{1}$$

   (b) Consider also a multipole expansion[2] of the radiated intensity according to a fixed, distant observer to show that (for $\eta^2 \ll 1$),

$$\frac{dU}{dt} = \frac{2e^4 E^2}{3m^2 c^3}\left(1 + \frac{7}{5}\eta^2 + \cdots\right)\,. \tag{2}$$

   Note that the 2^{nd} term in the expansion corresponds to radiation at frequency 2ω (and higher-order terms correspond to radiation at higher multiples of ω).[3]

[2] Recall Prob. 7, Set 8, http://kirkmcd.princeton.edu/examples/ph501set8.pdf

[3] From the quantum viewpoint, this corresponds to the absorption of two photons by the electron, with emission of one photon of double the energy of those in the incident wave.

3. Calculate the cross section for the scattering of an elliptically polarized plane electromagnetic wave in vacuum that is incident on a free electron of charge e and mass m. The electric field of the plane wave can be taken to be the real part of,

$$\mathbf{E} = (\mathbf{a} + \mathbf{b})\, e^{i(kz-\omega t)}, \tag{3}$$

where $\mathbf{a}$ is perpendicular to $\mathbf{b}$, and both are perpendicular to $\hat{\mathbf{z}}$, while $a \neq b$ in general.

Show that,

$$\frac{d\sigma}{d\Omega} = r_e^2 \frac{(\mathbf{a} \times \hat{\mathbf{n}})^2 + (\mathbf{b} \times \hat{\mathbf{n}})^2 - 2Re[(\mathbf{a} \cdot \hat{\mathbf{n}})(\mathbf{b} \cdot \hat{\mathbf{n}})]}{a^2 + b^2} \tag{4}$$

where $r_e = e^2/mc^2$ is the classical electron radius, with c as the speed of light in vacuum.

Note that for circularly polarized light, $\sigma_{\text{left}} = \sigma_{\text{right}} = \sigma_{\text{Thomson}}$, consistent with Prob. 2 above.

4. Calculate the cross section σ for the scattering of a linearly polarized plane electromagnetic wave in vacuum that is incident on a polar molecular of fixed electric-dipole moment **b**, supposing that the size of the molecule is small compared to the wavelength, *i.e.*, $\lambda \gg p/e$, and assuming that all directions of **p** are equally likely (as in a gas).

   Show that,

$$\sigma \approx \frac{16\pi p^4}{9c^4 I^2}\,, \tag{5}$$

   where I is the moment of inertia of the molecule about an axis perpendicular to **p** that contains the center of mass of the molecule, and we ignore terms that are second order in the velocity of the rotation of the dipole.

5. **Spectral Line Broadening**

A spectral "line" of central wavelength λ is always observed to have a finite width $\Delta\lambda$.

a) Consider the yellow sodium line with $\lambda = 5893$ Å.

What is the contribution to $\Delta\lambda/\lambda$ from the damping due to the radiation reaction?

b) **Doppler Broadening**

Suppose we observe the glowing sodium vapor at 600K.

Then, some of the glowing sodium atoms move towards us and some move away, and the observed radiation is Doppler shifted accordingly. How big is $\Delta\lambda/\lambda$ due to this effect? *There is no need to go into details of the Maxwell distribution.*

c) **Collision Broadening**

At atom starts glowing (becomes excited) due to a collision with another atom, which sets its "spring" oscillation. A second collision at time Δt later, which destroys the coherence of the oscillation. This limits the width of the pulse of radiation, and according to the "uncertainty" relation $\Delta\omega\,\Delta t \approx 1$, the spectrum of the pulse is broadened.

Let ν be the mean frequency of collisions [#/sec]. Then,

$$\nu = [\text{collison cross section}] \cdot [\#\ \text{atoms/volume}] \cdot [\text{mean relative velocity}]. \tag{6}$$

The probability of a collision during time dt is $\nu\,dt$, so the probability that no collision occurred between $t = 0$ and t is $e^{-\nu t}$. Averaging over many collisions, this means that the intensity of the radiation at time t after the beginning of emission is $I_0\,e^{-\nu t}$. This is similar to the effect of other damping mechanisms, for which $E = E_0\,e^{-\Gamma_{\rm other}t}$ and $I \propto E^2 = I_0\,e^{-2\Gamma_{\rm other}t}$. Hence, the effective damping constant Γ in the presence of collisions is,

$$\Gamma = \frac{\nu}{2} + \Gamma_{\rm other}. \tag{7}$$

At what pressure does,

$$\left.\frac{\Delta\lambda}{\lambda}\right|_{\rm collisions} = \left.\frac{\Delta\lambda}{\lambda}\right|_{\rm Doppler} ? \tag{8}$$

For discussion of "sharpening the line", see A.L. Schawlow, Phys. Today **35**(12), 46 (1982), http://kirkmcd.princeton.edu/examples/optics/schawlow_pt_35-12_46_82.pdf

6. **Optical Theorem**

On p. 208 of Lecture 17 on diffraction,[4] we noted that if we write the differential cross section for scattering of a plane electromagnetic wave as $d\sigma_{\rm scat}/d\Omega = |f(\theta)|^2$, then,

$$\sigma_{\rm total} = \frac{4\pi}{k} \left|{\rm Im} f(0)\right| \qquad \text{(Optical Theorem)}, \tag{9}$$

where k is the wave number of the incident wave (of angular frequency $\omega = kc$).

Give a suitable form of $f(\theta)$ for scattering off an electrons bound in an atom with natural frequency ω_0 and damping constant Γ_0 to show that the optical theorem indeed holds for the total cross section,

$$\sigma_{\rm total} = 4\pi r_e c \frac{\omega^2 \Gamma_0}{(\omega^2 - \omega_0^2)^2 + \omega^2 \Gamma_0^2}\,, \tag{10}$$

as found on p. 281 of Lecture 23.[5]

What physical effect saves the optical theorem for free electrons?

[4] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture17.pdf
[5] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture23.pdf

7. **Levinger-Bethe Sum Rule**

In a series of experiments, the scattering of gamma rays off copper nuclei was measured,[6] and the integrated cross section was determined to be,

$$\int \sigma_{\text{total}}\, dE = \int \sigma_{\text{total}}\, d(\hbar\omega) \approx 1.5 \times 10^{-24}\ \text{MeV} \cdot \text{cm}^2. \tag{11}$$

Calculate this integral, starting from p. 282 of Lecture 23 of the Notes,[7] assuming all the scattered radiation is due to oscillations of a dipole moment where:

(a) The entire nucleus moves as a whole.

(b) The neutrons in the nucleus remain fixed while only the protons move about.

(c) The protons move as a group, and the neutrons move as a separate group, which was claimed by Bethe[8] to lead to,

$$\int \sigma_{\text{total}}\, dE = 2\pi^2 \frac{NZ}{A} \alpha \lambda\!\!\!^{-2}_M M c^2, \tag{12}$$

where Z = no. of protons = 29 for copper, N = no. of neutrons, $A = N + Z = 63$ for copper, $\alpha = e^2/\hbar c = 1/137$, $M = M_p \approx M_n$, $\lambda\!\!\!^{-}_M = \hbar/Mc$.

Does the experimental result distinguished between cases (a)-(c)?

[6]G.C. Baldwin and G.S. Klaiber, *X-Ray Yield Curves for γ-n Reactions*, Phys. Rev. **73**, 1156 (1948), http://kirkmcd.princeton.edu/examples/EP/baldwin_pr_73_1156_48.pdf. The total cross section is dominated by the reaction $\gamma + \text{Cu}^{63} \to \text{Cu}^{62} + n$, which exhibits a "resonance" for incident gamma-ray energy around 25 MeV.

[7]http://kirkmcd.princeton.edu/examples/ph501/ph501lecture23.pdf

[8]J.S. Levinger and H.A. Bethe, *Dipole Transitions in the Nuclear Photo-Effect*, Phys. Rev. **78**, 115 (1950), http://kirkmcd.princeton.edu/examples/EP/levinger_pr_78_115_50.pdf

8. What is the minium angular frequency of an electromagnetic what that could propagate in "free" space without attenuation, according to the model of the index of refraction reviewed on p. 282 of Lecture 23 of the Notes,[9] supposing there is one (free) electron per cubic centimeter?

   For frequencies higher that the critical frequency considered above, is there actually "no attenuation"? For example, at optical frequencies the wavelength λ is small compared to average distance between the scattering centers postulated above. Then, it is not plausible that the scattered radiation adds coherently to produce "no attenuation". Rather, the scattered radiation is effectively lost to an observer looking at, say, a distant star. What is the attenuation length for light from a star that is "lost" in this manner?

   The concept of a scattering cross section is reviewed at
   https://en.wikipedia.org/wiki/Cross_section_(physics)

   The weak scattering of light by intergalactic electrons has been hard to measure accurately on Earth due to "foreground" scattering by "dust" in the Solar system. A satellite sent beyond Pluto to observe the scattered light has recently been reported to have seen about twice as much scattered light as can be attributed to stars in distant galaxies.[10]

---

[9] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture23.pdf

[10] T.R. Lauer *et al.*, *Anomalous Flux in the Cosmic Optical Background Detected with New Horizons Observations*, Ap. J. Lett. 927, L8 (2022),
http://kirkmcd.princeton.edu/examples/cosmology/lauer_apjl_927_L8_22.pdf

9. **Gravitational Redshift**

In 1911, Einstein gave a simple, approximate, non-quantum derivation of the gravitational redshift of light as it propagates away from a massive body. As in Lecture 24 of the Notes,[11] there are two key ingredients:

- The **principle of equivalence**, to related physics in a uniform gravitational field to that in a uniformly accelerated frame (without gravity).
- The use of **instantaneous inertial frames**, to relate physics in an accelerated frame to that in an inertial frame – for a short time.

As in the Notes, consider an accelerated frame A that coincides with inertial frame I$'$ at time $t = 0 = t'$ and has acceleration g (with respect to frame I$'$) along the z' axis.[12]

At time $t = 0$ light is emitted with frequency ν_0 from a source at rest at the origin in frame A. This light is detected by an observer at rest at $(0, 0, z_1)$ in frame A where it is found to have frequency ν_1.

Consider a second frame I$''$, which is the instantaneous inertial frame that coincides with frame A at the moment when the light is observed. Use a special-relativity analysis of the Doppler effect to show that,

$$\nu_1 = \frac{\nu_0}{1 + g z_1/c^2} + \mathcal{O}\left(\frac{g^2 z_1^2}{c^4}\right), \qquad \Rightarrow \qquad \nu_1 = \frac{\nu_0}{1 + \Phi(z_1)/c^2}\,, \tag{13}$$

in a uniform gravitational field with gravitational potential $\Phi(0) = 0$.

[11] http://kirkmcd.princeton.edu/examples/ph501/ph501set12.pdf

[12] You can ignore the distinction between (discussed on p. 268 of Lecture 22, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture22.pdf) between uniform acceleration with repect to frame A and that with respect to frame I$'$, whic distinction was not yet well understood in 1911.

Solutions

1. *This solution is abstracted from* http://kirkmcd.princeton.edu/examples/transmom2.pdf

The general sense of the answer has been given by Poynting,[13] who noted that an electromagnetic field can be said to contain a flux of energy (energy per unit area per unit time) given by,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}, \tag{14}$$

in Gaussian units, where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field (taken to be in vacuum throughout this paper) and c is the speed of light in vacuum.

Thomson[14,15,16] and Poincaré[17] noted that this flow of energy can also be associated with a momentum density given by,

$$\mathbf{p}_{\rm field} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} = \frac{(\mathbf{E}_{\rm wave} + \mathbf{E}_{\rm charge}) \times (\mathbf{B}_{\rm wave} + \mathbf{B}_{\rm charge})}{4\pi c}. \tag{15}$$

Hence, in the problem of a free electron in a plane electromagnetic wave we are led to seek an electromagnetic field momentum that is equal and opposite to the mechanical momentum of the electron. However, this field momentum should not include either of the self-momenta $(\mathbf{E}_{\rm wave} \times \mathbf{B}_{\rm wave})/4\pi c$ or $(\mathbf{E}_{\rm charge} \times \mathbf{B}_{\rm charge})/4\pi c$. The former is independent of the electron, while the latter can be considered as a part of the mechanical momentum of the electron according to the concept of "renormalization".

We desire to show that the **interaction** field momentum,

$$\mathbf{P}_{\rm int} = \int \mathbf{p}_{\rm int}\, d{\rm Vol} = \int d{\rm Vol} \frac{\mathbf{E}_{\rm wave} \times \mathbf{B}_{\rm charge} + \mathbf{E}_{\rm charge} \times \mathbf{B}_{\rm wave}}{4\pi c}, \tag{16}$$

is equal and opposite to the mechanical momentum of the electron.

We consider a plane electromagnetic wave that propagates in the $+z$ direction of a rectangular coordinate system. For linear polarization along $\mathbf{x}$,

$$\mathbf{E}_{\rm wave} = E_0 \cos(kz - \omega t)\, \hat{\mathbf{x}}, \qquad \mathbf{B}_{\rm wave} = E_0 \cos(kz - \omega t)\, \hat{\mathbf{y}}, \tag{17}$$

where $\omega = kc$ is the angular frequency of the wave, $k = 2\pi/\lambda$ is the wave number and $\hat{\mathbf{x}}$ is a unit vector in the x direction, *etc.*

[13] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884), http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[14] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf

[15] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893), http://kirkmcd.princeton.edu/examples/EM/thomson_recent_researches_sec_1-16.pdf

[16] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014), http://kirkmcd.princeton.edu/examples/thomson.pdf

[17] H. Poincaré, *Théorie de Lorentz et le Principe de la Réaction*, Arch. Neérl. **5**, 252 (1900), http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf

Transverse Momentum of the Electron in a Weak Wave

A free electron of mass m oscillates in this field such that its average position is at the origin. This simple statement hides the subtlety that our frame of reference is the average rest frame of the electron when inside the wave, and is not the lab frame of an electron that is initially at rest, but which is overtaken by a wave. If the velocity of the oscillating electron is small, we can ignore the $\mathbf{v}/c \times \mathbf{B}$ force, and take the motion to be entirely in the plane $z = 0$. Then, (also ignoring radiation damping) the equation of motion of the electron is,

$$m\ddot{\mathbf{x}} = e\mathbf{E}_{\rm wave}(0, t) = e\hat{\mathbf{x}}E_0 \cos\omega t. \tag{18}$$

Using eq. (17) we find the position of the electron to be,

$$\mathbf{x} = -\frac{e}{m\omega^2}\hat{\mathbf{x}}E_0 \cos\omega t. \tag{19}$$

and the mechanical transverse momentum of the electron is,

$$\mathbf{P}_{{\rm mech},\perp} = m\dot{\mathbf{x}} = \frac{e}{\omega}\hat{\mathbf{x}}E_0 \sin\omega t. \tag{20}$$

It is important to note that $\mathbf{P}_{{\rm mech},\perp}$ is proportional to the first power of the wave field strength.

Longitudinal Motion of the Electron

The root-mean-square (rms) transverse velocity of the electron is,

$$v_{\rm rms} = \sqrt{\langle \dot{x}^2 \rangle} = \frac{eE_{\rm rms}}{m\omega c}c. \tag{21}$$

The condition that $\mathbf{v}/c \times \mathbf{B}$ small is then,

$$\eta \equiv \frac{eE_{\rm rms}}{m\omega c} \ll 1, \tag{22}$$

where the dimensionless measure of field strength, η, is a Lorentz invariant. Similarly, the rms departure of the electron from the origin is,

$$x_{\rm rms} = \frac{eE_{\rm rms}}{m\omega^2} = \frac{\eta\lambda}{2\pi}. \tag{23}$$

Thus, condition (22) also insures that the extent of the motion of the electron is small compared to a wavelength, and so we may use the dipole approximation when considering the fields of the oscillating electron.

In the weak-field approximation, we can now use eq. (20) for the velocity to evaluate the second term of the Lorentz force,

$$e\frac{\mathbf{v}}{c} \times \mathbf{B} = \frac{e^2E_x^2}{2m\omega c}\hat{\mathbf{z}} \sin 2\omega t. \tag{24}$$

This term vanishes for circular polarization, in which case the motion is wholly in the transverse plane. However, for linear polarization the $\mathbf{v}/c \times \mathbf{B}$ force leads to oscillations along the z axis at frequency 2ω, as first analyzed in general by Landau.[18]

For polarization along the x-axis, the x-z motion has the form of a "figure 8", which for weak fields ($\eta \ll 1$) is described by,

$$x = -\frac{eE_x}{m\omega}\cos\omega t, \qquad z = -\frac{e^2 E_x^2}{8m^2\omega^3 c}\sin 2\omega t. \tag{25}$$

If the electron had been at rest before the arrival of the plane wave, then inside the wave it would move with an average drift velocity given by,

$$v_z = \frac{\eta^2/2}{1+\eta^2/2}c, \tag{26}$$

along the direction of the wave vector, as first deduced by McMillan.[19] In the present paper, we work in the frame in which he electron has no average velocity along the z axis. Therefore, prior to its encounter with the plane wave the electron had been moving in the negative z direction with speed given by eq. (26).

The Interference Term $\mathbf{P}_{\rm wave,static}$

The oscillating charge has oscillating fields, and the strength of those oscillating fields is proportional to the strength of the incident wave field. Hence, if we insert the oscillating fields of the charge into eq. (16), the interaction momentum will be quadratic in the wave field strength. This momentum cannot possibly balance the mechanical transverse momentum (20).

For the interaction momentum (16) to yield a result proportional to the wave field strength, we need to insert a field associated with the charge that is independent of the wave field. Thus, we are led to consider the static field of the charge.

Indeed, the fields associated with the electron can be regarded as the superposition of those of an electron at rest at the origin plus those of a dipole consisting of the actual oscillating electron and a positron at rest at the origin. Thus, we can write the electric field of the electron as $\mathbf{E}_{\rm static} + \mathbf{E}_{\rm osc}$, and the magnetic field as $\mathbf{B}_{\rm osc}$.

The interaction field momentum density can now be written,

$$\mathbf{p}_{\rm int} = \mathbf{p}_{\rm wave,static} + \mathbf{p}_{\rm wave,osc}, \tag{27}$$

where,

$$\mathbf{p}_{\rm wave,static} = \frac{\mathbf{E}_{\rm static} \times \mathbf{B}_{\rm wave}}{4\pi c}. \tag{28}$$

[18] L. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Pergamon Press, 1975), Prob. 2, § 47 and Prob. 2, § 49, http://kirkmcd.princeton.edu/examples/EM/landau_ctf_75.pdf
p. 112 of the 1941 Russian edition, http://kirkmcd.princeton.edu/examples/EM/landau_teoria_polya_41.pdf

[19] E.M. McMillan, *The Origin of Cosmic Rays*, Phys. Rev. **79**, 498 (1950), http://kirkmcd.princeton.edu/examples/accel/mcmillan_pr_79_498_50.pdf

and,

$$\mathbf{p}_{\rm wave,osc} = \frac{\mathbf{E}_{\rm wave} \times \mathbf{B}_{\rm osc} + \mathbf{E}_{\rm osc} \times \mathbf{B}_{\rm wave}}{4\pi c}. \tag{29}$$

We recall from eqs. (20) and (25) that the transverse mechanical momentum of the oscillating electron has pure frequency ω. Since the wave and the oscillating part of the electron's field each have frequency ω, the term $\mathbf{p}_{\rm wave,osc}$ contains harmonic functions of ω^2, which can be resolved into a static term plus ones in frequency 2ω. Hence, we have a second reason why we should not expect this term to cancel the mechanical momentum. Rather, we look to the term $\mathbf{p}_{\rm wave,static}$, since this has pure frequency ω. The term $\mathbf{p}_{\rm wave,osc}$ cancels the longitudinal momentum associated with the "figure-8" motion, and also includes a "hidden momentum" related to the fact that the average rest frame of an electron inside the wave is not the rest frame of the electron in the absence of the wave, as sketched in secs. 3-4.[20]

The static field of the electron at the origin is, in rectangular coordinates,

$$\mathbf{E}_{\rm static} = \frac{e}{r^3}(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}), \tag{30}$$

where r is the distance from the origin to the point of observation. Combing this with eq. (17) we have,

$$\mathbf{p}_{\rm wave,static} = \frac{e}{4\pi c r^3}\{-z\hat{\mathbf{x}} + x\hat{\mathbf{z}}\}E_0 \cos(kz - \omega t). \tag{31}$$

When we integrate this over all space to find the total field momentum, the term in $\hat{\mathbf{z}}$ vanishes as it is odd in x. Likewise, after expanding $\cos(kz-\omega t)$, the terms proportional to $z\cos kz$ vanish on integration. The remaining term is thus,

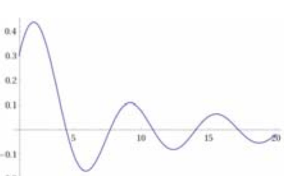
$$\begin{aligned} \mathbf{P}_{\rm wave,static} &= \int d{\rm Vol}\, \mathbf{p}_{\rm wave,static} \\ &= -\frac{e}{4\pi c}\hat{\mathbf{x}} E_0 \sin\omega t \int_V \frac{z \sin kz}{r^3} \\ &= -\frac{e}{\omega}\hat{\mathbf{x}} E_0 \sin \omega t = -\mathbf{P}_{\rm mech,\perp}, \end{aligned} \tag{32}$$

after an elementary volume integration (that involves integration by parts twice).[21]

[20] K.T. McDonald and K. Shmakov, *The Classical "Dressing" of a Free Electron in a Plane Electromagnetic Wave*; http://kirkmcd.princeton.edu/accel/dressing.pdf

[21]
$$\int_V \frac{z\sin kz}{r^3} = 2\pi \int_0^\infty dr \int_{-1}^1 du\, u\, \sin(kru) = 2\pi \int_0^\infty dr \left[-\left.\frac{u\,\cos(kru)}{kr}\right|_{-1}^1 + \int_{-1}^1 du\, \frac{\cos(kru)}{kr} \right] \tag{33}$$

$$= -4\pi \int_0^\infty dr\, \frac{\cos(kr)}{kr} + 4\pi \int_0^\infty dr\, \frac{\sin(kr)}{k^2r^2} = -4\pi \int_0^\infty dr\, \frac{\cos(kr)}{kr} - 4\pi \left.\frac{\sin(kr)}{k^2 r}\right|_0^\infty + 4\pi \int_0^\infty dr\, \frac{\cos(kr)}{kr} = \frac{4\pi}{k}.$$

$\sin(kr)/k^2r^2 - \cos(kr)/kr$ 

It is noteworthy that the integration is independent of any hypothesis as to the size of a classical electron. Indeed, the integrand of eq. (32) can be expressed as $\cos\theta \sin(kr\cos\theta)/r^2$ via the substitution $z = r\cos\theta$. Hence, the integral over a spherical shell varies as $\sin(kr)/k^2r^2 - \cos(kr)/kr$ (see footnote 10), and significant contributions to the integral occur for radii up to one wavelength of the electromagnetic wave. This contrasts with the self-momentum density of the electron which is formally divergent; if the integration is cut off at a minimum radius (the classical electron radius), the dominant contribution occurs within twice that radius.

The Momentum $\mathbf{P}_{\rm wave,osc}$

We could continue the analysis to consider the interaction field momentum $\mathbf{P}_{\rm wave,osc}$, but this leads to various subtleties. For discussion, see the link at the beginning of this solution.

2. A circularly polarized electromagnetic wave of angular frequency ω is incident on a free electron of charge e and rest mass m.

   Taking the origin to be at the time-average position of the electron, its steady motion in the circularly polarized wave is uniform circular motion at angular frequency ω in a circle of radius r in the plane perpendicular to the direction of the wave. Further, the radius vector to the electron is in line with $\mathbf{E}$, and hence perpendicular to $\mathbf{B}$, such that the electron's velocity is parallel to the magnetic field, Then, the equation of motion is,

$$F = eE = \gamma ma = \gamma m\omega^2 r = \gamma m\omega v = \gamma m\omega\beta c, \tag{34}$$

$$\frac{eE}{m\omega c} \equiv \eta = \gamma\beta, \qquad \eta^2 = \frac{\beta^2}{1-\beta^2}, \qquad \beta^2 = \frac{\eta^2}{1+\eta^2}, \qquad \gamma^2 = 1+\eta^2, \qquad r = \frac{eE}{\gamma m\omega^2}. \tag{35}$$

   We also note that the acceleration of the electron in this uniform circular motion is,

$$a = \frac{eE}{\gamma m}\left(= \omega\beta c = \frac{\omega\eta c}{\sqrt{1+\eta^2}} = \frac{\omega\eta c}{\gamma}\right). \tag{36}$$

   (a) The radiated intensity in the instantaneous rest frame of the electron is given by the Larmor formula,

$$\frac{dU^\star}{dt^\star} = \frac{2e^2 a^{\star 2}}{3c^3} = \frac{dU}{dt}, \tag{37}$$

   and this intensity is a Lorentz invariant, as discussed on p. 237, Lecture 20 of the Notes,[22] and so equals dU/dt, as measured at the charge in the lab frame. In the present case, the lab-frame acceleration $\mathbf{a}$ is perpendicular to the velocity $\mathbf{v}$, for which $a^\star = \gamma^2 a$, as discussed on p. 243, Lecture 20 of the Notes. Hence,

$$\frac{dU}{dt} = \frac{2e^2\gamma^4 a^2}{3c^3} = \frac{2e^4\gamma^2 E^2}{3m^2c^3} = \frac{2e^4E^2}{3m^2c^3}(1+\eta^2). \tag{38}$$

   (b) The leading terms in a multipole expansion of the radiated intensity according to a fixed, distant observer are those associated the electric dipole moment $\mathbf{p}$, the magnetic dipole moment $\mathbf{m}$, and the electric quadrupole (3-tensor) moment Q_{ij}. These were considered for the present example in Prob. 7, Set 8, http://kirkmcd.princeton.edu/examples/ph501set8.pdf, where it was noted that there is no magnetic dipole radiation, while eq. (25) of Set 8 says,

$$\frac{dU_{E1}}{dt} = \frac{2e^2r^2\omega^4}{3c^3} = \frac{2e^4E^2}{3\gamma^2m^2c^3}, \tag{39}$$

   and eqs. (26) and (162) of Set 8 imply that,

$$\frac{dU_{E_2}}{dt} = \frac{8e^2r^4\omega^6}{5c^3} = \frac{dU_{E1}}{dt}\frac{12r^2\omega^2}{5c^2} = \frac{dU_{E1}}{dt}\frac{12\eta^2}{5\gamma^2}. \tag{40}$$

[22] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture20.pdf

The multipole expansion for the radiated power can now be written as,

$$\frac{dU}{dt} = \frac{dU_{E1}}{dt} + \frac{dU_{E_2}}{dt} + \cdots = \frac{2e^4E^2}{3\gamma^2 m^2 c^3}\left(1 + \frac{12\eta^2}{5\gamma^2} + \cdots\right) \approx \frac{2e^4E^2}{3m^2c^3}\left(1 + \frac{7\eta^2}{5} + \cdots\right), \tag{41}$$

where the approximation holds for $\eta^2 \ll 1$, which is the realm for which it suffices to consider only the leading terms in the multipole expansion.[23]

As in Prob. 7, Set 8, we note that in this example the electric-dipole radiation varies with angular frequency ω, but the electric-quadrupole radiation varies with frequency 2ω.

[23]Comparison of eqs. (38) and (41) suggests that we should also consider magnetic-quadrupole radiation.

3. We consider an elliptically polarized plane electromagnetic wave in vacuum,

$$\mathbf{E}_{\rm in} = (\mathbf{a} + \mathbf{b})\, e^{i(kz-\omega t)}, \qquad \mathbf{B}_{\rm in} = \hat{\mathbf{z}} \times \mathbf{E}_{\rm in}, \tag{42}$$

where $a \neq b$ in general, while $\mathbf{a} \cdot \mathbf{b} = 0 = \mathbf{a} \cdot \hat{\mathbf{z}} = \mathbf{b} \cdot \hat{\mathbf{z}}$ and $\omega = kc$.

If this wave is incident on a free electron of charge e and mass m, leading to motion with speed $v \ll c$, where c is the speed of light in vacuum, the steady-state motion $\mathbf{x} = \mathbf{x}_0\, e^{i(kz-\omega t)}$ of the charge is approximately given by

$$m\ddot{\mathbf{x}} = \mathbf{F} = e \left(\mathbf{E}_{\rm in} + \frac{\mathbf{v}}{c} \times \mathbf{B}_{\rm in} \right) \approx e\, (\mathbf{a} + \mathbf{b})\; e^{i(kz-\omega t)}, \tag{43}$$

neglecting the very small term $\mathbf{v}/c \times \mathbf{B}_{\rm in}$ when $v \ll c$. Then, noting that the electric-dipole moment is $\mathbf{p} = e\,\mathbf{x}$, the time-average radiated intensity has the angular distribution, in the electric-dipole approximation (which is valid for $v \ll c$),

$$\begin{aligned} \left\langle \frac{d^2U}{dt\, d\Omega} \right\rangle &= \frac{1}{2} \frac{|\ddot{\mathbf{p}} \times \hat{\mathbf{n}}|^2}{4\pi c^3} = \frac{e^4}{8\pi c^3 m^2} \,|(\mathbf{a} + \mathbf{b}) \times \hat{\mathbf{n}}|^2 \\ &= \frac{e^4}{8\pi c^3 m^2} \left(|\mathbf{a} \times \hat{\mathbf{n}}|^2 + |\mathbf{b} \times \hat{\mathbf{n}}|^2 + 2\,|(\mathbf{a} \times \hat{\mathbf{n}}) \cdot (\mathbf{b} \times \hat{\mathbf{n}})| \right) \\ &= \frac{e^4}{8\pi c^3 m^2} \left(|\mathbf{a} \times \hat{\mathbf{n}}|^2 + |\mathbf{b} \times \hat{\mathbf{n}}|^2 - 2Re[(\mathbf{a} \cdot \hat{\mathbf{n}})(\mathbf{b} \cdot \hat{\mathbf{n}})] \right). \end{aligned} \tag{44}$$

recalling p. 186 of Lecture 16,[24] and using the vector identity that $(\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$.

The time-average incident flux of electromagnetic energy is given by the time-average Poynting vector,

$$\begin{aligned} \langle \mathbf{S}_{\rm in} \rangle &= \frac{1}{2} \frac{c}{4\pi} Re(\mathbf{E} \times \mathbf{B}^\star) = \frac{c}{8\pi} Re \left\{ [(\mathbf{a} + \mathbf{b}) \times [\hat{\mathbf{z}} \times (\mathbf{a}^\star + \mathbf{b}^\star)] \right\} \\ &= \frac{c}{8\pi} Re \left\{ (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a}^\star + \mathbf{b}^\star)\, \hat{\mathbf{z}} - [(\mathbf{a} + \mathbf{b}) \cdot \hat{\mathbf{z}}](\mathbf{a}^\star + \mathbf{b}^\star) \right\} \\ &= \frac{c}{8\pi} \left(|\mathbf{a}|^2 + |\mathbf{b}|^2 \right). \end{aligned} \tag{45}$$

Then, the scattering cross section is, recalling p. 275 of Lecture 23,[25]

$$\begin{aligned} \frac{d\sigma}{d\Omega} = \frac{\langle d^2U/dt\, d\Omega \rangle}{\langle S_{\rm in} \rangle} &= \frac{e^4}{m^2 c^4} \frac{|\mathbf{a} \times \hat{\mathbf{n}}|^2 + |\mathbf{b} \times \hat{\mathbf{n}}|^2 - 2Re[(\mathbf{a} \cdot \hat{\mathbf{n}})(\mathbf{b} \cdot \hat{\mathbf{n}})]}{|\mathbf{a}|^2 + |\mathbf{b}|^2} \\ &= r_e^2 \frac{|\mathbf{a} \times \hat{\mathbf{n}}|^2 + |\mathbf{b} \times \hat{\mathbf{n}}|^2 - 2Re[(\mathbf{a} \cdot \hat{\mathbf{n}})(\mathbf{b} \cdot \hat{\mathbf{n}})]}{|\mathbf{a}|^2 + |\mathbf{b}|^2}, \end{aligned} \tag{46}$$

where $r_e = e^2/mc^2$ is the classical electron radius.

[24] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture16.pdf
[25] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture23.pdf

If the incident electromagnetic wave is linearly polarized, with $\mathbf{b} = 0$, then the scattering cross section is,

$$\frac{d\sigma}{d\Omega} = r_e^2 \sin^2\alpha, \qquad \sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{8\pi r_e^2}{3}\,, \tag{47}$$

where α is the angle between $\mathbf{a}$ and $\hat{\mathbf{n}}$. This is the Thomson cross section found on p. 276 of Lecture 23.[26]

If the incident electromagnetic wave is circularly polarized, then $\mathbf{b} = \pm i\mathbf{a}$, and the scattering cross section is again given by eq. (47), *i.e.*, the Thomson cross section.

From eq. (41) above, for $v \ll c$ we have $\eta \ll 1$ and,

$$\frac{dU}{dt} \approx \frac{2e^4 E^2}{3m^2 c^3}\,. \tag{48}$$

Then, the total cross section (for a circularly polarized incident wave) is,

$$\sigma = \frac{dU/dt}{S_{\rm in}} \approx \frac{2e^4 E^2/3m^2c^3}{cE^2/4\pi} = \frac{8\pi e^4}{3m^2c^4} = \frac{8\pi r_e^2}{3}\,, \tag{49}$$

as in eq. (47).

[26] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture23.pdf

4. We consider a linearly polarized plane electromagnetic wave in vacuum that is incident on an axially symmetric polar molecular of fixed electric-dipole moment $\mathbf{p}$, supposing that the size of the molecule is small compared to the wavelength, *i.e.*, $\lambda \gg p/e$, and assuming that all directions of $\mathbf{p}$ are equally likely (as in a gas).

The electromagnetic fields of the incident wave can be written as,

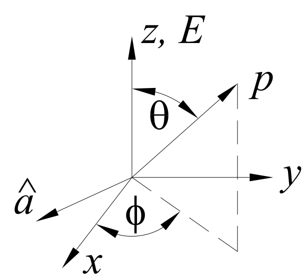
$$\mathbf{E} = E_0 \cos(kx - \omega t)\, \hat{\mathbf{z}}, \qquad \mathbf{B} = -E_0 \cos(kx - \omega t)\, \hat{\mathbf{y}}, \tag{50}$$

where $\omega = kc$ and c is the speed of light in vacuum. This wave exerts torque,

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} = \mathbf{p} \times E_0 \cos \omega t\, \hat{\mathbf{z}}, \tag{51}$$

on the molecule, taking it to be at the origin, and supposing that the molecule is small compared to the wavelength of the incident wave. Hence, molecule rotates about the constant vector,

$$\hat{\mathbf{a}} = \frac{\mathbf{p} \times \hat{\mathbf{z}}}{|\mathbf{p} \times \hat{\mathbf{z}}|}. \tag{52}$$



That is, azimuthal angle ϕ remains constant while polar angle θ varies, and since $|\mathbf{p}|$ is constant,

$$\frac{d\mathbf{p}}{dt} = \dot{\mathbf{p}} = \dot{\theta}\, \hat{\mathbf{a}} \times \mathbf{p}, \qquad \ddot{\mathbf{p}} = \ddot{\theta}\, \hat{\mathbf{a}} \times \mathbf{p} + \dot{\theta}\, \hat{\mathbf{a}} \times \dot{\mathbf{p}} = \ddot{\theta}\, \hat{\mathbf{a}} \times \mathbf{p} - \dot{\theta}^2 \mathbf{p} \approx \ddot{\theta}\, \hat{\mathbf{a}} \times \mathbf{p}, \tag{53}$$

where we neglect the term in $\dot{\theta}^2$ as second order in the velocity of the rotating dipole. The radiated intensity is, recalling p. 186 of Lecture 16 of the Notes,[27]

$$\frac{dU}{dt} = \frac{2\,|\ddot{\mathbf{p}}|^2}{3c^3} \approx \frac{2p^2\, \ddot{\theta}^2}{3c^3}. \tag{54}$$

Taking I to be the moment of inertia of the (axially symmetric) molecule about any axis (such as $\hat{\mathbf{a}}$) perpendicular to $\mathbf{p}$, and supposing that the molecule is not rotating in the absence of the incident wave, its angular momentum is $\mathbf{L} = I\, \dot{\theta}\, \hat{\mathbf{a}}$. Then, the torque equation of motion is, for a molecule centered on the origin,

$$\frac{d\mathbf{L}}{dt} = I\, \ddot{\theta}\, \hat{\mathbf{a}} = \boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}, \qquad \ddot{\theta} = \frac{|\mathbf{p} \times \mathbf{E}|}{I} = \frac{pE_0 \cos \omega t \sin \theta}{I}. \tag{55}$$

[27] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture16.pdf

Hence,

$$\frac{dU}{dt} \approx \frac{2p^4 E_0^2 \cos^2 \omega t \sin^2 \theta}{3c^3 I^2} \,. \tag{56}$$

For a collection of molecules with random orientation of $\mathbf{p}$, as in a gas, the average radiated intensity is,

$$\left\langle \frac{dU}{dt} \right\rangle \approx \frac{2p^4 E_0^2 \cos^2 \omega t \left\langle \sin^2 \theta \right\rangle}{3c^3 I^2} = \frac{4p^4 E_0^2 \cos^2 \omega t}{9c^3 I^2} \,. \tag{57}$$

The incident flux of energy is, at the origin,

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} = \frac{cE^2}{4\pi} \,\hat{\mathbf{x}} = \frac{cE_0^2 \cos^2 \omega t}{4\pi} \,\hat{\mathbf{x}}, \tag{58}$$

so the (average) scattering cross section is,

$$\sigma = \frac{\langle dU/dt \rangle}{S} \approx \frac{16\pi p^4}{9c^4 I^2} \,. \tag{59}$$

This problem was first considered by Lord Rayleigh, Phil. Mag. **35**, 373 (1918),
http://kirkmcd.princeton.edu/examples/EM/rayleigh_pm_35_373_18.pdf
See also Prob. 3, p. 220 of http://kirkmcd.princeton.edu/examples/EM/landau_ctf_75.pdf

5. **Spectral Line Broadening**

This is Prob. 16.12, p. 772 of http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

A spectral "line" of central wavelength λ is always observed to have a finite width $\Delta\lambda$.

a) As discussed on pp. 272-275 of Lecture 23 of the Notes,[28] an electron oscillating with natural angular frequency ω is subject to damping due to the radiation reaction, which leads to a spread of frequencies of full width at half maximum in the frequency spectrum of $\Delta\omega = \Gamma_r/2 = \omega^2 r_e/3c$, where $\Gamma_r = 2\omega^2 r_e/3c$ is the radiation-damping constant, $r_e = e^2/mc^2$ is the classical electron radius, e and m are the charge and mass of the electron, and c is the speed of light in vacuum.[29]

Then,

$$\omega = \frac{2\pi c}{\lambda}, \qquad \Delta\omega = \frac{2\pi c \Delta\lambda}{\lambda^2}, \tag{60}$$

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda\Delta\omega}{2\pi c} = \frac{\Delta\omega}{\omega} = \frac{\omega r_e}{3c} = \frac{2\pi r_e}{3\lambda} \approx \frac{2r_e}{\lambda} \approx \frac{2 \cdot 2.8 \times 10^{-13}\ \text{cm}}{6 \times 10^{-5}\ \text{cm}} \approx 10^{-8}, \tag{61}$$

for the sodium line with $\lambda = 5893$ Å, and $\omega \approx 3 \times 10^{15}\ \text{s}^{-1}$.

b) **Doppler Broadening**

If glowing sodium vapor has temperature 600K the typical velocity $\bar{v}$ of the sodium atoms is related by $M\bar{v}^2/2 = 3kT/2$, where $M = 23 \cdot 1.7 \times 10^{-24}$ gm is the mass of a sodium atom (of atom mass 23) and $k = 1.4 \times 10^{-16}$ erg/K is Boltzmann's constant.

From p. 215 of Lecture 18 of the Notes,[30] the relativistic Doppler shift is $\omega' = \gamma\omega(1 + \hat{\mathbf{n}} \cdot \mathbf{v}/c)$. Hence, the distribution of frequencies ω' emitted by the (slow) moving sodium atoms with random directions is flat between $\omega(1-v/c)$ and $\omega(1+v/c)$, with $\Delta\omega_{\text{Doppler}} \approx 2\omega v/c$. Then,

$$\frac{\Delta\lambda_{\text{Doppler}}}{\lambda} = \frac{\Delta\omega_{\text{Doppler}}}{\omega} \approx \frac{2\bar{v}}{c} \approx \frac{2}{c}\sqrt{\frac{3kT}{M}} \approx \frac{2}{3 \times 10^{10}}\sqrt{\frac{3 \cdot 1.4 \times 10^{-16} \cdot 600}{23 \cdot 1.7 \times 10^{-24}}} \approx 6 \times 10^{-6}, \tag{62}$$

about 600 times that due to radiation damping.

We note that the typical velocity of the sodium atoms is $\bar{v} \approx 8 \times 10^4$ cm/ s (so $\gamma = 1/\sqrt{1 - \bar{v}^2/c^2} \approx 1$), and that the damping constant for Doppler broadening at 600K is related by,

$$\Gamma_{\text{Doppler}} = 2\Delta\omega = 2\frac{\Delta\omega}{\omega}\omega \approx 2 \cdot \cdot 6 \times 10^{-6} 3 \times 10^{15} \approx 4 \times 10^{10}\ \text{s}^{-1}. \tag{63}$$

[28] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture23.pdf

[29] On p. 273 of Lecture 15, the symbol $\Delta\omega$ is used to denote the (very small) shift in the central frequency of the oscillation due to radiation damping. Here, $\Delta\omega$ is the width of the frequency spectrum.

[30] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture18.pdf

c) **Collision Broadening**

At atom starts glowing (becomes excited) due to a collision with another atom, which sets its "spring" oscillation. A second collision at time Δt later, which destroys the coherence of the oscillation. This limits the width of the pulse of radiation, and according to the "uncertainty" relation $\Delta\omega\,\Delta t \approx 1$, the spectrum of the pulse is broadened.

Let ν be the mean frequency of collisions [#/sec]. Then,

$$\nu = [\text{collison cross section}] \cdot [\#\ \text{atoms/volume}] \cdot [\text{mean relative velocity}]. \tag{64}$$

The collision cross section in sodium vapor is around 10^{14} cm^2, and the mean relative velocity of colliding atoms is approximately their rms velocity $\bar{v}$ found is part b) above. The no. of atoms per unit volume follow from the ideal gas law,

$$\frac{N}{V} = \frac{P}{kT} = P[\text{atm}]\frac{10^6[\text{dyne/cm}^2/\text{atm}]}{1.4 \times 10^{-16} \cdot 600} \approx 1.1 \times 10^{20} \cdot P[\text{atm}], \tag{65}$$

where P is the gas pressure. Then,

$$\nu \approx 10^{-14} \cdot 1.1 \times 10^{20} \cdot P[\text{atm}] \cdot 8 \times 10^4 \approx 10^{11} \cdot P[\text{atm}]. \tag{66}$$

The probability of a collision during time dt is $\nu\, dt$, so the probability that no collision occurred between $t = 0$ and t is $e^{-\nu t}$. Averaging over many collisions, this means that the intensity of the radiation at time t after the beginning of emission is $I_0\, e^{-\nu t}$. This is similar to the effect of other damping mechanisms, for which $E = E_0\, e^{-\Gamma_{\text{other}} t}$ and $I \propto E^2 = I_0\, e^{-2\Gamma_{\text{other}} t}$. Hence, the effective damping constant in the presence of collisions is,

$$\Gamma = \frac{\nu}{2} + \Gamma_{\text{other}}. \tag{67}$$

The contributions to the width of the frequency spectrum from collision broadening and Doppler broadening are equal when,

$$\frac{\nu}{2} \approx 5 \times 10^{10} \cdot P[\text{atm}] = \Gamma_{\text{Doppler}} \approx 4 \times 10^{10}, \qquad P \approx 0.8 \text{ atm}. \tag{68}$$

At this pressure,

$$\left.\frac{\Delta\lambda}{\lambda}\right|_{\text{collisions}} = \left.\frac{\Delta\lambda}{\lambda}\right|_{\text{Doppler}}. \tag{69}$$

6. **Optical Theorem**

   This is Prob. 9, p. 424 of kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

   The equation of motion of an electron of charge e and mass m that is bound to atom and subject to a weak, linearly polarized, incident plane electromagnetic wave can be taken as,

$$m(\ddot{\mathbf{x}} + \Gamma_0 \dot{\mathbf{x}} + \omega_0^2 \mathbf{x}) = e\mathbf{E} = eE_0 \mathrm{Re}\, e^{i(kz-\omega t)}\, \hat{\mathbf{x}}, \tag{70}$$

   where ω_0 is the natural frequency of oscillation of the electron, Γ_0 is the damping constant of that oscillation, $\omega = kc$ is the angular frequency of the incident wave, and we neglect the $\mathbf{v} \times \mathbf{B}$ force for a weak incident wave. Then,

$$\mathbf{x} = \frac{e\mathbf{E}}{m(\omega_0^2 - \omega^2 - i\omega\Gamma_0)}, \qquad \ddot{\mathbf{x}} = \frac{\omega^2 e\mathbf{E}}{m(\omega^2 - \omega_0^2 + i\omega\Gamma_0)}. \tag{71}$$

   The differential scattering cross section is, in the electric-dipole approximation,

$$\frac{d\sigma_{\mathrm{scat}}}{d\Omega} = \frac{d^2U/dt\, d\Omega}{|\mathbf{S}|} = \frac{|e\ddot{\mathbf{x}} \times \hat{\mathbf{n}}|^2 / 4\pi c^3}{c\, |\mathbf{E}|^2 / 4\pi} = \left| \frac{e^2 \omega^2\, \hat{\mathbf{x}} \times \hat{\mathbf{n}}}{mc^2(\omega^2 - \omega_0^2 + i\omega\Gamma_0)} \right|^2 \equiv |f(\theta)|^2, \tag{72}$$

   where θ is the angle between $\hat{\mathbf{n}}$ and the z-axis.

   There is damping in the present model, which implies that the atom can absorb energy as well as scatter it. On p. 281 of Lecture 23 of the Notes,[31] we found that,

$$\sigma_{\mathrm{total}} = 4\pi r_e c \frac{\omega^2 \Gamma_0}{(\omega^2 - \omega_0^2)^2 + \omega^2 \Gamma_0^2}. \tag{73}$$

   According to the optical theorem, and using eq. (72),

$$\sigma_{\mathrm{total}} = \frac{4\pi}{k} |\mathrm{Im} f(0)| = \frac{4\pi c}{\omega} \frac{r_e\, \omega^3 \Gamma_0}{(\omega^2 - \omega_0^2)^2 + \omega^2 \Gamma_0^2}, \tag{74}$$

   in agreement with eq. (73), noting that in the forward scattering direction, $\theta = 0$ and $\hat{\mathbf{n}} = \hat{\mathbf{z}}$.

   For a free electron, $\omega_0 = 0$, but there still exists damping due to the radiation reaction, $\Gamma_0 \to \Gamma_r = 2\omega^3 r_e / 3c$, the radiation-damping constant at angular frequency ω. The total cross section is the same as the total scattering cross section, which follows from eq. (72) as,[32]

$$\sigma_{\mathrm{total}} = \int \frac{d\sigma_{\mathrm{scat}}}{d\Omega}\, d\Omega = \frac{8\pi r_e^2}{3} \frac{\omega^4}{\omega^4 + \omega^2 \Gamma_r^2} = 8\pi c r_e \frac{\omega^2 \Gamma_r}{\omega^4 + \omega^2 \Gamma_r^2}, \tag{75}$$

   in agreement with eq. (72) for $\omega_0 = 0$ and $\Gamma_0 \to \Gamma_r$.

[31] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture23.pdf

[32] In a spherical coordinate system with $\hat{\mathbf{x}}$ as the polar axis, and α as the polar angle of $\hat{\mathbf{n}}$, the differential cross section varies as $\sin^2 \alpha$ and quickly integrates to give eq. (75).

7. **Levinger-Bethe Sum Rule**

This is Prob. 7, p. 424 of kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

On p. 282 of Lecture 23,[33] we found that for a charge e of mass m bound to an "atom" by an "oscillator" with a natural angular frequency ω_0 and associated damping constant Γ_0, the total cross section obeys a "sum rule",

$$\int \sigma_{\rm total}\, d\omega = \int_0^\infty \frac{8\pi r_e^2}{3} \frac{\Gamma_0}{\Gamma_r} \frac{\omega^2\omega_0^2}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma_0^2}\, d\omega = \frac{2\pi^2 e^2}{mc}\,. \tag{76}$$

Expressing this in terms of the energy $E = \hbar\omega$ of quanta of the incident electromagnetic radiation,

$$\int \sigma_{\rm total}\, dE = \int \sigma_{\rm total}\, d(\hbar\omega) = \frac{2\pi^2 e^2\hbar}{mc} = 2\pi^2 \frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2 mc^2 = 2\pi^2\alpha\lambda\!\!\!^{-}{}_m^2 mc^2, \tag{77}$$

where $\alpha = e^2/\hbar c = 1/137$ for e as the charge of the electron or proton, and $\lambda\!\!\!^{-}_m = \hbar/mc$ is the reduced Compton wavelength for a particle of mass m.

We now consider three models for the response of a nucleus with Z protons and N neutrons, each of mass $M \approx 940$ MeV$/c^2$. In all of these, $e \to Ze$ and the α in eq. (77) goes to $Z^2\alpha$. The reduced Compton wavelength of the proton is $\lambda\!\!\!^{-}_p \approx 2\times 10^{-14}$ cm.

(a) The entire nucleus moves as a whole.[34]

Then, $m \to AM$ where $A = N + Z$, and,

$$\int \sigma_{\rm total}\, dE = 2\pi^2 Z^2\alpha \frac{\lambda\!\!\!^{-}{}_M^2}{A^2} AMc^2 = \frac{Z^2}{A} 2\pi^2\alpha\lambda\!\!\!^{-}{}_M^2 Mc^2 = \frac{Z^2}{A} 5.4\times 10^{-26}\ {\rm MeV\cdot cm^2}. \tag{78}$$

For copper, with $Z = 29$ and $A = 63$, $Z^2/A = 13.3$ and,

$$\int \sigma_{\rm total}\, dE = 0.78 \times \times 10^{-24}\ {\rm MeV\cdot cm^2}. \tag{79}$$

(b) The neutrons in the nucleus remain fixed while only the protons move about.[35]

Then, $m \to ZM$, and,

$$\int \sigma_{\rm total}\, dE = 2\pi^2 Z^2\alpha \frac{\lambda\!\!\!^{-}{}_M^2}{Z^2} ZMc^2 = Z\cdot 2\pi^2\alpha\lambda\!\!\!^{-}{}_M^2 Mc^2 = 1.6 \times \times 10^{-24}\ {\rm MeV\cdot cm^2}. \tag{80}$$

[33] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture23.pdf

[34] Between eqs. (4) and (5) of the paper of Levinger and Bethe, it was claimed that if the nucleus moves as a whole, photons cannot be absorbed. This seems wrong, as the reaction $\gamma + Cu^{63} \to Cu^{62} + n$ involves absorption of a photon, and could proceed with Cu^{63} and Cu^{62} as entities without substructure.

[35] In the "liquid drop" model of nuclei advocated by Bohr,
http://kirkmcd.princeton.edu/examples/nuclear/bohr_pr_55_418_39.pdf
http://kirkmcd.princeton.edu/examples/nuclear/bohr_pr_56_426_39.pdf, the nucleons have strong interactions, but can move freely with respect to one another, as in a liquid. This model seems consistent with scenario (b).

(c) The protons move as a group, and the neutrons move as a separate group, which was claimed by Bethe[36] to lead to,

$$\int \sigma_{\rm total}\, dE = 2\pi^2 \frac{NZ}{A} \alpha \lambda\!\!\!^{-2}_M Mc^2 = 0.85 \times \times 10^{-24}\ {\rm MeV} \cdot {\rm cm}^2, \qquad (81)$$

with $NZ/A = 15.7$ for copper.

The experimental data[37] indicate that,

$$\int \sigma_{\rm total}\, dE \approx 1.5 \times 10^{-24}\ {\rm MeV} \cdot {\rm cm}^2. \qquad (82)$$

which seems to favor model (b), that the protons move under the influence of the incident electromagnetic wave while the neutrons are unaffected.

[36] J.S. Levinger and H.A. Bethe, *Dipole Transitions in the Nuclear Photo-Effect*, Phys. Rev. **78**, 115 (1950), http://kirkmcd.princeton.edu/examples/EP/levinger_pr_78_115_50.pdf

[37] G.C. Baldwin and G.S. Klaiber, *X-Ray Yield Curves for γ-n Reactions*, Phys. Rev. **73**, 1156 (1948), http://kirkmcd.princeton.edu/examples/EP/baldwin_pr_73_1156_48.pdf. The total cross section is dominated by the reaction $\gamma + {\rm Cu}^{63} \to {\rm Cu}^{62} + n$, which exhibits a "resonance" for incident gamma-ray energy around 25 MeV.

8. *This is Prob. 2, p. 423 of* kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

According to the model of the index of refraction reviewed on p. 282 of Lecture 23 of the Notes,[38] in a dilute gas of free electrons with number density N,

$$n^2 = 1 - \frac{4\pi N e^2}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\,, \tag{83}$$

where the plasma frequency of the electron gas is,

$$\omega_p = \sqrt{\frac{4\pi N e^2}{m}}\,, \qquad \hbar\omega_p = mc^2\sqrt{\frac{4\pi N e^2\hbar^2}{m^3c^4}} = mc^2\sqrt{4\pi N\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^3} = mc^2\sqrt{4\pi N\alpha\lambda\!\!\!^-{}_C^3}\,, \tag{84}$$

with $\alpha = e^2/\hbar c = 1/137$ and $\lambda\!\!\!^-{}_C = \hbar/mc = 3.87 \times 10^{-11}$ cm is the reduced Compton wavelength of an electron.

For $N = 1/\mathrm{cm}^2$, the plasma frequency is,

$$\omega_p = \sqrt{\frac{(4.8 \times 10^{10})^2}{9.1 \times 10^{-28}}} = 1.6 \times 10^{24}\ \mathrm{s}^{-1}, \tag{85}$$

and the corresponding photon energy is,

$$\hbar\omega_p = 5.11 \times 10^5\ \mathrm{eV}\ \sqrt{\frac{4\pi}{137}(3.87 \times 10^{-11})^3} = 3.7 \times 10^{-12}\ \mathrm{eV}. \tag{86}$$

In this model, the index of refraction is real for $\omega > \omega_p$, in which case these electromagnetic waves propagate with "no attenuation".

However, the electromagnetic waves scatter off the free electrons, and the scattered photons are effectively lost to an observer of, say, a distant star.

For photon energy small compared to the mass of an electron, the scattering cross section is $\sigma_{\mathrm{Thomson}} = 8\pi r_e^2/3$, where $r_e = e^2/mc^2$ is the classical electron radius. We recall that the concept of the scattering cross section is related to the probability that a scattering occurs along a path of length l through the scattering centers of number density N according to,[39]

$$P_{\mathrm{scattering}} = N\sigma l. \tag{87}$$

Hence, the probability that no scatters occured over this distance is,

$$P_{\mathrm{no\ scattering}} = e^{-N\sigma l}, \tag{88}$$

and the attenuation length L due to scattering is therefore,

$$L = \frac{1}{N\sigma}\,. \tag{89}$$

For $N = 1/\mathrm{cm}^3$ and $\sigma_{\mathrm{Thomson}} = 8\pi r_e^2/3 = 6.6 \times 10^{-25}$ cm^2, the attenuation length is $L = 1.5 \times 10^{24}$ cm $\approx 1.5 \times 10^6$ lightyears, about half the distance from Earth to the Andromeda galaxy.

[38] http://kirkmcd.princeton.edu/examples/ph501/ph501lecture23.pdf

[39] See, for example, p. 12 of http://kirkmcd.princeton.edu/examples/ph529/ph529l1.pdf

9. **Gravitational Redshift**

This problem follows A. Einstein, Ann. d. Phys. **35**, 898 (1911),
http://kirkmcd.princeton.edu/examples/GR/einstein_ap_35_898_11.pdf
http://kirkmcd.princeton.edu/examples/GR/einstein_ap_35_898_11_english2.pdf

We consider an accelerated frame A (in zero gravity) that coincides with inertial frame I′ at time $t = 0 = t'$ and has acceleration g along the z' axis.[40]

At time $t = 0 = t'$ light is emitted with frequency ν_0 from a source at rest at the origin in frame A (and so is also considered to have frequency $\nu' = \nu_0$ in frame I′). This light is detected by an observer at rest at time $(0, 0, z_1)$ at time $t_1$in frame A where it is found to have frequency ν_1. To a first approximation, $t_1 = z_1/c \approx t'_1$, where t'_1 is the time of observation of the light according to frame I′, when frame A has velocity $v'_1 = gt'_1 \approx gz_1/c$ with respect to frame I′.[41]

We now consider a second frame I″ which is the instantaneous inertial frame that coincides with frame A at time t'_1. Then, frame I″ has velocity v'_1 with respect to frame I′.

Recalling the discussion of the Doppler effect on p. 215 of Lecture 18 of the Notes,[42] an observer in frame I″ finds the light to have frequency,

$$\nu''_1 = \nu' \frac{1 + \hat{\mathbf{n}} \cdot \mathbf{v}'_1/c}{\sqrt{1 - v'^2_1/c^2}} \approx \nu_0(1 - v'_1/c) \approx \frac{\nu_0}{1 + v'_1/c} \approx \frac{\nu_0}{1 + gz_1/c^2} \,. \tag{91}$$

We argue that ν''_1 in the is the same as the frequency ν_1 observed in the accelerated frame A, as frame I″ is the instantaneous inertial frame of the observer in frame A.

Finally, we invoke the equivalence principle to identify

$$\nu''_1 = \nu_1 \approx \frac{\nu_0}{1 + gz_1/c^2} \approx \frac{\nu_0}{1 + \Phi(z_1)/c^2} \,, \tag{92}$$

as the (redshifted) frequency observed in a uniform gravitational field at height z_1 above the point where the light was emitted with frequency ν_0.

---

[40]We ignore the distinction between (discussed on p. 268 of Lecture 22, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture22.pdf) between uniform acceleration with repect to frame A and that with respect to frame I′, whic distinction was not yet well understood in 1911.

[41]We could also note that (with respect to frame I′) the origin of frame A moved distance $gt'^2_1/2$ during time t'_1, while the light traveled distance ct'_1. The distance between these two is,

$$\Delta z' = ct'_1 - \frac{gt'^2_1}{2} \,, \qquad t'_1 = \frac{c}{g} \pm \sqrt{\frac{c^2}{g^2} - \frac{2\Delta z'}{g}} \approx \frac{\Delta z'}{c} \,. \tag{90}$$

Then, to a first approximation, $\Delta z' = z_1$ and $v'_1 = gt'_1 = gz_1/c$.

[42]http://kirkmcd.princeton.edu/examples/ph501/ph501lecture18.pdf