Princeton University

Ph501

Electrodynamics

Problem Set 3

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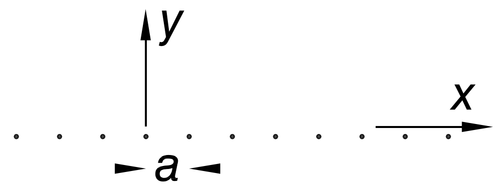
1. A grid of infinitely long wires is located in the (x, y) plane at $y = 0$, $x = \pm na$, $n = 0, 1, 2, \ldots$. Each line carries charge λ per unit length.

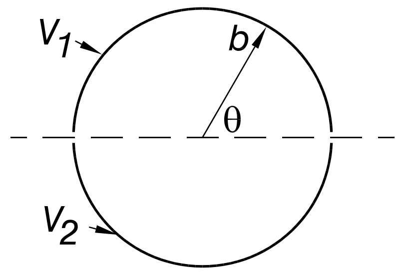


Obtain a series expansion for the potential $\phi(x, y)$. Show that for large y the field is just that due to a plane of charge density λ/a. By noting that $\sum_{n=1}^{\infty} \frac{z^n}{n} = -\ln(1-z)$ for z complex or real, sum the series to show

$$\begin{aligned} \phi(x,y) &= -\lambda \left[\frac{2\pi y}{a} + \ln\left(1 - 2e^{-2\pi y/a} \cos\frac{2\pi x}{a} + e^{-4\pi y/a}\right) \right] \\ &= -\lambda \ln\left[2\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right)\right] . \end{aligned} \tag{1}$$

Show that the equipotentials are circles for small x and y, as if each wire were alone.

2. Two halves of a long, hollow conducting cylinder of inner radius b are separated by small lengthwise gaps, and kept at different potentials V_1 and V_2.

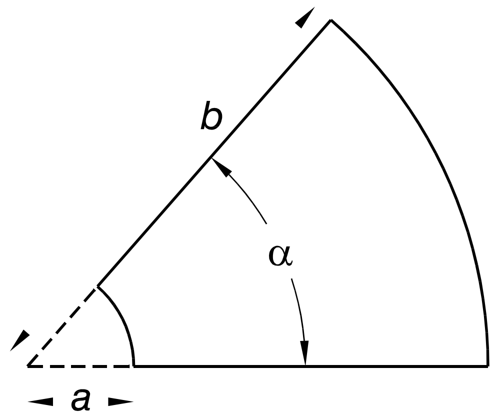


Give a series expansion for the potential $\phi(r,\theta)$ inside, and sum the series to show

$$\phi(r,\theta) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1}\left(\frac{2br\sin\theta}{b^2 - r^2}\right) \tag{2}$$

Note that $\sum_{n \text{ odd}} \frac{z^n}{n} = \frac{1}{2}\ln\frac{1+z}{1-z}$, and $Im \ln z = \text{phase}(z)$.

3. The two dimensional region $a < r < b$, $0 \leq \theta \leq \alpha$ is bounded by conducting surfaces held at ground potential, except for the surface at $r = b$.

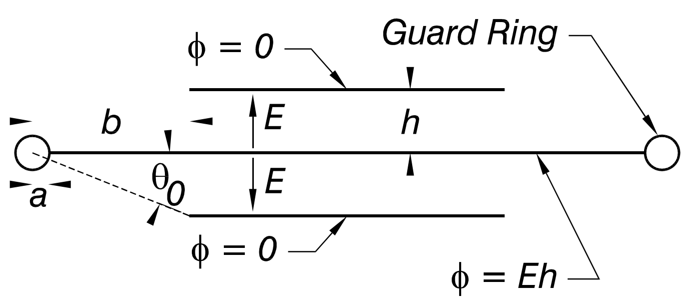


Give an expression for $\phi(r, \theta)$ satisfying these boundary conditions.

Give the lowest order terms for E_r and E_θ on the surfaces $r = a$, and $\theta = 0$.

As an application of the case $\alpha = 2\pi$, consider a double gap capacitor designed for use at very high voltage (as in "streamer chamber" particle detectors):

The central electrode is extended a distance b beyond the ground planes, and is terminated by a cylinder of radius $a \ll b$. Calculate the maximal electric field on the guard cylinder compared to the field E inside the capacitor, keeping only the first-order term derived above.



You may approximate the boundary condition at $r = b$ as

$$\phi(r = b) \simeq \begin{cases} Eh(1 - \theta/\theta_0), & |\theta| < \theta_0, \\ 0, & \theta_0 < |\theta|\pi, \end{cases} \tag{3}$$

where $\theta_0 \approx h/b \ll 1$ and h is the gap height. Note that the surfaces $r = a$ and $\theta = 0$ are not grounded, but are at potential Eh.

Answer: $E_{\max} \simeq \frac{2Eh}{\pi\sqrt{ab}}$.

4. Find the potential distribution inside a spherical region of radius a bounded by two conducting hemispheres at potential $\pm V/2$ respectively. Do the integrals to evaluate the two lowest-order nonvanishing terms.

5. Find the potential both inside and outside a spherical volume of charge of radius a in which the charge density varies linearly with the distance from some equatorial plane ($Q_{\text{tot}} = 0$).

6. A uniform field E_0 is set up in an infinite dielectric medium of dielectric constant ϵ. Show that if a spherical cavity is created, then the field inside the cavity is:

$$E = \frac{3\epsilon}{2\epsilon + 1} E_0 \,. \tag{4}$$

This problem differs from our discussion of the "actual" field on a spherical molecule in that the field inside the remaining dielectric can change when the cavity is created. The result could be rewritten as

$$E_{\text{cavity}} = E_0 + \frac{4\pi P}{2\epsilon + 1} \,, \tag{5}$$

where P is the dielectric polarization. A Clausius-Mosotti relation based on this analysis is, however, less accurate experimentally than the one discussed in the lectures.

7. A spherical capacitor consists of two conducting spherical shells of radii a and b, $a < b$, but with their centers displaced by a small amount $c \ll a$. Take the center of the sphere a as the origin. Show that the equation of the surface of sphere b in spherical coordinates with $\mathbf{z}$ along the line of centers is

$$r = b + cP_1(\cos\theta) + \mathcal{O}(c^2)\,. \tag{6}$$

Suppose sphere a is grounded and sphere b is at potential V. Show that the electric potential is

$$\phi(r,\theta) = V\left[\frac{r-a}{b-a}\left(\frac{b}{r}\right) - \frac{abc}{r^2(b-a)}\left(\frac{r^3-a^3}{b^3a^3}\right)P_1(\cos\theta) + \mathcal{O}(c^2)\right]. \tag{7}$$

What is the capacitance, to order c?

8. a) A charge Q is distributed uniformly along a line from $z = -a$ to $z = a$ at $x = y = 0$. Show that the electric potential for $r > a$ is

$$\phi(r, \theta) = \frac{Q}{r} \sum_n \left(\frac{a}{r}\right)^{2n} \frac{P_{2n}(\cos\theta)}{2n+1} . \tag{8}$$

b) A flat circular disk of radius a has charge Q distributed uniformly over its area. Show that the potential for $r > a$ is

$$\phi(r, \theta) = \frac{Q}{r} \left[1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{8}\left(\frac{a}{r}\right)^4 P_4 - \frac{5}{64}\left(\frac{a}{r}\right)^6 P_6 + \ldots\right] . \tag{9}$$

For both examples, also calculate the potential for $r < a$.

9. A conducting disk of radius a carrying charge Q has surface charge density

$$\rho(r) = \frac{Q}{2\pi a\sqrt{a^2 - r^2}} \tag{10}$$

(both sides combined).

a) Show that the potential in cylindrical coordinates is

$$\phi(r, z) = \frac{Q}{a} \int_0^\infty e^{-k|z|} J_0(kr) \frac{\sin ka}{k} dk \,. \tag{11}$$

See section 5.302 of the notes on Bessel functions for a handy integral.

b) Show that the potential in spherical coordinates is ($r > a$):

$$\phi(r, \theta) = \frac{Q}{r} \sum_n \frac{(-1)^n}{2n+1} \left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\theta) \,. \tag{12}$$

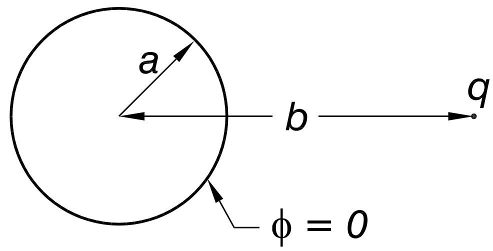
Note the relation for $P_{2n}(0)$ (see sec. 5.157) to obtain a miraculous cancelation.

10. A semi-infinite cylinder of radius a about the z axis ($z > 0$) has grounded conducting walls. The disk at $z = 0$ is held at potential V. The "top" of the cylinder is open. Show that the electric potential inside the cylinder is

$$\phi(r, z) = \frac{2V}{a} \sum_l \frac{e^{-k_l z}}{k_l} \frac{J_0(k_l r)}{J_1(k_l a)}. \tag{13}$$

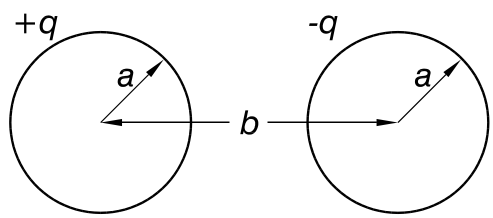
Refer to the notes on Bessel functions for the needed relations.

11. a) Calculate the electric potential ϕ everywhere outside a grounded conducting cylinder of radius a if a thin wire located at distance $b > a$ from the center of the cylinder carries charge q per unit length.



Use separation of variables. Interpret your answer as a prescription for the image method in two dimensions.

b) Use the result of a) to calculate the capacitance per unit length between two conducting cylinders of radius a, whose centers are distance b apart.



Answer (in Gaussian units, the capacitance per unit length is dimensionless):

$$C = \frac{1}{4\ln\left(\frac{b+\sqrt{b^2-4a^2}}{2a}\right)}. \tag{14}$$

c) [A bonus.] You have also solved the problem: What is the resistance between two circular contacts of radius a separated by distance b on a sheet of conductivity σ?

Apply voltage V between the contacts. Field $\mathbf{E}$ appears, and current density $\mathbf{J} = \sigma\mathbf{E}$ arises as well. The total current is

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \sigma \int \mathbf{E}\cdot d\mathbf{S} = 4\pi\sigma Q_{\rm in}\,, \tag{15}$$

by Gauss' law, and $Q_{\rm in}$ is the charge on one of the contacts needed to create the field $\mathbf{E}$. But if t is the thickness of the sheet, then the disk is like a length t of the cylindrical conductor considered in part a). Therefore, $Q_{\rm in}/t = VC$ with capacitance C as in part b), assuming that $\mathbf{J}$ and $\mathbf{E}$ are two-dimensional. Hence, $I = 4\pi\sigma tVC = V/R$ by Ohm's law, and (15) leads to

$$R = \frac{1}{4\pi\sigma tC} = \frac{R_\Box}{4\pi C}. \tag{16}$$

where

$$R_\Box = \frac{1}{\sigma t} = \frac{L}{\sigma tL} \tag{17}$$

is the resistance of a square of any size on the sheet.

Solutions

1. This problem is 2-dimensional, and well described in rectangular coordinates (x, y). We try separation of variables:

$$\phi(x,y) = \sum X(x)Y(y)\,. \tag{18}$$

Away from the surface $y = 0$, Laplace's equation, $\nabla^2\phi = 0$, holds, so one of X and Y can be oscillatory and the other exponential. The X functions must be periodic with period a, and symmetric about $x = 0$. This suggests that we choose

$$X_n(x) = \cos k_n x, \qquad \text{with} \qquad k_n = \frac{2n\pi}{a}\,. \tag{19}$$

We first consider the regions $y > 0$ and $y < 0$ separately, and then match the solutions at the boundary The Y functions are exponential, and should vanish far from the plane $y = 0$. Hence we consider

$$Y_n(y) = \begin{cases} e^{-k_n y}, & y > 0, \\ e^{k_n y}, & y < 0. \end{cases} \tag{20}$$

However, we must remember that the case of index $n = 0$ is special in that the separated equations are $X_0'' = 0 = Y_0''$, so that we can have $X_0 = 1$ or x, and $Y_0 = 1$ or y. In the present case, $X_0 = 1$ is the natural extension of (19) for nonzero n, so we conclude that $Y_0 = \pm y$ is the right choice; otherwise $X_0Y_0 = 1$, which is trivial. Then the potential $\phi = \pm y$ will be associated with a constant electric field in the y direction, which is to be expected far from the grid of wires.

Combining X and Y, our series solution thus far is

$$\phi(x,y) = \begin{cases} a_0 y + \sum_{n>0} a_n \cos(2n\pi x/a)\, e^{-2n\pi y/a}, & y > 0, \\ -a_0 y + \sum_{n>0} a_n \cos(2n\pi x/a)\, e^{2n\pi y/a}, & y > 0, \end{cases} \tag{21}$$

where we have used continuity of the potential at $y = 0$ to use the same a_n for both $y > 0$ and $y < 0$. Note, however, the sign change for a_0, corresponding to the constant electric field that points away from the wire plane.

At the boundary, $y = 0$, the surface charge density is

$$\begin{aligned} \sigma &= \lambda \sum_n \delta(x - na) = \frac{1}{4\pi}(E_y(0^+) - E_y(0^-)) = \frac{1}{4\pi}\left(-\frac{\partial\phi(x,0^+)}{\partial y} + \frac{\partial\phi(x,0^-)}{\partial y}\right) \\ &= -\frac{a_0}{2\pi} + \frac{1}{a}\sum_n n a_n \cos(2n\pi x/a). \end{aligned} \tag{22}$$

We evaluate the a_n by considering the interval $[-a/2 < x < a/2]$. Multiplying by $\cos(2n\pi x/a)$ and integrating, we find

$$a_0 = -\frac{2\pi\lambda}{a}, \qquad \text{and} \qquad a_n = \frac{2\lambda}{n}. \tag{23}$$

The potential is then,

$$\phi(x,y) = -\frac{2\pi\lambda|y|}{a} + 2\lambda \sum_{n>0} \frac{1}{n} \cos(2n\pi x/a)\, e^{-2n\pi|y|/a}. \tag{24}$$

To sum the series, we write it as

$$\begin{aligned}
\phi(x,y) &= -\frac{2\pi\lambda|y|}{a} + 2\lambda Re \sum_{n>0} \frac{1}{n} e^{2n\pi x/a}\, e^{-2n\pi|y|/a} \\
&= -\frac{2\pi\lambda|y|}{a} + 2\lambda Re \sum_{n>0} \frac{1}{n} (e^{2\pi ix/a}\, e^{-2\pi|y|/a})^n \\
&= -\frac{2\pi\lambda|y|}{a} - 2\lambda Re \ln(1-z),
\end{aligned} \tag{25}$$

where

$$z = e^{2\pi ix/a}\, e^{-2\pi|y|/a}. \tag{26}$$

To take the real part, we note that if

$$\ln(1-z) \equiv u + iv, \qquad \text{then} \qquad 1 - z = e^u\, e^{iv}, \qquad |1-z| = e^u, \tag{27}$$

and

$$\begin{aligned}
Re \ln(1-z) &= u = \ln|1-z| \\
&= \ln|1 - e^{-2\pi|y|/a}[\cos(2\pi x/a) + i\sin(2\pi x/a)] \\
&= \ln\sqrt{1 - 2\cos(2\pi x/a)e^{-2\pi|y|/a} + e^{-4\pi|y|/a}} \\
&= \frac{1}{2}\ln[1 - 2\cos(2\pi x/a)e^{-2\pi|y|/a} + e^{-4\pi|y|/a}].
\end{aligned} \tag{28}$$

The potential is now

$$\begin{aligned}
\phi(x,y) &= -\lambda \ln e^{2\pi|y|/a} - \lambda\ln[1 - 2\cos(2\pi x/a)e^{-2\pi|y|/a} + e^{-4\pi|y|/a}] \\
&= -\lambda\ln[e^{2\pi|y|/a} - 2\cos(2\pi x/a) + e^{-2\pi|y|/a}] \\
&= -\lambda\ln[2\cosh(2\pi|y|/a) - 2\cos(2\pi x/a)] \\
&= -\lambda\ln[\cosh(2\pi|y|/a) - \cos(2\pi x/a)] - \lambda\ln 2.
\end{aligned} \tag{29}$$

For x, y small:

$$\cosh\frac{2\pi|y|}{a} - \cos\frac{2\pi x}{a} \approx 1 + \frac{1}{2}\left(\frac{2\pi y}{a}\right)^2 + \ldots - 1 + \frac{1}{2}\left(\frac{2\pi x}{a}\right)^2 + \ldots = \frac{1}{2}\left(\frac{2\pi r}{a}\right)^2 \tag{30}$$

where $r^2 = x^2 + y^2$. Thus, at small x, y,

$$\phi(x,y) \to -2\lambda \ln\frac{2\pi r}{a}, \tag{31}$$

which is just the potential for an individual line charge λ. Close to each wire, the equipotentials are cylinders around this wire.

2. For a 2-dimensional potential problem with cylindrical boundaries, it is appropriate to use polar coordinates (r, θ). In general, the potential could be expanded in a series of terms in $r^{\pm n} \cos n\theta$ and $r^{\pm n} \sin n\theta$. For a bounded potential in the region $r < b$, only factors of r^n can occur.

   In the present problem, we measure θ from the plane that separates the two half cylinders, and take $0 < \theta < \pi$ on the half cylinder at potential V_1. Since θ varies over the full range $[0, 2\pi]$, n must be an integer. The average potential is $(V_1 + V_2)/2$, and the variable part of the potential has the symmetries $\phi(r, -\phi) = -\phi(r, \phi)$, and $\phi(r, \pi - \theta) - \phi(r, \theta)$. The first implies that only factors of $\sin n\theta$ can occur, and the second tells us that n must be odd.

   Thus, the potential has the form:

$$\phi(r, \theta) = \frac{V_1 + V_2}{2} + \sum_{n \text{ odd}} a_n r^n \sin n\theta. \tag{32}$$

   To fix the coefficients a_n, we use the boundary conditions at $r = b$, which can be written as

$$\sum_{n \text{ odd}} a_n b^n \sin n\theta = \frac{V_1 - V_2}{2} \text{sign}(\theta), \tag{33}$$

   where

$$\text{sign}(\theta) \equiv \begin{cases} +1, & 0 < \theta < \pi, \\ -1, & -pi < \theta < 0. \end{cases} \tag{34}$$

   Thus, we have to learn how to decompose the function sign(θ) in Fourier series.

   For a straightforward evaluation of the Fourier coefficients a_n, multiply (34) by $\sin n\theta$ and integrate from 0 to 2π:

$$\pi a_n b^n = (V_1 - V_2) \int_0^\pi \sin n\theta \, d\theta = \frac{2}{n}(V_1 - V_2). \tag{35}$$

   Thus,

$$\begin{aligned} \phi(r, \theta) &= \frac{V_1 + V_2}{2} + \frac{2}{\pi}(V_1 - V_2) \sum_{n \text{ odd}} \frac{r^n}{nb^n} \sin n\theta \\ &= \frac{V_1 + V_2}{2} + \frac{2}{\pi}(V_1 - V_2) Im \sum_{n \text{ odd}} \frac{(re^{i\theta}/b)^n}{n} \\ &= \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} Im \ln \frac{1 + re^{i\theta}/b}{1 - re^{i\theta}/b} \\ &= \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} Im \ln \frac{1 - (r/b)^2 + 2i(r/b)\sin\theta}{1 + (r/b)^2 - 2(r/b)\cos\theta} \\ &= \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1}\left(\frac{2br \sin\theta}{b^2 - r^2}\right). \end{aligned} \tag{36}$$

In the above, we used the facts about logarithms stated in the problem; the second of which follows from (27).

As a sidelight, we can compare (33) with (36) to learn that

$$\mathrm{sign}(\theta) = \frac{4}{\pi} \sum_{n\ \mathrm{odd}} \frac{\sin n\theta}{n}. \tag{37}$$

This is, of course, also the famous Fourier expansion of a square wave, since that is the result of periodically extending the definition (33).

3. The general possibilities for a series expansion for this problem are similar to those of problem 2. Since $\phi(r,0) = 0 = \phi(r,\alpha)$, the angular factors can only be $\sin(n\pi\theta/\alpha)$. Then, since the radial extent includes neither the origin nor ∞, factors of both $r^{n\pi/\alpha}$ and $r^{-n\pi/\alpha}$ can occur. Thus, the potential can be written

$$\phi(r,\theta) = \sum_{n=1}^{\infty} \left[a_n \left(\frac{r}{a}\right)^{\pi n/\alpha} + b_n \left(\frac{a}{r}\right)^{\pi n/\alpha} \right] \sin \frac{\pi n \theta}{\alpha}. \tag{38}$$

The use of factors r/a and a/r is convenient for enforcing the boundary condition $\phi(a,\theta) = 0$, since this simply requires $b_n = -a_n$.

For the electric field, we get:

$$\begin{aligned} E_r &= -\frac{\partial \phi}{\partial r} = -\sum_n \frac{n\pi}{\alpha} a_n \left[\frac{1}{r}\left(\frac{r}{a}\right)^{\pi n/\alpha} + \frac{1}{r}\left(\frac{a}{r}\right)^{\pi n/\alpha} \right] \sin \frac{\pi n \theta}{\alpha}, &(39) \\ E_\theta &= -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\sum_n \frac{n\pi}{\alpha} a_n \left[\frac{1}{r}\left(\frac{r}{a}\right)^{\pi n/\alpha} - \frac{1}{r}\left(\frac{a}{r}\right)^{\pi n/\alpha} \right] \cos \frac{\pi n \theta}{\alpha} &(40) \end{aligned}$$

At $r = a$, $E_\theta = 0$. At $\theta = 0$, $E_r = 0$, as expected. At $r = a$,

$$E_r = -\frac{2\pi}{a\alpha} \sum_n n a_n \sin \frac{\pi n \theta}{\alpha}. \tag{41}$$

At $\theta = 0$,

$$E_\theta = -\frac{\pi}{\alpha r} \sum_n n\, a_n \left[\left(\frac{r}{a}\right)^{\pi n/\alpha} - \left(\frac{a}{r}\right)^{\pi n/\alpha} \right]. \tag{42}$$

To obtain this, we ignored the small terms proportional to $(a/b)^{n/2}$ compared to the terms proportional to $(b/a)^{n/2}$ in $\phi(r = b)$.

We cannot determine the coefficients a_n until the boundary condition at $r = b$ is specified.

For the example of a "Streamer Chamber", $\alpha = 2\pi$. The surfaces $r = a$ and $\theta = 0, 2\pi$ are at potential Eh rather than 0, but we can accommodate this by simply adding Eh to (38). To evaluate a_n, we use the boundary condition (3) at $r = b$. Since $a \ll b$, we can approximate the potential there as

$$\phi(r = b) \approx Eh + \sum_n a_n \left(\frac{b}{a}\right)^{n/2} \sin \frac{n\theta}{2} = \begin{cases} Eh(1 - \theta/\theta_0), & |\theta| < \theta_0, \\ 0, & \theta_0 < |\theta|. \end{cases} \tag{43}$$

Subtracting Eh from both sides, we find

$$\sum_n a_n \left(\frac{b}{a}\right)^{n/2} \sin \frac{n\theta}{2} = \begin{cases} -Eh\theta/\theta_0, & |\theta| < \theta_0, \\ -Eh, & \theta_0 < |\theta|. \end{cases} \tag{44}$$

On multiplying by $\sin(n\theta/2)$ and integrating from 0 to π, we find

$$n\, a_n = \frac{2Eh}{\pi} \left(\frac{a}{b}\right)^{n/2} \left(\cos \frac{n\pi}{2} - \frac{2}{n\theta_0} \sin \frac{n\theta_0}{2} \right). \tag{45}$$

From (41), the field on the surface of the guard cylinder is

$$E_r(r=a) = -\frac{2Eh}{\pi a}\sum_n \left(\frac{a}{b}\right)^{n/2}\left(\cos\frac{n\pi}{2} - \frac{2}{n\theta_0}\sin\frac{n\theta_0}{2}\right)\sin\frac{n\theta}{2}\,. \tag{46}$$

Since a/b is very small, it suffices to keep only the first term, which is maximal at $\theta = \pi$:

$$E_{r,\max}(r=a) \simeq \frac{2Eh}{\pi\sqrt{ab}}. \tag{47}$$

For reasonable values of a, b and h, we have $E_{r,\max} \lesssim E$.

A sign that our approximations are somewhat delicate is obtained by evaluating $E_\theta(r = b)$ using (42). If we keep only the first term, we find that $E_\theta(r = b) \approx Eh/\pi b$, instead of E. However, because of the form (45) of the a_n, the leading term at each order n in series (42) does not have any factors of a/b, and this series converges much more slowly than does (41). The terms are of similar magnitude until $n\theta_0 \approx \pi$, *i.e.*, until $n \approx \pi b/h$, and $E_\theta(r = b)$ sums to E.

4. This problem involves a spherical boundary, so we seek a solution in spherical coordinates (r, θ, φ). Since the problem has axial symmetry, the potential will be independent of φ, and of the general form

$$\phi(r,\theta) = \sum_n \left[A_n \left(\frac{r}{a}\right)^n + B_n \left(\frac{a}{r}\right)^{n+1} \right] P_n(\cos\theta). \tag{48}$$

The region of interest contains the origin, so we must have $B_n = 0$ for a finite potential there.

To find the A_n, we use the boundary condition at $r = a$: multiply (48) by P_n and integrate over $\cos\theta$ to find

$$A_n = \frac{2n+1}{2} \int\limits_0^1 \phi(a,\theta) P_n(\cos\theta)\ d\cos\theta = \frac{2n+1}{4} V \left[\int_0^1 (P_n(z) - P_n(-z))\ dz \right]. \tag{49}$$

Since $P_n(z) = (-1)^n P_n(-z)$, we get $A_n = 0$ for even n. For odd n, we get

$$A_n = \frac{2n+1}{2} V \int_0^1 P_n(z)\ dz. \tag{50}$$

Using the explicit expressions for the polynomials P_n, we find for the first 2 nonvanishing terms:

$$A_1 = \frac{3}{2} V \int_0^1 z dz = \frac{3}{4} V, \tag{51}$$

$$A_3 = \frac{7}{2} V \int_0^1 \frac{1}{2}(5z^2 - 3z) dz = -\frac{7}{16} V. \tag{52}$$

The potential is:

$$\phi(r,\theta) = \frac{3}{4} V \frac{r}{a} P_1(\cos\theta) - \frac{7}{16} V \left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \ldots \tag{53}$$

5. This problem has an axially symmetric charge distribution $\rho(r)$, so we can evaluate the potential via the multipole expansion. This is the sum of two series, one for contributions from charge at radius $r' < r$ and the other for charge at $r' > r$:

$$\begin{aligned}\phi(r,\theta) &= \sum_n \frac{P_n(\cos\theta)}{r^{n+1}} \int_0^r 2\pi r'^2\, dr' \int_{-1}^1 d\cos\theta' \rho(r') r'^n P_n(\cos\theta') \\ &\quad + \sum_n r^n P_n(\cos\theta) \int_r^\infty 2\pi r'^2\, dr' \int_{-1}^1 d\cos\theta' \rho(r') \frac{P_n(\cos\theta')}{(r')^{n+1}}. \end{aligned} \tag{54}$$

In the present problem, the charge distribution is nonzero only for $r < a$, where it has the form $\rho = \rho_0 z = \rho_0 r\cos\theta = \rho_0 r P_1(\cos\theta)$.

We first evaluate the potential outside the sphere of radius a, for which we need only the first series of (54). The integral is

$$\int_0^a 2\pi r'^2\, dr' \int_{-1}^1 d\cos\theta' \rho_0 r' P_1(\cos\theta') r'^n P_n(\cos\theta') = \begin{cases} \frac{2\pi a^5}{5}\frac{2}{3}\rho_0, & n = 1, \\ 0, & n \neq 1. \end{cases}, \tag{55}$$

using the orthogonality relation

$$\int_{-1}^1 d\cos\theta\ P_n(\cos\theta) P_m(\cos\theta) = \frac{2\delta_{nm}}{2n+1}. \tag{56}$$

Thus,

$$\phi(r > a, \theta) = \frac{4\pi a^5 \rho_0}{15} \frac{\cos\theta}{r^2}, \tag{57}$$

which the potential due to a dipole of strength $p = 4\pi a^4 \rho_0/15$.

The total charge in the upper hemisphere is

$$Q_0 = \int_0^a 2\pi r^2\, dr \int_0^1 d\cos\theta\ \rho_0 r P_1(\cos\theta) = 2\pi \frac{a^4}{4}\frac{\rho_0}{2} = \frac{\pi a^4 \rho_0}{4}, \tag{58}$$

with $-Q_0$ in the lower hemisphere. The effective height z_0 of this charge is such that the dipole moment is $p = 2Q_0 z_0$, so $z_0 = 8a/15$.

For the potential inside the sphere, we must evaluate both series in (54), but we see in each case that only the $n = 1$ term survives the angular integration. Therefore,

$$\begin{aligned}\phi(r < a, \theta) &= \frac{P_1(\cos\theta)}{r^2} \int_0^r 2\pi r'^2\, dr' \frac{2}{3}\rho_0 r'^2 + r P_1(\cos\theta) \int_r^a 2\pi r'^2\, dr' \frac{2}{3}\rho_0 \frac{r'}{r'^2} \\ &= \pi\rho_0 \cos\theta \left(\frac{2a^2 r}{3} - \frac{2r^3}{5}\right). \end{aligned} \tag{59}$$

Expressions (57) and (59) give the same value at $r = 1$, as expected.

For possible instruction, we give a second solution for the potential inside the sphere, where Poisson's equation applies:

$$\nabla^2\phi = -4\pi\rho = -4\pi\rho_0 r P_1(\cos\theta). \tag{60}$$

Since this is a linear partial differential equation, a solution can be found in terms of a particular solution $\phi_{\rm p}$ to (60) plus the general solution to the homogeneous equation, which is Laplace's equation $\nabla^2\phi_{\rm h} = 0$ in the present case. Also, we must match our solution for $r < a$ to that found for $r > a$.

This problem has axial symmetry, so the general solution to the homogeneous equation for $r < a$ can be written

$$\phi_{\rm h}(r,\theta) = \sum_n A_n r^2 P_n(\cos\theta). \tag{61}$$

Since the solution for $r > a$ involves only P_1, we expect that the solution for $r < a$ will also. Then, $\phi_{\rm h} = Ar\cos\theta$.

Returning to Poisson's equation, writing $\mu = \cos\theta$, we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right] = -4\pi\rho_0 r\mu. \tag{62}$$

We hope for a simple power law solution in r, and expect the angular function to be just $P_1 = \mu$. That is, we try $\phi = Br^n\mu$. Inserting this into (62), we learn that the particular solution is $\phi_{\rm p} = -2\pi\rho_0 r^3\mu/5$. The complete solution then has the form

$$\phi(r < a,\theta) = \phi_{\rm h} + \phi_{\rm p} = Ar\cos\theta - \frac{2\pi\rho_0 r^3}{5}\cos\theta. \tag{63}$$

Matching this to (57) at $r = a$ requires:

$$Aa - \frac{2\pi\rho_0 a^3}{5} = \frac{4\pi\rho_0 a^3}{15}, \tag{64}$$

which gives $A = \frac{2}{3}\pi\rho_0 a^2$, and hence the solution (59).

6. Let the potential be given by the function $\phi_1(r,\theta)$ inside the sphere $(r < a)$ and $\phi_2(r,\theta)$ outside. The boundary conditions for the potential are

$$\phi_1(a,\theta) = \phi_2(a,\theta), \tag{65}$$

$$\frac{\partial}{\partial r}\phi_1(a,\theta) = \epsilon\frac{\partial}{\partial r}\phi_2(a,\theta), \tag{66}$$

where ϵ is the dielectric constant of the medium at $r > a$.

Since the asymptotic electric field at $r \to \infty$ is $\mathbf{E_0} = E_0\hat{\mathbf{z}}$, the potential ϕ_2 approaches

$$\phi_2(r \to \infty) \to -E_0 z = -E_0 r P_1(\cos\theta), \tag{67}$$

even after we have created the cavity at the origin. Hence, it is clear that the decomposition of ϕ in spherical harmonics should contain only terms proportional to $P_1(\cos\theta)$. Recalling the general form (48), we expect

$$\phi_1(r,\theta) = A\frac{r}{a}P_1(\cos\theta), \tag{68}$$

$$\phi_2(r,\theta) = \left[-E_0 r + B\left(\frac{a}{r}\right)^2\right]P_1(\cos\theta). \tag{69}$$

From the boundary conditions at $r = a$, we get

$$A = -E_0 a + B = \epsilon(-E_0 a - 2B). \tag{70}$$

Solving for A and B, we get

$$\phi_1 = -\frac{3\epsilon}{2\epsilon+1}E_0 r\cos\theta = -\frac{3\epsilon}{2\epsilon+1}E_0 z, \tag{71}$$

$$\mathbf{E}(r < a) = \frac{3\epsilon}{2\epsilon+1}E_0\hat{\mathbf{z}}. \tag{72}$$

The asymptotic value of polarization is related to $\mathbf{E}_0$ via

$$\mathbf{P} = \frac{\epsilon-1}{4\pi}\mathbf{E}_0. \tag{73}$$

Thus, we may rewrite $\mathbf{E}(r < a)$ in terms of $\mathbf{E}_0$ and asymptotic value of $\mathbf{P}$ as

$$\mathbf{E}(r < a) = \mathbf{E}_0 + \frac{4\pi\mathbf{P}}{2\epsilon+1}. \tag{74}$$

7. The equation of the surface of the sphere of radius b and center at $z = c$ is

$$b^2 = r^2 - 2rc\cos\theta + c^2 = r^2\left(1 - 2\frac{c}{r}\cos\theta\right) + c^2, \tag{75}$$

where r is measured from the origin in a spherical coordinate system. For small c, we approximate this as

$$r = b + c\cos\theta + \mathcal{O}(c^2) = bP_0(\cos\theta) + cP_1(\cos\theta) + \mathcal{O}(c^2). \tag{76}$$

Between the spherical shells of radii a and b, the potential ϕ has the general axially symmetric form (48). From the boundary condition $\phi(r = a) = 0$, we conclude that

$$\phi(r,\theta) = \sum_n A_n\left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^{n+1}\right]P_n(\cos\theta). \tag{77}$$

The boundary condition on the outer shell can expressed via (76) in terms of $P_0(\cos\theta)$ and $P_1(\cos\theta)$. Hence, it is plausible that only A_0 and A_1 are nonzero in (77), which then reads

$$\phi = A_0\left(1 - \frac{a}{r}\right) + A_1\left(\frac{r}{a} - \frac{a^2}{r^2}\right)P_1. \tag{78}$$

[For a discussion that does not make this leap, see eqs.]

The boundary condition on the outer shell now implies that

$$\begin{aligned}\phi(r = b + cP_1) &= V \\ &= A_0\left(1 - \frac{a}{b + cP_1}\right) + A_1\left(\frac{b + cP_1}{a} - \frac{a^2}{(b + cP_1)^2}\right)P_1 \\ &\approx A_0\left(1 - \frac{a}{b}\right) + \left\{A_0\frac{ac}{b^2} + A_1\left(\frac{b}{a} - \frac{a^2}{b^2}\right)\right\}P_1\end{aligned} \tag{79}$$

where we have dropped terms in P_1^2 in the last line of (79) as these lead to a correction to A_0 of order c^2. Hence, the constant term on the last line of (79) equals V, while the coefficient of P_1 must be zero. This determines the values of A_0 and A_1, and the potential is

$$\phi(r,\theta) = V\left[\frac{r - a}{b - a}\frac{b}{r} - \frac{abc}{r^2(b - a)}\left(\frac{r^3 - a^3}{b^3 - a^3}\right)P_1(\cos\theta) + \mathcal{O}(c^2)\right]. \tag{80}$$

To find the capacitance $C = Q/V$, we have to find the charge $\pm Q$ on the spherical shells. It is simpler to calculate this for the inner shell which is at $r = q$.

$$Q = \int \sigma\, dS = \int \frac{E_r(r = a)}{4\pi}\, dS = -\frac{a^2}{2}\int_{-1}^{1} d\cos\theta \frac{\partial\phi(a,\theta)}{\partial r}. \tag{81}$$

The terms in the integrand proportional to $P_1(\cos\theta)$ will integrate to zero, so the capacitance is unchanged by a small offset c.

For the record, $Q = abV/(b-a) = CV$ so the capacitance is $C = ab/(b-a)$ (which is a length, as always in Gaussian units).

As a footnote, we show how the boundary condition on the outer shell could be implemented without immediately assuming that only coefficients A_0 and A_1 are important in (77). Neglecting terms of $\mathcal{O}(c^2)$, we find

$$\begin{aligned}
&\phi(r = b + c\cos\theta) = V \\
&\approx \sum_n A_n \left\{ \left(\frac{b}{a}\right)^n \left(1 + n\frac{c}{b}\cos\theta\right) - \left(\frac{b}{a}\right)^{-n-1} \left(1 - (n+1)\frac{c}{b}\cos\theta\right) \right\} P_n(\cos\theta) \\
&= \sum_n A_n \left[\left(\frac{b}{a}\right)^n - \left(\frac{b}{a}\right)^{-n-1} \right] P_n(\cos\theta) \\
&\quad + \frac{c}{b}\sum_n A_n \left[n\left(\frac{b}{a}\right)^n + (n+1)\left(\frac{b}{a}\right)^{-n-1} \right] P_1(\cos\theta) P_n(\cos\theta). \qquad (82)
\end{aligned}$$

At this point, we invoke the recurrence relation

$$P_1(\cos\theta)P_n(\cos\theta) = \frac{n+1}{2n+1}P_{n+1}(\cos\theta) + \frac{n}{2n+1}P_{n-1}(\cos\theta). \qquad (83)$$

The constants A_n are determined from the requirement that the coefficients of P_n in (82) should be zero for $n > 0$, and the coefficient of $P_0 = 1$ is V. This leads to recurrence relations for A_n of the following form (schematically):

$$(\cdots)A_n + c(\cdots)A_{n+1} + c(\cdots)A_{n-1} = 0, \qquad n > 0. \qquad (84)$$

By iteration, we find a solution with the property $A_n \propto \mathcal{O}(c^n)$. It is straightforward to find the first two coefficients A_0 and A_1, which again gives (80).

8. The problem concerns two examples of specified, axially symmetry charge distributions. Hence, the multipole expansion (54) can be used to calculate the electric potential.

   a) The charge distribution is $\rho = Q/2a$ along the line from $z = -a$ to $z = a$. Thus, there is charge only for $\cos\theta = 1$ and -1, and the charge distribution is symmetric in $\cos\theta$. Since $P_n(-1) = (-1)^n P_n(1)$, the integrals in (54) will be nonzero only for even n. For an observer at $r > a$, the multipole expansion simplifies to

$$\phi(r > a, \theta) = \frac{Q}{2a} \sum_n \frac{P_{2n}(\cos\theta)}{r^{2n+1}} \cdot 2 \int_0^a dz' (z')^{2n} P_{2n}(1) = \frac{Q}{r} \sum_n \left(\frac{a}{r}\right)^{2n} \frac{P_{2n}(\cos\theta)}{2n+1}. \quad (85)$$

   Similarly,

$$\begin{aligned} \phi(r < a, \theta) &= \frac{Q}{2a} \sum_n \frac{P_{2n}(\cos\theta)}{r^{2n+1}} \cdot 2 \int_0^r dz' (z')^{2n} P_{2n}(1) \\ &\quad + \frac{Q}{2a} \sum_n r^{2n} P_{2n}(\cos\theta) \cdot 2 \int_r^a \frac{dz'}{(z')^{2n+1}} P_{2n}(1) \\ &= \frac{Q}{a} \sum_n P_{2n}(\cos\theta) \left\{ \frac{1}{2n+1} + \frac{1}{2n} \left[1 - \left(\frac{r}{a}\right)^{2n} \right] \right\}. \end{aligned} \quad (86)$$

   As expected (85) and (86) agree at $r = a$.

   b) As in example a), the charge distribution is symmetric in $\cos\theta$, so only even n will contribute to the multipole expansion of the potential. The charge distribution on the disc $r < a$, $\cos\theta = 0$ is $\rho = Q/\pi a^2$. Hence, the potential for $r > a$ is

$$\begin{aligned} \phi(r > a, \theta) &= \frac{Q}{\pi a^2} \sum_n \frac{P_{2n}(\cos\theta)}{r^{2n+1}} \int_0^a 2\pi r' \, dr' \, (r')^{2n} P_{2n}(0) \\ &= \frac{Q}{r} \sum_n \frac{(-1)^n (2n-1) P_{2n}(\cos\theta)}{2^n (n+1)} \left(\frac{a}{r}\right)^{2n}, \end{aligned} \quad (87)$$

   which gives (9), noting that $P_{2n}(0) = (-1)^2 (2n-1)/2^n$. Similarly,

$$\begin{aligned} \phi(r < a, \theta) &= \frac{Q}{\pi a^2} \sum_n \frac{P_{2n}(\cos\theta)}{r^{2n+1}} \int_0^r 2\pi r' \, dr' (r')^{2n} P_{2n}(0) \\ &\quad + \frac{Q}{\pi a^2} \sum_n r^{2n} P_{2n}(\cos\theta) \int_r^a 2\pi r' \, dr' \frac{P_{2n}(0)}{(r')^{2n+1}}. \\ &= \frac{Qr}{a^2} \sum_n \frac{(-1)^n P_{2n}(\cos\theta)}{2^n} \left[\frac{4n+1}{n+1} - 2\left(\frac{r}{a}\right)^{2n-1} \right]. \end{aligned} \quad (88)$$

   Again, these two expressions agree at $r = a$.

   The charged surfaces in these examples are not conductors, so those surfaces are not equipotentials.

   If you had forgotten the multipole expansion, you could have proceeded by first solving the simpler problem of the potential on the axis. For example, in the case of the charged disk,

$$\begin{aligned}\phi(z > a) &= \frac{Q}{\pi a^2}\int\limits_0^a \frac{2\pi r\ dr}{\sqrt{r^2+z^2}} = \frac{2Q}{a^2}\left[\sqrt{z^2+a^2}-z\right] \\ &= \frac{Q}{z}\left[1-\frac{1}{4}\left(\frac{a}{z}\right)^2+\frac{1}{8}\left(\frac{a}{z}\right)^4-\frac{5}{64}\left(\frac{a}{z}\right)^6+\ldots\right]. \qquad (89)\end{aligned}$$

The potential $\phi(r,\theta)$ is obtained from this simply by replacing z by r and multiplying the term in $1/r^n$ by $P_n(\cos\theta)$, *etc.*

9. a) We expect any solution of Laplace's equation having axial symmetry in cylindrical coordinates (r, θ, z) to be a sum of expressions of the form

$$e^{\pm kz} J_0(kr). \tag{90}$$

In the example of a conducting disk, there are no boundary surfaces, so the separation constant k will be continuous. The problem is symmetric about the plane $z = 0$, so we look for a solution of the form

$$\phi(r, z) = \int_0^\infty dk\ f(k) e^{-k|z|} J_0(kr). \tag{91}$$

To find the Fourier coefficients $f(k)$, we note that the electric field experiences a jump across the conducting disk at $z = 0$:

$$\frac{\partial}{\partial z}\phi(r < a, z = 0^+) - \frac{\partial}{\partial z}\phi(r < a, z = 0^-) = -4\pi\sigma. \tag{92}$$

Hence,

$$-2\int_0^\infty k f(k) J_0(kr)\ dk = -\frac{2Q}{a}\frac{\theta(a-r)}{\sqrt{a^2 - r^2}}, \tag{93}$$

using expression (10) for the charge density σ. Given the integral relation

$$\int_0^\infty \sin ka J_0(kr)\ dk = \left\{ \begin{array}{ll} 0, & r > a \\ \frac{1}{\sqrt{a^2-r^2}}, & r < a \end{array} \right\} = \frac{\theta(a-r)}{\sqrt{a^2 - r^2}}. \tag{94}$$

we find that $f(k) = Q\sin(ka)/ka$, and the potential is given as in (11).

b) To give a solution in spherical coordinates for the potential due to a specified, axially symmetric charge distribution, we again use the multiple expansion (54). The present problem is quite similar to problem 8b, so we write

$$\begin{aligned} \phi(r > a, \theta) &= \frac{Q}{2\pi a}\sum_n \frac{P_{2n}(\cos\theta)}{r^{2n+1}} \int_0^a 2\pi r'\ dr'\ \frac{(r')^{2n}}{\sqrt{a^2 - r'^2}} P_{2n}(0) \\ &= \frac{\sqrt{\pi}Q}{a}\sum_n \frac{P_{2n}(0)\Gamma(n+1)}{(2n+1)\Gamma(n+3/2)}\left(\frac{a}{r}\right)^{2n+1} P_{2n}(\cos\theta), \\ &= \frac{Q}{a}\sum_n \frac{(-1)^n}{2n+1}\left(\frac{a}{r}\right)^{2n+1} P_{2n}(\cos\theta), \end{aligned} \tag{95}$$

10. This problem has boundary conditions on the potential that $\phi(r = a) = 0$, and $\phi(z = 0) = V$, so a solution in cylindrical coordinates (r, θ, z) for $r < a$, $z > 0$ will have the form
$$\phi = \sum_l A_l e^{-k_l z} J_0(k_l r). \tag{96}$$
where $J_0(k_l a) = 0$. At $z = 0$, we have
$$V = \sum_l A_l J_0(k_l r). \tag{97}$$
The $\{J_0(k_l r)\}$ are an orthogonal set of functions on the interval $[0, a]$ upon integration with respect to dr^2 rather than dr. Hence, we can evaluate the Fourier coefficients A_l by multiplying (97) by $J_0(k_m r)$ and integrating:
$$\begin{aligned}\sum_l A_l \int_0^a r\ dr\ J_0(k_l r) J_0(k_m r) &= A_m \frac{a^2}{2}[J_1(k_m a)]^2 \\ &= V \int_0^a r\ dr\ J_0(k_m r) = V \frac{a}{k} J_1(k_m a),\end{aligned} \tag{98}$$
using 5.297(3) and 5.294(7) of Smythe. With this, we obtain the expansion
$$\phi(r, \theta, z) = \frac{2V}{a} \sum_l \frac{e^{-k_l z}}{k_l} \frac{J_0(k_l r)}{J_1(k_l a)}. \tag{99}$$

11. This two dimensional problem involves cylinders about the z axis, so we use cylindrical coordinates to discuss the potential $\phi(r, \theta)$. The first conducting cylinder of radius a has its axis along the z.

a) A wire at $(r, \theta) = (b, 0)$ carries charge density q per unit length.

We present three related solutions; the first two use Fourier series, where the first decomposes the potential into $\phi = \phi_{\rm wire} + \phi_{\rm cylinder}$, while the second does not; the third solution is more elementary.

In all cases, we have the symmetry $\phi(-\theta) = \phi(\theta)$, so the Fourier expansion for the potential contains terms in $\cos n\theta$, but not $\sin n\theta$.

The potential due to the wire has the general form

$$\phi_{\rm wire}(r, \theta) = \begin{cases} a_0 + \sum_{n=1} a_n \left(\frac{r}{b}\right)^n \cos n\theta, & r < b, \\ a_0 + b_0 \ln \frac{r}{b} + \sum_{n=1} a_n \left(\frac{b}{r}\right)^n \cos n\theta, & r > b, \end{cases} \tag{100}$$

since this should not blow up at the origin, should be continuous at $r = b$, and can have a logarithmic divergence at infinity.

The potential due to the conducting cylinder has the form

$$\phi_{\rm cylinder}(r > a, \theta) = A_0 + B_0 \ln r + \sum_{n=1} \frac{A_n}{r^n} \cos n\theta. \tag{101}$$

The cylinder is grounded, so the total potential (and not $\phi_{\rm cylinder}$) obeys $\phi(r = a) = 0$. Hence,

$$a_0 + A_0 + B_0 \ln a = 0, \qquad A_n = -a_n \left(\frac{a^2}{b}\right)^n, \tag{102}$$

and so the potential due to the cylinder is

$$\phi_{\rm cylinder}(r > a, \theta) = -a_0 + B_0 \ln \frac{r}{a} - \sum_{n=1} a_n \left(\frac{a^2/b}{r}\right)^n \cos n\theta, \tag{103}$$

where coefficient B_0 is not yet determined.

Comparing (103) to the form (100) of the potential due to the wire for $r > b$, we see that these are the same for terms with $n > 0$, except for an overall $-$ sign, and the substitution $b \to a^2/b$. Since we are still free to choose the value of B_0, we set it to $-b_0 \ln b/a$, and the potential of the cylinder becomes

$$\phi_{\rm cylinder}(r > a, \theta) = -a_0 - b_0 \ln \frac{r}{a^2/b} - \sum_{n=1} a_n \left(\frac{a^2/b}{r}\right)^n \cos n\theta, \tag{104}$$

and now the $n = 0$ terms are also related to those of eq. (100) in the same way as the terms with $n > 0$.

This suggests that the potential due to a grounded, conducting cylinder of radius a in the presence of a line charge density q at $r = b$ is the same that due to a line charge density $-q$ at $r = a^2/b$. This is the desired image method for cylindrical geometry.

There was no need to evaluate the Fourier coefficients a_n to reach this conclusion!

In the second solution, we do not separate the potential into two parts, and we carry out the evaluation of the Fourier coefficients.

The cylinder $r = a$ is at zero potential, so the most general form that satisfies these conditions for $a < r < b$ is

$$\phi(r,\theta) = b_0 \ln r - b_0 \ln a + \sum_{n=1}^{\infty} A_n \left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right] \cos n\theta \qquad (a < r < b). \tag{105}$$

Beyond the wire at $r = b$, we can only have the form

$$\phi(r,\theta) = c_0 + d_0 \ln r + \sum_{n=1}^{\infty} B_n \left(\frac{b}{r}\right)^n \cos n\theta \qquad (r > b). \tag{106}$$

As this problem is meant to represent a real 2-wire system, the energy per unit length must be finite. Therefore, we must have $d_0 = 0$, so that no field lines from the wire escape to infinity. This also means that the charge on the cylinder at $r = a$ must be $-q$.

The potential is continuous at $r = b$, which leads to the conditions

$$c_0 = b_0 \ln \frac{b}{a}, \qquad B_n = A_n \left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right]. \tag{107}$$

The remaining condition is obtained by considering a Gaussian surface (of unit length in z) that surrounds the cylindrical surface (b, θ):

$$4\pi q_{\rm in} = \int \mathbf{E} \cdot d\mathbf{S} = \int b \, d\theta \, (E_{r^+} - E_{r^-}). \tag{108}$$

For this we learn that

$$\begin{aligned} 4\pi q \delta(\theta) &= b(E_{r^+} - E_{r^-}) = b\left(-\frac{\partial\phi(b^+)}{\partial r} + \frac{\partial\phi(b^-)}{\partial r}\right) \\ &= b_0 + \sum_n n \cos n\theta \left\{B_n + A_n \left[\left(\frac{b}{a}\right)^n + \left(\frac{a}{b}\right)^n\right]\right\}. \end{aligned} \tag{109}$$

Multiply this by $\cos n\theta$ and integrate over θ to find

$$b_0 = 2q, \qquad B_n + A_n \left[\left(\frac{b}{a}\right)^n + \left(\frac{a}{b}\right)^n\right] = \frac{4q}{n}. \tag{110}$$

Combining this with (107), we learn that

$$c_0 = 2q \ln \frac{b}{a}, \qquad A_n = \frac{2q}{n}\left(\frac{a}{b}\right)^n, \qquad B_n = \frac{2q}{n}\left[1 - \left(\frac{a}{b}\right)^{2n}\right]. \tag{111}$$

For $a < r < b$ we now have

$$
\begin{aligned}
\phi(r,\theta) &= 2q\ln\frac{r}{a} + 2q\sum_n \frac{1}{n}\left(\frac{a}{b}\right)^n \left[\left(\frac{r}{a}\right)^n - \left(\frac{a}{r}\right)^n\right]\cos n\theta \\
&= 2q\ln\frac{r}{a} + 2qRe\sum_n \frac{1}{n}\left[\left(\frac{r}{b}\right)^n - \left(\frac{a^2}{br}\right)^n\right](e^{i\theta})^n \\
&= 2q\ln\frac{r}{a} - 2qRe\left[\ln\left(1 - \frac{re^{i\theta}}{b}\right) - \ln\left(1 - \frac{a^2e^{i\theta}}{br}\right)\right] \\
&= 2q\ln\frac{r}{a} - 2q\ln\left|1 - \frac{re^{i\theta}}{b}\right| + 2q\ln\left|1 - \frac{a^2e^{i\theta}}{br}\right| \\
&= 2q\ln\frac{b}{a} - 2q\ln\left|b - re^{i\theta}\right| + 2q\ln\left|r - \frac{a^2e^{i\theta}}{b}\right| \qquad (112) \\
&= 2q\ln\frac{b}{a} - 2q\ln\sqrt{r^2 - 2br\cos\theta + b^2} + 2q\ln\sqrt{r^2 - 2r\frac{a^2}{b}\cos\theta + \left(\frac{a^2}{b}\right)^2}.
\end{aligned}
$$

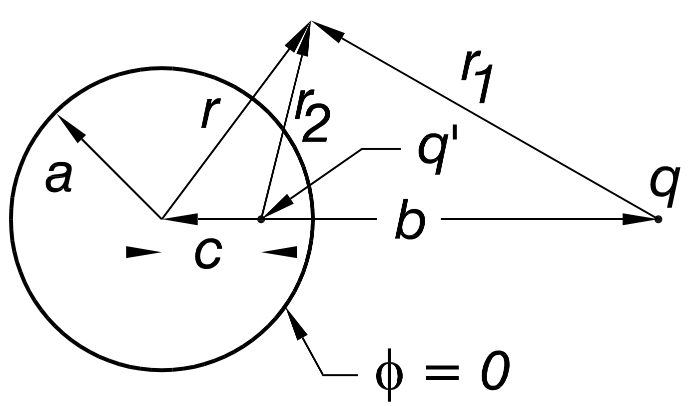
Recall that the potential at distance R from a line of charge q per unit length is $\phi = -2q\ln R + \text{const}$. Thus the second term of the last line of (112) is the potential due to the line of charge density q at $(r,\theta) = (b,0)$. The second term is equal to the potential due to a line of charge density $-q$ at $(r,\theta) = (a^2/b, 0)$.

Hence, we have demonstrated the cylindrical image method: the image in a conducting cylinder of radius a of a line of charge density q at radius b is a line of charge density $-q$ at radius a^2/b. In terms of distances r_1 and r_2 shown in the figure below, the potential is then,

$$\phi(r,\theta) = 2q\ln\frac{br_2}{ar_1}. \qquad (113)$$

[We skip the demonstration that this prescription also works for $r > b$.]

For a third solution, we suppose that there exists an image wire carrying charge density q' at distance c from the center of the conducting cylinder, as shown in the figure.



Then the potential at an aribtrary point (r,θ) due to the two wires is

$$
\begin{aligned}
\phi(r,\theta) &= \phi_q + \phi_{q'} = K - 2q\ln r_1 - 2q'\ln r_2 \\
&= K - q\ln(r^2 + b^2 - 2br\cos\theta) - q'\ln(r^2 + c^2 - 2cr\cos\theta). \qquad (114)
\end{aligned}
$$

The cylinder is grounded, so

$$\phi(r = a, \theta) = 0 = K - q \ln(a^2 + b^2 - 2ab\cos\theta) - q' \ln(a^2 + c^2 - 2ac\cos\theta). \tag{115}$$

This can be arranged by putting $q' = -q$, so that (115) simplifies to

$$K = q \ln \frac{a^2 + b^2 - 2ab\cos\theta}{a^2 + c^2 - 2ac\cos\theta}. \tag{116}$$

If we take $c = a^2/b$ (inspired by our knowledge of the spherical image method), we find that the argument of the logarithm becomes b^2/a^2, which is independent of θ, as desired. Hence, $K = 2q\ln(b/a)$, and the potential is again (113).

b) Suppose we have two parallel conducting cylinders of radius a each, carrying charge $+q$ and $-q$ per unit length, whose axes are distance b apart. We want to find locations for two line charge densities $+q$ and $-q$ such that the fields from these lines charges are the same as those due to the two conducting cylinders.

Clearly, these lines charges should be placed symmetrically in the plane containing the axes of the cylinders, say distance c apart. Then the first line charge is distance $(b+c)/2$ from the center of the second cylinder. Its image charge would then be located distance $2a^2/(b+c)$ from the center of the second cylinder. We want the image charge to be the same as the second line charge, which is at distance $(b-c)/2$ from the center of the second cylinder. Equating the two distances, we find that

$$c = \sqrt{b^2 - 4a^2}. \tag{117}$$

To find the capacitance $C = q/\Delta V$, we need the potential difference ΔV between the two cylinders. We can calculate the potential on a conducting cylinder at any convenient point. For example, consider the point on one cylinder closest to the other. This point is distance $a - (b-c)/2$ from one line charge, and distance $(b+c)/2 - a$ from the second line charge. The potential at this point is therefore

$$\begin{aligned} V &= -2q\ln[a - (b-c)/2] + 2q\ln[(b+c)/2 - a] = 2q \ln \frac{c + b - 2a}{c - (b - 2a)} \\ &= 2q \ln \frac{b+c}{2a}. \end{aligned} \tag{118}$$

The corresponding point on the second cylinder is at potential $-V$, so $\Delta V = 2V$, and the capacitance is

$$C = \frac{1}{4 \ln \frac{b + \sqrt{b^2 - 4a^2}}{2a}}. \tag{119}$$