PRINCETON UNIVERSITY

# Ph501
# Electrodynamics
# Problem Set 4

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1. a) **Child's Law.**  Before the transistor era, a common device was a vacuum diode. This is a parallel plate capacitor with a potential difference $V$ across a gap $d$, all of which is inside a vacuum tube. The cathode (at $\phi = 0$) is heated, so electrons can jump off and flow to the anode (at $\phi = V$). Positive charges have very low probability of leaving the anode and flowing to the cathode. The resulting one way flow of charge from cathode to anode is the diode action.

Consider a steady situation in which a constant current density $j = \rho(x)v(x)$ flows, and where the electrons leave the cathode with velocity $v(0) = 0$. Here, $\rho(x)$, $0 \le x \le d$, is the electron charge density.

Solve for the potential $\phi(x)$ via Poisson's equation,

$$\nabla^2 \phi = -4\pi\rho. \tag{1}$$

Show that

$$\phi(x) = V\left(\frac{x}{d}\right)^{4/3}, \qquad \text{and} \qquad J = -\frac{1}{9\pi}\frac{V^{3/2}}{d^2}\sqrt{\frac{2e}{m}}. \tag{2}$$

where $e$ and $m$ are the magnitudes of the charge and mass of the electron, respectively.

Note that since the current density $J = nev$ is constant, and $v \to 0$ near the cathode, the charge density $n \to \infty$ there. The field due to this large "space charge" distribution near the cathode opposes the field due to the capacitor alone, and cancels it completely very close to the cathode. That is, $E(x) \propto x^p$ with $p > 0$. Then, $\phi(x) = -\int E\,dx \propto x^{1+p}$ rises more quickly than the simple linear relation for an ordinary capacitor.

b) **Laser Driven Vacuum Photodiode.**  A vacuum photodiode is constructed in the form of a parallel-plate capacitor of area $A$, plate separation $d$. A battery maintains constant potential $V$ between the plates. A short laser pulse illuminates that cathode at time $t = 0$ with energy sufficient to liberate all of the surface-electron charge density. This charge moves across the capacitor gap as a sheet until it is collected at the anode at time $T$. Then another laser pulse strikes the cathode, and the cycle repeats.

Estimate the average current density $\langle j \rangle$ that flows onto the anode from the battery, ignoring the recharging of the cathode as the charge sheet moves away. Then calculate the current density and its time average when this effect is included.

You may suppose that the laser photon energy is equal to the work function of the cathode, so the electrons leave the cathode with zero velocity.

2. Obtain a Legendre series expansion for the potential inside a conducting sphere of radius $a$ and conductivity $\sigma$ when a current $I$ enters at one pole through a fine wire of radius $b \ll a$, and leaves through the other pole via a similar fine wire.

   Define the potential as $\phi = 0$ on the equator.

   By noting that $P_n(-\mu) = (-1)^n P_n(\mu)$, and referring to the expansion of $1/R$ given on p. 57 of the notes, show that

   $$\phi(r < a, \theta) = \frac{I}{2\pi\sigma} \left[ \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{2} \int_0^r \left( \frac{1}{r_1} - \frac{1}{r_2} \right) d\ln r' \right], \tag{3}$$

   where $r_{1,2}$ is the distance from the "north" ("south") pole to the point $(r, \theta, \varphi)$ in spherical coordinates. The integrals can be found in tables if desired.
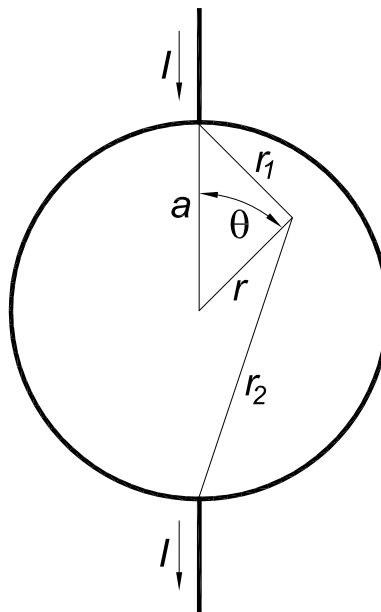
   

   Finally, suppose the wires have radius $b \ll a$, and their surface of contact with the sphere is an equipotential. Show that the resistance of the sphere is that of a piece of wire roughly $b$ long, if that wire had conductivity $\sigma$.

   Hint: Express the radial current density at $r = a$ in terms of delta functions, $\delta(\cos\theta - 1)$ and $\delta(\cos\theta + 1)$.

3. **Charge Distributions in a Wire that carries a Steady Current**

a) A wire of circular cross-section carries a current $I$ which is uniformly distributed across the wire. We consider this current to be due to a number density $\rho$ of free electrons moving with average drift velocity $v$. (In a typical situation, $v \ll$ 1cm sec$^{-1}$!) Let $\rho_0$ be the uniform number density of positive ions in the wire. For steady current flow, there must be no radial force on the electrons. Use the Lorentz force law,

$$\mathbf{F} = q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \tag{4}$$

to find the relation between $\rho_0$ and $\rho$ such that the force vanishes. (As a check, you may wish to do this problem via special relativity, but try it using Maxwell's equations and the Lorentz force.)

b) A resistor of resistance $R$, length $l$ and cross-sectional area $A$ carries a current $I$, delivered by fine lead wires. Calculate the charge that accumulates on the end faces of the resistor in order to produce the field $\mathbf{E}$ which drives the current according to Ohm's law. Suppose that the current in the wire varies with time. Show that the conduction current inside the resistor is different from that in the lead wires, but that Maxwell's concept of displacement current restores the continuity of "total current". You may assume that $l \ll \sqrt{A}$ so that the current density $\mathbf{J}$ is uniform inside the wire.

4. A straw tube chamber is a low cost version of a proportional counter, which latter is a descendent of the Geiger counter.[1]

   These devices consist of a pair of coaxial conducting cylinders with the region between the cylinders is filled with a gas such as argon. The inner cylinder of radius $a$ is the anode, and is held at potential $V$; the outer cylinder of radius $b$ is the cathode, and is grounded.

   If a penetrating charged particle passes through the chamber, it will ionize about two gas molecules per mm of path length. The ionization electrons are pulled by the electric field towards the anode. Close to the anode, the field is strong enough that the electrons gain enough energy during one mean free path to ionize the molecule they hit next, liberating one or more additional electrons. In a proportional chamber, the field is kept low enough that the resulting Townsend avalanche[2] involves $10^4$-$10^6$ molecules.

   What is the time dependence, $I(t)$, of the current that flows off the anode due to the avalanche of a single initial electron?

   What is the spatial dependence, $q(z)$ of the charge distribution induced on the anode during the time when the current is large, where the $z$ axis is the chamber axis? You may restrict your attention to values of $z$ far from the ends of the tube of length $l$.

   Measurement of the charge distribution via a segmented cathode permits localization in $z$ of the ionization, and hence, of the initiating charged particle.[3]

   You may ignore the tiny current that flows while the electron drifts towards the anode. The avalanche takes place so close to the anode, that the small remaining drift time for the electrons to reach the anode may also be ignored. In this approximation, the situation at $t = 0$ is that electrons of total charge $-q_0$ reside on the anode in close proximity to positive ions of total charge $+q_0$. Current flows off the anode only when some of the field lines from the positive ions detach from the electrons on the anode, and extend to the cathode where charge is induced to terminate these field lines. This occurs only as the positive ions move away from the anode, with velocity related by

$$v = \mu E, \tag{5}$$

   where $\mu$ is the positive-ion mobility.

[1]E. Rutherford and H. Geiger, *An Electrical Method of Counting the Number of $\alpha$-Particles from Radioactive Substances*, Proc. Roy. Soc. London A **81**, 141 (1908),
http://kirkmcd.princeton.edu/examples/detectors/rutherford_prsla_81_141_08.pdf
H. Geiger and W. Müller, *Elektronenzählrohr zur Messung schwächster Aktivit/"aten*, Naturw. **16**, 617 (1928), http://kirkmcd.princeton.edu/examples/detectors/geiger_naturw_16_617_28.pdf
[2]J.S. Townsend, *The conductivity in Gases produced by Motion of Negatively-charged Ions*, Nature **62**, 340 (1900), http://kirkmcd.princeton.edu/examples/detectors/townsend_nature_62_340_00.pdf
Phil. Mag. **6**, 198 (1901), http://kirkmcd.princeton.edu/examples/detectors/townsend_pm_6_198_01.pdf
[3]C. Leonidopoulos, C. Lu and A.J. Schwartz, *Development of a Straw Tube Chamber with Pickup-Pad Readout*, Nucl. Instr. and Meth. A427, 465 (1999),
http://kirkmcd.princeton.edu/examples/detectors/leonidopoulos_nim_a427_465_99.pdf

### $I(t)$ **via Reciprocity and Weighting Fields**

This problem can be solved by an application of Green's reciprocation theorem, which states that if a set of fixed conductors is at potentials $V_i$ when carrying charges $Q_i$, and at potentials $V_i'$ when carrying charges $Q_i'$, then

$$\sum_i V_i Q_i' = \sum_i V_i' Q_i. \tag{6}$$

To see this, we label the 3-dimensional potential distribution associated with charges $Q_i$ by $\phi(\mathbf{r})$, and that associated with charges $Q_i'$ by $\phi'$. The space outside the conductors is charge free and with dielectric constant $\epsilon = 1$. Then $\nabla^2 \phi = 0 = \nabla^2 \phi'$ outside the conductors.

We invoke Green's theorem (p. 37 of the Notes),

$$\int (\phi \nabla^2 \phi' - \phi' \nabla^2 \phi) d\text{vol} = \oint (\phi \boldsymbol{\nabla} \phi' - \phi' \boldsymbol{\nabla} \phi) \cdot d\mathbf{S}, \tag{7}$$

where we take the bounding surface $S$ to be that of the set of conductors. Hence,[4]

$$0 = \sum_i \oint (V_i \boldsymbol{\nabla} \phi_i' - V_i' \boldsymbol{\nabla} \phi_i) \cdot d\mathbf{S}_i = -4\pi \sum_i (V_i Q_i' - V_i' Q_i), \tag{8}$$

using Gauss' Law (in Gaussian units) that

$$4\pi Q_i = \oint \mathbf{E}_i \cdot d\mathbf{S}_i = -\oint \boldsymbol{\nabla} \phi_i \cdot d\mathbf{S}_i. \tag{9}$$

In the present problem, we have a small charge $q_0$ at position $\mathbf{r}_0(t)$ that moves under the influence of the field due to conductors $i = 1, ..., n$ that are held at potentials $V_i$. The charges $Q_i$ on the conductors obey $Q_i \gg q_0$, so the motion of charge $q_0$ is determined, to a very good approximation by the charges $Q_i$ on the conductors when $q_0 = 0$. Hence, the problem can be considered as the superposition of two situations:

A: charge $q_0$ absent; conductors $i = 1, ...n$ at potentials $V_i$.

B: charge $q_0$ present; conductors $i = 1, ...n$ grounded, with charges $\Delta Q_i$ on them.
We are particularly interested in the charge on electrode 1, whose time rate of change is the desired current $I(t)$.

To use the reciprocation theorem, we suppose that in case B the charge resides on a tiny conductor at position $\mathbf{r}_0$ that is at the potential $V_0 = \phi_A(\mathbf{r}_0)$ obtained from case A. Then, the charges and potentials in case B can be summarized as

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[4]Green's theorem (7) first appeared as eq. (12), p. 26 of G. Green, *Mathematical Papers* (1828), http://kirkmcd.princeton.edu/examples/EM/green_papers.pdf.
The reciprocation theorem (7) was hinted at on pp. 33-39, but may have first been explicitly stated in Art. 85b, p. 105 of J.C Maxwell, *A Treatise on Electricity and Magnetism*, 2nd ed. (1878), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_04.pdf.
That this theorem also holds for conductors in a medium of uniform dielectric constant is shown in sec. 3.09, p. 54 of W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, 1968).

B: $\{q_0, \ V_0; \ \Delta Q_i, \ V_i = 0, \ i = 1, ..., n\}$.

We solve the electrostatics problem for a third case,

C: $\{q_0' = 0, \ V_0'(\mathbf{r}_0); \ Q_1, \ V_1' = 1; \ \Delta Q_i = 0, \ V_i' = 0, \ i = 2, ..., n.\}$,
in which conductor 1 is held at unit potential, the charges on all other conductors at zero, and all other conductors are grounded except for the tiny conductor at position $\mathbf{r}_0$. Again, we solve this problem as in case A, first ignoring the tiny conductor, then evaluating $V_0'$ as $\phi_C(\mathbf{r}_0)$.

The reciprocation theorem (6) applied to cases B and C implies that

$$0 = q_0 V_0' + \Delta Q_1 \cdot 1. \tag{10}$$

The current that moves off electrode 1 in case B is therefore,

$$I_1 = -\frac{d\Delta Q_1}{dt} = q_0 \frac{dV_0'(\mathbf{r}_0)}{dt} = q_0 \boldsymbol{\nabla} V_0'(\mathbf{r}_0) \cdot \frac{d\mathbf{r_0}}{dt} = -q_0 \mathbf{E}_{\mathrm{w}} \cdot \mathbf{v}, \tag{11}$$

where the velocity $\mathbf{v}$ of the charge is determined using the fields from case A, and

$$\mathbf{E}_{\mathrm{w}} = -\boldsymbol{\nabla} V_0'(\mathbf{r}_0) = -\boldsymbol{\nabla} \phi_C(\mathbf{r}_0) \tag{12}$$

is called the **weighting field**.[5] For the case of two conductors (plus charge $q_0$) one of which is grounded, the weighting field is the same as the field from case A, but in general they are distinct.

As the present problem involves only two conductors, you may wish to find a solution that does not appear to use the initially cumbersome machinery of the reciprocation theorem.

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[5]The result (11) was first deduced by the present method by W. Shockley, *Currents to Conductors Induced by a Moving Point Charge*, J. Appl. Phys. **9**, 635 (1939),
http://kirkmcd.princeton.edu/examples/EM/shockley_jap_9_635_39.pdf,
and by S. Ramo, *Currents Induced by Electron Motion*, Proc. I.R.E. **27**, 584 (1939),
http://kirkmcd.princeton.edu/examples/EM/ramo_pire_27_584_39.pdf.
That this result also follows from an energy argument was pointed out by C.K. Jen, *On the Induced Current and Energy Balance in Electronics*, Proc. I.R.E. **29**, 349 (1941),
http://kirkmcd.princeton.edu/examples/EM/jen_pire_29_345_41.pdf.
For discussion that the weighting-field method holds for multiple charges (space charge) and for a uniform dielectric medium, see, for example, L.A. Hamel and M. Julien, *Generalized demonstration of Ramo's theorem with space charge and polarization effects*, Nucl. Instr. Meth. A **597**, 207 (2008),
http://kirkmcd.princeton.edu/examples/EM/hamel_nim_a597_207_08.pdf.
See also, K.T. McDonald, *Does Space Charge or the Dielectric Constant Affect Induced Charge in a Liquid Argon Detector?* (June 4, 2016), http://kirkmcd.princeton.edu/examples/induced.pdf.

5. **Resistance of a Disk with Edge Contacts**

Calculate the resistance between two contacts on the rim of a disk of radius $a$, thickness $t \ll a$, and conductivity $\sigma$, when each (perfectly conducting) contact extends for a small distance $\delta$ around the circumference, and the distance along the chord between the contacts is $d \gg \delta$.
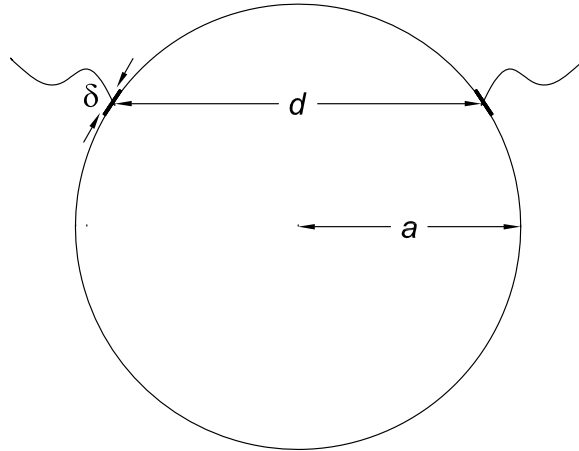
6. Some biological systems consist of two "phases" of nearly square fiber bundles of differing thermal and electrical conductivities. Consider a circular region of radius $a$ near a corner of such a system as shown below.
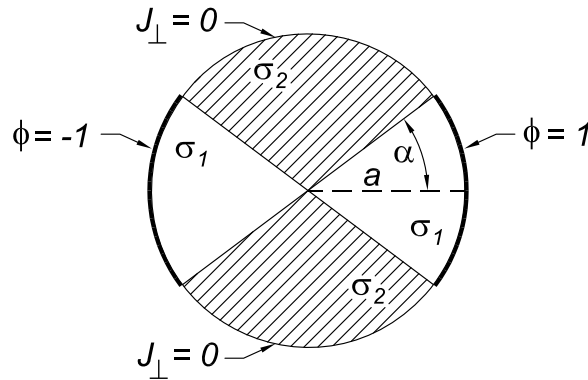


Phase 1, with electrical conductivity $\sigma_1$, occupies the "bowtie" region of angle $\pm\alpha$, while phase 2, with conductivity $\sigma_2 \ll \sigma_1$, occupies the remaining region.

Deduce the approximate form of lines of current density **J** when a background electric field is applied along the symmetry axis of phase 1. What is the effective conductivity $\sigma$ of the system, defined by the relation $I = \sigma\Delta\phi$ between the total current $I$ and the potential difference $\Delta\phi$ across the system?

It suffices to consider the case that the boundary arc $(r = a, |\theta| < \alpha)$ is held at electric potential $\phi = 1$, while the arc $(r = a, \pi - \alpha < |\theta| < \pi)$ is held at electric potential $\phi = -1$, and no current flows across the remainder of the boundary.

Hint: When $\sigma_2 \ll \sigma_1$, the electric potential is well described by the leading term of a series expansion.

7. A rectangular loop of size $2a$ by $2b$ carries a current $I'$, and is free to rotate about an axis that bisects the sides of length $2b$. The axis is parallel to and distance $d$ from a wire that carries current $I$. If the plane of the loop makes angle $\theta$ to the plane containing the wire and the axis, and if the currents in the wire and in the side (of length $2a$) of the loop closest to the wire flow in the same direction, show that the magnitude of the torque on the loop is

$$N = \frac{8abdII'}{c} \frac{(b^2 + d^2)\sin\theta}{b^4 + d^4 - 2b^2d^2\cos 2\theta} \ . \tag{13}$$

What is its direction?
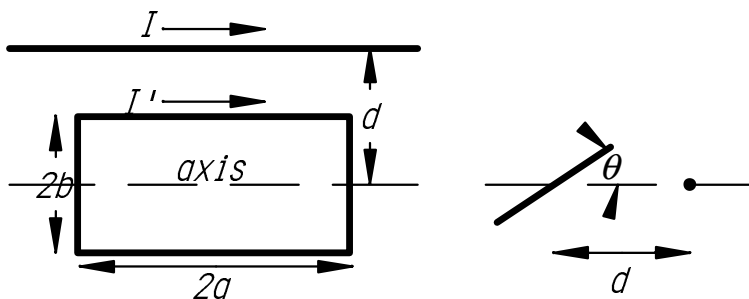
8. **Helmholtz Coils**

a) Each of a pair of parallel, coaxial "Helmholtz" coils has radius $a$ and carries a current $I$ in the same sense. Their centers are at $z = \pm b$, where the $z$ axis is the common axis of the coils. Calculate the magnetic field along the axis, and determine the separation $2b$ such that the first, second and third derivatives of $B_z$ with respect to $z$ all vanish at the mid' axis. Thus, the field is very uniform at the center of the Helmholtz coils.

b) Suppose we desire an even more uniform field at the origin. Add a second pair of Helmholtz coils of radius $a' = a/2$. What current $I'$ should flow in the second pair so as to cancel the 4rth derivative of $B_z$ of the first pair? What fraction of the original central field is lost in this configuration?

c) In some applications, it is more important that the field **outside** the coils be as small as possible, rather than the field inside be highly uniform.

Give an expansion for the field along the axis of a set of Helmholtz coils as a function of $u = 1/z$ for $z \gg a,\ b$. Identify the first two nonvanishing multipoles, and find the value of $b$ for which the second of these can be made to vanish.

To cancel the leading multipole as well, add a second coil pair with $a' = 2a$. What current $I'$ should flow in this pair? What fraction of the central field of the first pair is lost? What is the order of the first remaining nonzero multipole?

[See, E.M. Purcell, Am. J. Phys. **57**, 18 (1989).[6]]

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[6]http://kirkmcd.princeton.edu/examples/EM/purcell_ajp_57_18_88.pdf

9. **Expansion of an Axially Symmetric Magnetic Field in Terms of the Axial Field**

Suppose a magnetic field in a current-free region is rotationally symmetric about the $z$-axis. Then,

$$\mathbf{B} = B_r(r, z)\hat{\mathbf{r}} + B_z(r, z)\hat{\mathbf{z}} \tag{14}$$

in cylindrical coordinates. The axial field $B_z(0, z)$ is often relatively easy to calculate. If we write

$$B_z(r, z) = \sum_{n=0}^{\infty} a_n(z) r^n, \qquad \text{and} \qquad B_r(r, z) = \sum_{n=0}^{\infty} b_n(z) r^n , \tag{15}$$

then $a_0(z) = B_z(0, z)$. Use $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and $\boldsymbol{\nabla} \times \mathbf{B} = 0$ to show that

$$B_z(r, z) = \sum_n (-1)^n \frac{a_0^{(2n)}(z)}{(n!)^2} \left(\frac{r}{2}\right)^{2n} , \tag{16}$$

and

$$B_r(r, z) = \sum_n (-1)^{n+1} \frac{a_0^{(2n+1)}(z)}{(n+1)(n!)^2} \left(\frac{r}{2}\right)^{2n+1} , \tag{17}$$

where

$$a_0^{(n)} = \frac{d^n a_0}{dz^n} . \tag{18}$$

This magnetic field can also be deduced from the vector potential whose only nonzero component is

$$A_\phi(r, z) = \sum_n (-1)^n \frac{a_0^{(2n)}(z)}{(n+1)(n!)^2} \left(\frac{r}{2}\right)^{2n+1} . \tag{19}$$

For the example of Helmholtz coils, prob. 5, we know that

$$B_z(0, z) = B_0 + B_4 z^4 + \ldots \tag{20}$$

Give $B_z$ and $B_r$ correct to fourth order in $r$ and $z$.

Show also that, for small $r$, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ leads to the relation

$$B_r(r, z) \approx -\frac{r}{2} \frac{\partial B_z(0, z)}{\partial z} . \tag{21}$$

**Remark.** An electrostatic field with azimuthal symmetry about the $z$ axis can also be expanded according to eqs. (16)-(17). For example, consider a capacitor with circular plates centered about $(r, \theta, z) = (0, 0, 0)$. Then we can expand

$$E_z(0, 0, z) \approx E_z(0, 0, 0) + \frac{z^2}{2} \frac{d^2 E_z(0, 0, 0)}{dz^2} + \ldots \tag{22}$$

and

$$E_z(r, 0, 0) \approx E_z(0, 0, 0) - \frac{r}{2} \frac{d^2 E_z(0, 0, 0)}{dz^2} + \ldots \tag{23}$$

Thus, if $E_z$ has a maximum with respect to $z$ at the origin, it is at a minimum with respect to $r$, or vice versa. The field $\mathbf{E}$ cannot be at a maximum with respect to both $r$ and $z$, as shown in general in prob. 1(c) of set 1.

10. **Nonaxially Symmetric Magnetic Field in Terms of the Axial Field**

In the previous problem, it was demonstrated how knowledge of a static, axial magnetic field leads to a complete characterization of the field if that field is axially symmetric.

A variant on the electro- or magnetostatic boundary value problem arises in accelerator physics, where a specified field, say $\mathbf{B}(0, 0, z)$, that is not axially symmetric is desired along the $z$ axis. In general there exist static fields $\mathbf{B}(x, y, z)$ that reduce to the desired field on the axis, but the "boundary condition" $\mathbf{B}(0, 0, z)$ is not sufficient to insure a unique solution.

For example, find a field $\mathbf{B}(x, y, z)$ that reduces to

$$\mathbf{B}(0, 0, z) = B_0 \cos kz\,\hat{\mathbf{x}} + B_0 \sin kz\,\hat{\mathbf{y}} \tag{24}$$

on the $z$ axis. In this, the magnetic field rotates around the $z$ axis as $z$ advances.

Show that the use of rectangular or cylindrical coordinates leads "naturally" to different forms for $\mathbf{B}$ off the $z$ axis.

One 3-dimensional field extension of (24) is the so-called helical wiggler, which obeys the auxiliary requirement that the field at $z + \delta$ be the same as the field at $z$, but rotated by angle $k\delta$. Show that this field pattern can be realized by a current-carrying wire that is wound in a helix of period $\lambda = 2\pi/k$.

See, B.M. Kincaid, *A short-period helical wiggler as an improved source of synchrotron radiation*, J. Appl. Phys. **48**, 2684-2691 (1977);[7] J.P. Blewett and R. Chasman, *Orbits and fields in the helical wiggler*, J. Appl. Phys. **48**, 2692-2698 (1977).[8]

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[7]http://kirkmcd.princeton.edu/examples/EM/kincaid_jap_48_2684_77.pdf
[8]http://kirkmcd.princeton.edu/examples/EM/blewett_jap_48_2692_77.pdf

11. **Axial Field of a Solenoid Magnet**

A solenoidal coil of radius $a$ and length $l$ has $n$ turns per unit length and carries a current $I$ (in each turn). On the axis, show that

$$B_z(0, z) = \frac{2\pi n I}{c}(\cos\theta_1 + \cos\theta_2), \qquad (25)$$

where $\theta_1$ and $\theta_2$ are the angles between the axis and the ends of the solenoid at the observation point. Near the midpoint of the solenoid $(z = 0)$, show

$$B_r(r, z) \approx \frac{288\pi n I a^2 r z}{c l^4} . \qquad (26)$$

At the end of the coil $(z = l/2)$, show that

$$B_z \approx \frac{2\pi n I}{c} \approx \frac{B_z(0,0)}{2}, \qquad (27)$$

and

$$B_r \approx \frac{\pi n I r}{ac} . \qquad (28)$$

If one is interested in the fields near the end of a long solenoid $(l \gg a)$, it is often sufficient to approximate the coil as semi-infinite, for which (25) leads to

$$B_z(0, z) = \frac{2\pi n I}{c}\left(1 + \frac{z}{\sqrt{z^2 + a^2}}\right), \qquad (29)$$

where $z = 0$ at the end of coil.

12. a) A coil is wound on the surface of a sphere such that the magnetic field inside the sphere will be uniform. How should the turns be distributed?

b) What is the effective magnetic dipole moment of a sphere of uniform surface charge density $\sigma$ which rotates with constant angular velocity $\omega$ about an axis of the sphere?

An electron has a permanent magnetic dipole moment of magnitude

$$\mu = \frac{e\hbar}{2mc} .$$

(30)

Suppose the electron is a rotating spherical shell of charge with radius

$$a = \frac{e^2}{mc^2} ,$$

(31)

the "classical electron radius". What is the velocity at the equator?

Compare the magnetic field energy of the rotating shell of charge with its electric field energy.

13. **Saturation and Hysteresis**

a) A piece of iron saturates in a magnetic field of $\approx 20,000$ Gauss, when all available electron magnetic moments are aligned. The density of iron is 8 g/cm$^3$. How many electrons per iron atom have been aligned to produce this field?

b) You can understand some aspects of the hysteresis curve of a ferromagnet via a model consisting of two permanent dipoles separated by a fixed distance $d$, but free to rotate. In the absence of any external field, what is the equilibrium orientation and energy of the two dipoles?

Suppose a magnetic field **B** is applied at right angles to the line of centers of the dipoles. What is the minimum field strength needed to align the dipoles along **B**? Higher fields produce no further change – saturation has occurred.

Suppose the dipoles were originally aligned parallel to their line of centers, and then a field **B** is applied antiparallel to the dipoles. What is the minimum value of $B$ needed to flip the dipoles?

If the dipoles flip and later **B** is reduced to zero, the dipoles do not unflip – hysteresis has occurred.

14. **Magnetic Field Mapping.** You may be familiar with the method of mapping the equipotentials in 2-dimensional electrostatic problems using conducting paper:
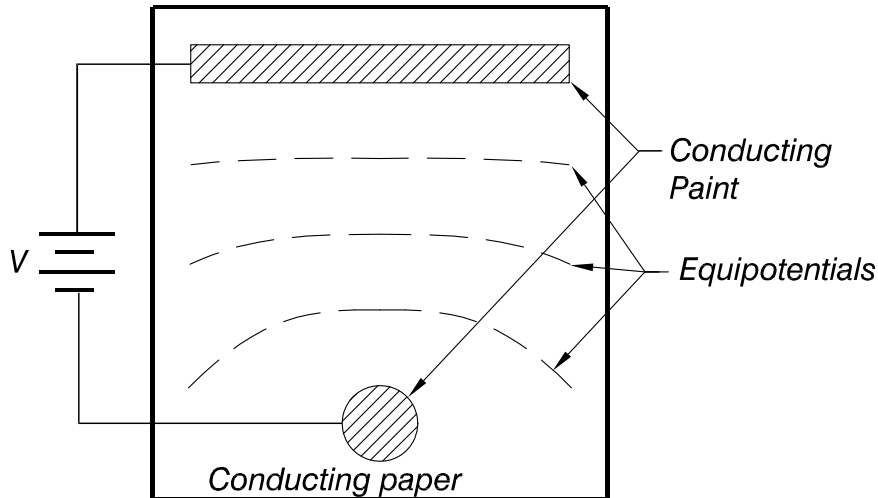


On the paper, $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{J}$ is the two dimensional current density (= current per unit length perpendicular to $\mathbf{J}$), $\sigma$ is the surface conductivity of the resistive paper, and $\mathbf{E}$ is the electric field in the paper. Outside of the sources and sink of current in the patches of conducting paint, we have $\mathbf{\nabla}\cdot\mathbf{J} = 0$, so $\mathbf{\nabla}\cdot\mathbf{E} = 0$ also. The currents and fields are steady, so $\mathbf{\nabla}\times\mathbf{E} = 0$, and the electric field can be derived from a potential, $\mathbf{E} = -\mathbf{\nabla}\phi$ that obeys Laplace's equation, $\nabla^2\phi = 0$. The value of the potential $\phi$ at any point on the paper can be read directly with a voltmeter.

The boundary conditions are that

- $\mathbf{J}$ and $\mathbf{E}$ are perpendicular to the boundaries of the patches of conducting paint.
- $\mathbf{J}$ and $\mathbf{E}$ are parallel to the edges of the paper (where there is no conducting paint. Thus, the region outside the paper is like a dielectric with constant $\epsilon = 0$. This is not very physical, so make the paper much larger than the region used to model the problem of interest (if the boundaries are not entirely conducting).

The conducting paper technique can also be used to model 2-dimensional magnetostatic problems due to current distributions that are normal to the paper.

Imagine that the regions of conducting paint represent the cross sections of infinite conductors that are perpendicular to the paper. Let $\hat{\mathbf{z}}$ label the unit vector normal to the paper. Then, the vector potential due to our imagined currents would be

$$\mathbf{A} = \frac{1}{c}\int\frac{\mathbf{J}}{r}d\mathrm{Vol} = A_z\hat{\mathbf{z}}. \tag{32}$$

(Here, $\mathbf{J}$ is due to the imagined current normal to the paper, not the surface currents in the paper.)

Show that the observed potential $\phi$ on the paper, when a battery feeds current into and out of the conductors on the paper, is proportional to the vector potential of the imagined situation:

$$A_z = k\phi, \qquad k = \text{ constant.} \tag{33}$$

Your analysis might include the following:

- Relate the magnetic field $\mathbf{B}$ of the imagined 2-dimensional current distribution to $\phi$ on the paper. When $\mathbf{A} = A_z\hat{\mathbf{z}}$, $\mathbf{B}$ is perpendicular to $z$. Show that lines of $\mathbf{B}$ exactly follow equipotentials of $\phi$.

- Relate $\mathbf{B}$ to $\mathbf{E}$ on the paper.

- Suppose current $I$ from a battery enters a region of conducting paint on the paper, causing current density $\mathbf{J}$ to flow outwards:
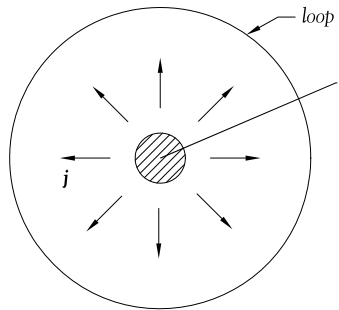


Consider $\oint \mathbf{J} \times d\mathbf{l}$ for a loop enclosing the region of conducting paint to determine the constant $k$ in (33).

- Show that the voltage difference between any two points on the paper is proportional to the magnetic flux passing between these points.

- Show that the boundary conditions at the edge of the paper are such that we may consider the region outside the paper as being iron of a very large permeability $\mu$.

As an example, consider a long electromagnetic with an iron yoke:
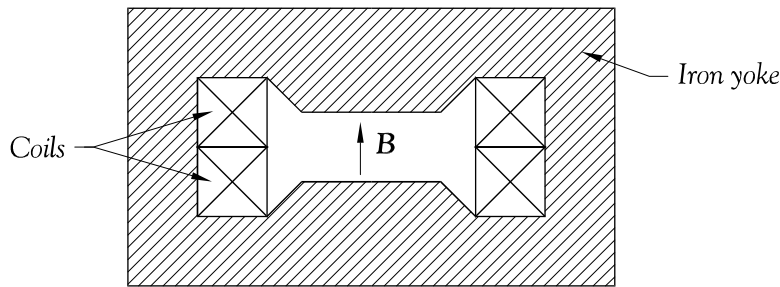


Invoking symmetry, we could map this with an arrangement like:
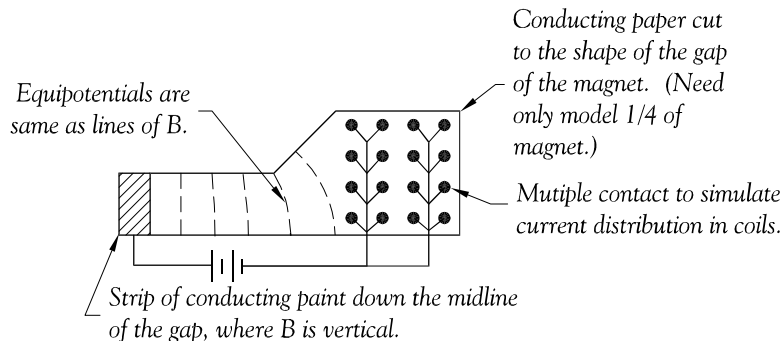


Try it in the lab sometime!

## Solutions

1. a) The problem is 1-dimensional, so Poisson's equation is

$$\frac{d^2\phi}{dx^2} = -4\pi\rho(x). \tag{34}$$

Since an electron leaves the cathode at $v = 0$, when it reaches position $x$, it has energy $e\phi(x) = mv^2/2$, and velocity

$$v = \sqrt{\frac{2e\phi}{m}}. \tag{35}$$

Since the current density $J = \rho v$ is constant, eq. (34) becomes

$$\frac{d^2\phi}{dx^2} = -4\pi\frac{J}{v} = -4\pi J\sqrt{\frac{m}{2e}}\phi^{-1/2}. \tag{36}$$

We try (pray for) a power law solution, $\phi = ax^p$, which quickly leads $p = 4/3$. Then, since $\phi(d) = V$, the potential is

$$\phi(x) = V\left(\frac{x}{d}\right)^{4/3}. \tag{37}$$

Equation (36) can now be rearranged as

$$J = -\frac{\phi''}{4\pi}\sqrt{\frac{2e}{m}}\phi^{1/2} = -\frac{1}{9\pi}\frac{V^{3/2}}{d^2}\sqrt{\frac{2e}{m}} = -\frac{1}{6.36\pi}\frac{V^{3/2}}{d^2}\sqrt{\frac{e}{m}}. \tag{38}$$

The electric **space charge** density $\rho(x)$ follows from eqs. (34) and (37),

$$\rho(x) = -\frac{\phi''}{4\pi} = \frac{V}{9\pi d\sqrt[3]{dx^2}}, \tag{39}$$

which is very large close to the cathode at $x = 0$.

b) The initial electric field in the capacitor is $E = V/d$, so the initial surface charge density on the cathode is

$$\sigma = -E/4\pi = -V/4\pi d. \tag{40}$$

The laser liberates this charge density at $t = 0$.

The average current density that flows onto the anode from the battery is

$$\langle J \rangle = -\frac{\sigma}{T} = \frac{V}{4\pi dT}, \tag{41}$$

where $T$ is the transit time of the charge across the gap $d$. We first estimate $T$ by ignoring the effect of the recharging of the cathode as the charge sheet moves away from it. In this approximation, the average field on the charge sheet is always $E/2 = V/2d$,

so the acceleration of an electron is $a = eV/2dm$, and the time to travel distance $d$ is $T = \sqrt{2d/a} = 2d\sqrt{m/eV}$. Hence,

$$\langle J \rangle = \frac{1}{8\pi} \frac{V^{3/2}}{d^2} \sqrt{\frac{e}{m}}. \tag{42}$$

This is close to Child's Law (38).

[This sign difference between (38) and (42) is because the former is the current flowing off the anode, while the latter is the current flowing onto it.]

We now make a detailed calculation, including the effect of the recharging of the cathode, which will reduce the average current density somewhat.

At some time $t$, the charge sheet is at distance $x(t)$ from the cathode, and the anode and cathode have charge densities $\sigma_A$ and $\sigma_C$, respectively. All the field lines that leave the anode terminate on either the charge sheet or on the anode, so

$$\sigma + \sigma_C = -\sigma_A. \tag{43}$$

The magnitude of the electric field strength in the region I between the anode and the charge sheet is

$$E_I = 4\pi\sigma_A, \tag{44}$$

and that in region II between the charge sheet and the cathode is

$$E_{II} = -4\pi\sigma_C. \tag{45}$$

The voltage between the capacitor plates is therefore,

$$V = E_I(d - x) + E_{II}x = 4\pi\sigma_A d - V\frac{x}{d}, \tag{46}$$

using (40) and (43-45). Thus,

$$\sigma_A = \frac{V}{4\pi d}\left(1 + \frac{x}{d}\right), \qquad \sigma_C = -\frac{Vx}{4\pi d^2}, \tag{47}$$

and the time-dependent current density flowing onto the anode is

$$J(t) = \dot{\sigma}_A = \frac{V\dot{x}}{4\pi d^2}. \tag{48}$$

This differs from the average current density (41) in that $\dot{x}/d \neq T$, since $\dot{x}$ varies with time.

To find the velocity $\dot{x}$ of the charge sheet, we consider the force on it, which is due to the average field set up by charge densities on the anode and cathode,

$$E_{\text{on } \sigma} = 2\pi(-\sigma_A + \sigma_C) = -\frac{V}{2d}\left(1 + \frac{2x}{d}\right). \tag{49}$$

The equation of motion of an electron in the charge sheet is

$$m\ddot{x} = -eE_{\text{on } \sigma} = \frac{eV}{2d}\left(1 + \frac{2x}{d}\right), \tag{50}$$

or

$$\ddot{x} - \frac{eV}{md^2}x = \frac{eV}{2md}. \tag{51}$$

With the initial conditions that the electron starts from rest, $x(0) = 0 = \dot{x}(0)$, we readily find that

$$x(t) = \frac{d}{2}(\cosh kt - 1), \tag{52}$$

where

$$k = \sqrt{\frac{eV}{md^2}}. \tag{53}$$

The charge sheet reaches the anode at time

$$T = \frac{1}{k}\cosh^{-1}\frac{3}{2} = \frac{0.96}{k}, \tag{54}$$

compared to $T = 1/k$ as found above without the battery. anode-current density is, using (41) and (54),

$$\langle J \rangle = \frac{V}{4\pi dT} = \frac{V^{3/2}}{4\pi \cosh^{-1}(3/2) \ d^2}\sqrt{\frac{e}{m}} = \frac{V^{3/2}}{12.09 \ \pi d^2}\sqrt{\frac{e}{m}}. \tag{55}$$

The electron velocity is

$$\dot{x} = \frac{dk}{d}\sinh kt, \tag{56}$$

so the anode-current density (48) is

$$J = \frac{1}{8\pi}\frac{V^{3/2}}{d^2}\sqrt{\frac{e}{m}}\sinh kt \qquad (0 < t < T). \tag{57}$$

2. **Potential inside a Resistive Sphere**

Although current is flowing inside the resistive sphere, it remains electrically neutral. Hence, the electromagnetic scalar potential $\phi$ satisfies Laplace's equation, $\nabla^2 \phi = 0$.

We analyze the problem in spherical coordinates $(r, \theta, \varphi)$, with the origin at the center of the sphere of radius $a$, and $\theta = 0$ and $\pi$ at the points of contact with the wires. The problem has axial symmetry, so $\phi$ will be independent of $\varphi$. We require the potential to be well behaved at the origin, so it can be expressed in a Legendre series,

$$\phi(r < a) = \sum_{n=0}^{\infty} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{58}$$

The convention that $\phi = 0$ at the equator, $\theta = \pi/2$, implies that $A_n = 0$ for $n$ even. Therefore, we can write

$$\phi(r < a) = \sum_{n \text{ odd}} A_n \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{59}$$

To complete the solution inside the sphere, we need a boundary condition on $\phi$ at the surface of the sphere, $r = a$. We know that the radial component of the current density, $J_r$ is zero at the surface, except for the contact points where the current enters and exits. Since $\mathbf{J} = \sigma\mathbf{E} = -\sigma\nabla\phi$, we obtain a condition on the derivative of the potential at the boundary,

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a^-} = -E_r(r = a^-) = -\frac{J_r(r = a^-)}{\sigma}. \tag{60}$$

In the limit of very fine wires, the current density $J_r(r = a^-)$ is zero except at the poles, so we can express it in terms of Dirac $\delta$ functions. The current $dI$ that crosses an annular region on the surface of the sphere of angular extent $d\cos\theta$ centered on angle $\theta$ is given by

$$dI = 2\pi a^2 J_r(a^-, \theta) d\cos\theta. \tag{61}$$

Current $I$ enters at $\cos\theta = 1$, and exits at $\cos\theta = -1$. Hence, the form

$$J_r(a^-, \theta) = \frac{I}{2\pi a^2}\left[-\delta(\cos\theta - 1) + \delta(\cos\theta + 1)\right]. \tag{62}$$

describes the entrance and exit currents upon integration of eq. (61).

Combining eqs. (59)-(60) and (62), we have

$$\sum_{n \text{ odd}} \frac{nA_n}{a} P_n(\cos\theta) = \frac{I}{2\pi a^2}[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)]. \tag{63}$$

As usual, to evaluate the Fourier coefficients $A_n$, we multiply eq. (63) by $P_n(\cos\theta)$ and integrate over $d\cos\theta$ to find
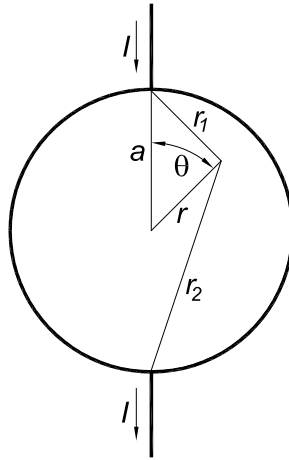
$$\frac{nA_n}{a}\int_{-1}^{1} P_n^2(\cos\theta)\,d\cos\theta = \frac{2nA_n}{(2n+1)a} = \frac{2I}{2\pi a^2\sigma}. \tag{64}$$

Thus, the Legendre series expansion for the potential is

$$\phi(r < a, \theta) = \frac{I}{2\pi a\sigma} \sum_{n \text{ odd}} \left(2 + \frac{1}{n}\right) \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{65}$$

To express this series in closed form, we utilize the expansion for the distance $r_1$ between the point $(a, 0)$ and $(r, \theta)$ given on p. 57 of the notes:

$$\frac{1}{r_1} = \frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n P_n(\cos\theta), \tag{66}$$



Similarly, the distance $r_2$ between the point $(a, \pi)$ and $(r, \theta)$ is

$$\frac{1}{r_2} = \frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n P_n(\cos(\theta-\pi)) = \frac{1}{a} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n P_n(-\cos\theta) = \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{67}$$

Hence,

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{2}{a} \sum_{n \text{ odd}} \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{68}$$

It follows that, on integration along the radius from the origin to the point $(r, \theta)$,

$$\int_0^r \left(\frac{1}{r_1} - \frac{1}{r_2}\right) \frac{dr'}{r'} = \frac{2}{a} \sum_{n \text{ odd}} \frac{1}{n} \left(\frac{r}{a}\right)^n P_n(\cos\theta). \tag{69}$$

Then, eqs. (65) and (68-69) combine to give to the alternative form (3) for $\phi$,

$$\phi(r < a, \theta) = \frac{I}{2\pi\sigma b} \left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{2} \int_0^r \left(\frac{1}{r_1} - \frac{1}{r_2}\right) d\ln r'\right]. \tag{3}$$

As we approach the "north" pole, $r_1 \to 0$, (we claim; details given later) the first term in eq. (3) dominates. That is, the potential $\phi$ diverges at the poles for the case of very fine wires.

When considering actual wires of radius $b$, we suppose that our solution holds outside the region of contact between the wire and the sphere. Indeed, we expect that the potential close to the wire, and outside the conducting sphere, to be constant in planes perpendicular to the axis of the wire, so that the interface between the wire and the sphere is an equipotential. This cuts off the formal divergence in eq. (3) near the poles.

In this way, the potential at the upper interface is obtained from (3) on putting $r_1 = b$ and neglecting all but the first term,

$$\phi_{\text{interface}} \equiv V \approx \frac{I}{2\pi\sigma b}. \tag{70}$$

The potential difference across the sphere is twice this;

$$\Delta V = 2V \approx \frac{I}{\pi\sigma b} = I\frac{b}{\sigma\pi b^2} \equiv IR. \tag{71}$$

Thus, the effective resistance of the sphere is

$$R \approx \frac{1}{\pi\sigma b} = \frac{b}{\sigma\pi b^2}, \tag{72}$$

which is also the resistance of a piece of wire of radius $b$, length $b$, and conductivity $\sigma$.

To verify the claim that the first term of eq. (3) dominates for small $r_1$, we consider the point $(r, \theta) = (a - b, 0)$ for $b \ll a$. Then, the first term of eq. (3) is $1/b$, and the second term is $1/(2a - b)$ which is negligible compared to the first. Inside the integral term of eq. (3), we have $R_1 = a - r$ and $R_2 = a + r$, so the integral is

$$\int_0^{a-b} \left( \frac{1}{r_1} - \frac{1}{r_2} \right) d\ln r = \int_0^{a-b} \frac{2}{a^2 - r^2} \, dr = \frac{1}{a} \ln \frac{2a - b}{b} \approx \frac{1}{a} \ln \frac{2a}{b}. \tag{73}$$

The ratio of the integral term to the first term of (3) is therefore,

$$\frac{b}{2a} \ln \frac{2a}{b}, \tag{74}$$

which goes to zero as $b$ becomes small.

### Magnetic Field and Poynting Vector

The power dissipated by the resistive bead is, according to eq. (72),

$$P = I^2 R \approx \frac{I^2}{\pi\sigma b}. \tag{75}$$

As a check on the solution (65) for the potential, we consider whether the dissipated power equals the integral of the Poynting flux, $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$ (in Gaussian units), normal to the surface of the bead, where $c$ is the speed of light in vacuum.

For this, we need the magnetic field $\mathbf{B}$ at $r = a$, due to the electric currents in the problem. This field is azimuthal, because of the azimuthal symmetry of the problem.

Then, the magnetic field at the surface of the resistive bead follows easily from Ampère's law,

$$2\pi r_\perp B_\phi(r = a^-) = 2\pi a \sin\theta B_0 \; = \; -\frac{4\pi}{c}I, \qquad B_\phi(r = a^-) = -\frac{2I}{c\,a\sin\theta}, \qquad (76)$$

The radial component of the Poynting vector also depends on the electric field component,

$$E_\theta(r = a^- = -\frac{1}{a}\frac{\partial\phi(r = a^-)}{\partial\theta}. \qquad (77)$$

Then, the radial component of the Poynting vector at the surface of the bead is

$$\begin{aligned}
S_r(r = a^-, \theta) & = \; \frac{c}{4\pi}E_\theta(r = a^-)B_\phi(r = a^-) = \frac{I}{2\pi a^2}\frac{1}{\sin\theta}\frac{\partial\phi(r = a^-)}{\partial\theta} \\
& = \; -\frac{I}{2\pi a^2}\frac{\partial\phi(r = a^-)}{\partial\cos\theta} = -\frac{I}{2\pi a^2}\frac{I}{2\pi a\sigma}\sum_{n\text{ odd}}\left(2 + \frac{1}{n}\right)P_n'(\cos\theta). \, (78)
\end{aligned}$$

The integral of the radial component of the Poynting vector over the surface of the bead is

$$\begin{aligned}
P_{\text{into bead}} & = \; -2\pi a^2\int_{-1}^1 S_r(r = a^-, \theta)\,d\cos\theta = \frac{I^2}{2\pi a\sigma}\sum_{n\text{ odd}}\left(2 + \frac{1}{n}\right)\int_{-1}^1 P_n'(\cos\theta)\,d\cos\theta \\
& = \; \frac{I^2}{\pi a\sigma}\sum_{n\text{ odd}}\left(2 + \frac{1}{n}\right) \approx \frac{b}{a}I^2 R\sum_{n\text{ odd}}\left(2 + \frac{1}{n}\right). \qquad (79)
\end{aligned}$$

Formally, the result (79) diverges, which corresponds to infinite power dissipation at the points of contact of the wires with the bead, in the limit of zero radius of these wires. For wires of finite radius $b$, the power dissipated is finite, $P = I^2 R$ for resistance $R$ as approximated in eq. (72), but then the formal solutions (3) and (65) are only approximate. Since the sum of the first $N$ terms of the series $\sum_{N\text{ odd}}(2 + 1/n)$ is roughly $N^2$, we infer that the form (65) for the potential inside the bead in case of wires of radius $b$ is a reasonable approximation if we keep only the first $N \approx \sqrt{a/b}$ terms.

In practice, one would be content to keep many fewer terms than this.

### Potential outside the Resistive Sphere

We now consider the problem outside the sphere, for which one model is that the wires are perfect conductors extending from $r = a$ to distance $d$, where they are attached to perfectly conducting hemispheres of radius $d > a$, with a ring-shaped battery of potential difference $2V$ between them at location $(r, \theta) = (c, \pi/2)$.
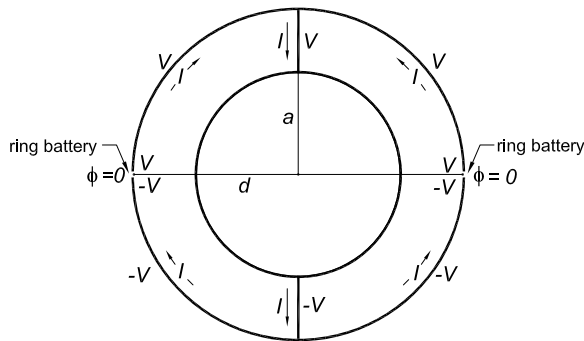
In the region $a < r < d$ and outside the wires at $\theta = 0$ and $\pi$, the potential obeys $\nabla^2 \phi = 0$, is azimuthally symmetric, and symmetric about the plane $\theta = \pi/2$, so it can be expanded as

$$\phi(a < r < d) = \sum_{n \text{ odd}} \left[ B_n \left( \frac{r}{a} \right)^n + C_n \left( \frac{a}{r} \right)^{n+1} \right] P_n(\cos\theta). \tag{80}$$

Continuity of the potential at $r = a$ requires, recalling eqs. (59) and (64), that

$$B_n + C_n = A_n = \frac{(2n+1)I}{2\pi a n \sigma} = \frac{(2n+1)bV}{an}. \tag{81}$$

The constant potential $V$ on the upper hemisphere requires that

$$\phi(r = d, 0 < \cos\theta < 1) = V = \sum_{n \text{ odd}} \left[ B_n \left( \frac{d}{a} \right)^n + C_n \left( \frac{a}{d} \right)^{n+1} \right] P_n(\cos\theta), \tag{82}$$

and the constant potential $V$ on the upper wire requires that

$$\phi(a < r < d, \theta = 0) = V = \sum_{n \text{ odd}} \left[ B_n \left( \frac{r}{a} \right)^n + C_n \left( \frac{a}{r} \right)^{n+1} \right]. \tag{83}$$

If we multiply eq. (82) by $P_n(\cos\theta) = P_n(x)$ and integrate over $x$ from 0 to 1, we obtain (using Wolfram Alpha with `integrate Legendre P(n,x) from x=0 to 1`),[9]

$$V \int_0^1 P_n(x)\, dx = \frac{\sqrt{\pi}\, V}{2\Gamma(1 - \frac{n}{2})\Gamma(\frac{n+3}{2})} \equiv K_n V = \frac{1}{2n+1} \left[ B_n \left( \frac{d}{a} \right)^n + C_n \left( \frac{a}{d} \right)^{n+1} \right], \tag{84}$$

$K_1 = 1/2$, $K_3 = -1/8$, $K_5 = 1/16$, $K_7 = -5/128$, $K_9 = 7/256$, $K_{11} = -21/1024$, ...
Combining this with eq. (81), we find

$$B_n = \frac{(2n+1)V}{(\frac{d}{a})^n - (\frac{a}{d})^{n+1}} \left[ K_n - \frac{b}{an} \left( \frac{a}{d} \right)^{n+1} \right], \qquad C_n = \frac{(2n+1)V}{(\frac{d}{a})^n - (\frac{a}{d})^{n+1}} \left[ \frac{b}{an} \left( \frac{d}{a} \right)^n - K_n \right]. \tag{85}$$

It is obvious how well the $B_n$ and $C_n$ of eq. (85) satisfy the condition (83), but a numerical example suggests that they do so. For example, suppose that $d = 2a$,

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[9]Jackson uses Rodrigues' formula and integration by parts $n$ times to find $K_n = (-1/2)^{(n-1)/2}(n-2)!!/2[(n+1)/2]!$, his eq. (3.26).

$b/a = 0.03$ and $V = 1$. Then, using only the first six terms of eq. (83) yields the following plot,
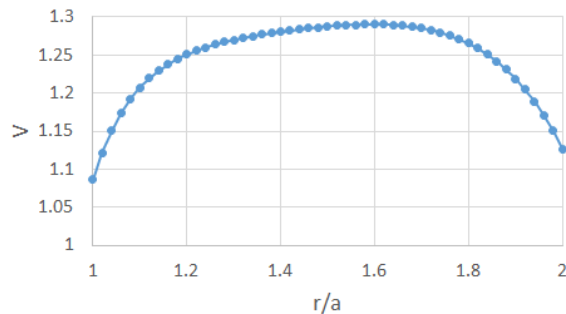


We also infer that the surface $r = a$ of the sphere supports electric charge density,

$$\varsigma(\theta) = \frac{E_r(r = a^+) - E_r(r = a^-)}{4\pi} = -\frac{\partial}{\partial r}[\phi(r = a^+) - \phi(r = a^-)] \tag{86}$$

$$= \sum_{n \text{ odd}} \frac{-nB_n + (n+1)C_n + nA_n}{a}P_n(\cos\theta) = \sum_{n \text{ odd}} \frac{(2n+1)C_n}{a}P_n(\cos\theta).$$

This illustrates the general result that current-carrying conductors (of finite conductivity $\sigma$) have nonzero surface charge density, as needed to shape the electric field $\mathbf{E} = \mathbf{J}/\sigma$ which drives the current inside the conductor.

3. a) Take the axis of the wire to be the $z$ axis of a cylindrical coordinate system, $(r, \theta, z)$. By rotational symmetry, there is no azimuthal component to the electric field, and by translation symmetry, the axial and radial components can only depend on $r$. Similarly, there is no radial or axial component to the magnetic field, and its azimuthal component is only dependent on $r$.

Consider a cylindrical portion of the wire, of radius $r$ and length $l$. The charge contained in this cylinder is then,

$$Q = e(\rho_0 - \rho)\pi r^2 l, \tag{87}$$

where $e$ is the magnitude of the charge of an electron. From Gauss' Law and (87), we have

$$4\pi Q = 4\pi^2 e r^2 l(\rho_0 - \rho) = \oint \mathbf{E} \cdot d\mathbf{S} = 2\pi r l E_r(r), \tag{88}$$

since the contributions from the flat end surfaces are be equal and opposite. Hence,

$$\mathbf{E} = 2\pi e(\rho_0 - \rho)r\hat{\mathbf{r}} + E_z\hat{\mathbf{z}}. \tag{89}$$

The resulting radial force on the free electrons must be opposed by magnetic effects associated with the electron current, which we presume flows only in the $z$ direction

The current density in the wire is

$$\mathbf{J} = -\rho e v\hat{\mathbf{z}}, \tag{90}$$

so from Ampère's law,

$$2\pi r B_\theta = -\frac{4\pi}{c}\rho e v\pi r^2, \qquad \text{and} \qquad \mathbf{B} = -\frac{2\pi}{c}e\rho v r\hat{\boldsymbol{\theta}}. \tag{91}$$

The radial component of the Lorentz force on an electron, which must vanish if there is to be no current in radial direction, is then,

$$F_r = -e\left(E_r - \frac{v_z}{c}B_\theta\right) = -e\left[2\pi e(\rho_0 - \rho)r + 2\pi e\rho\frac{v^2}{c^2}r\right] = 0, \tag{92}$$

using (89) and (91). Hence,

$$\rho_0 = \rho\left(1 - \frac{v^2}{c^2}\right), \tag{93}$$

and the positive charge density is less than the negative by one part in $10^{21}$ for $v = 1$ cm/s.

b) In this problem, we treat the resistor as a kind of conductive capacitor. Since the current density is uniform, the electric field $E$ is also. To maintain this electric field, surface charge $\pm Q$ must reside at the ends of the resistor. From Gauss' Law,

$$Q = \frac{EA}{4\pi}, \tag{94}$$

where $A$ is the cross-sectional area of the resistor.

If the current $I$ into to resistor varies with time, then part of it goes to changing the charge $Q$ at the ends of the resistor, and part of it appears as the conduction current $I_C$ across the resistor. Thus, $I_C$ is less than $I$ according to

$$I_C = I - \dot{Q}. \tag{95}$$

However, Maxwell advises us that inside the resistor we should also consider the displacement current,

$$I_D = \frac{\dot{D}}{4\pi} = \frac{\epsilon \dot{E}}{4\pi} = \epsilon \dot{Q} = \dot{Q}, \tag{96}$$

using (94), in a medium whose dielectric constant $\epsilon$ is unity.

Combining (95) and (96), the "total" current inside the resistor is thus,

$$I_{\text{total}} = I_C + I_D = I, \tag{97}$$

which illustrates Maxwell's notion that "total" currents are conserved.

4. The form of the current $I(t)$ in a cylindrical "straw tube" can also be found without using the reciprocation theorem, so we illustrate that first.

**Elementary Solution for $I(t)$**

The current that flows off the anode is equal to minus the rate of change of the charge $q(t) < 0$ that remains on the anode as the positive ions of total charge $q_0$ move outward according to $r(t)$.

The key to an elementary solution is that although the positive ions occupy a very small volume around the point $(r, \theta, z) = (r(t), 0, 0)$ in cylindrical coordinates, the charge they induce on the cathode is exactly the same as if those ions were uniformly spread out over a cylinder of radius $r$.

Because the superposition principle holds in electrostatics, the problem of the chamber with voltage $V$ on the anode plus ions at a fixed position between the anode and cathode can be separated into two parts. First, an empty chamber with voltage $V$ on the anode, and second, a grounded chamber with positive ions inside. [That is, we decompose the problem into cases A and B of the discussion of the reciprocation theorem, even though we won't use that theorem here.]

For the second part, the radial electric field in the region $a < r < r(t)$ can be calculated from the charge $q$ on the anode as

$$E(r) = \frac{2q(t)}{rl}, \tag{98}$$

using Gauss' Law, where $l \gg b$ is the length of the cylinder. Similarly, the electric field in the region $r(t) < r < b$ is

$$E(r) = \frac{2(q_0 + q(t))}{rl}. \tag{99}$$

The potential difference between the inner and outer cylinder must be zero. Hence,

$$0 = \frac{2q(t)}{l} \int_a^{r(t)} \frac{dr}{r} + \frac{2(q_0 + q(t))}{l} \int_{r(t)}^b \frac{dr}{r} = \frac{2q_0}{l} \ln \frac{b}{r(t)} + \frac{2q(t)}{l} \ln \frac{b}{a}, \tag{100}$$

and so

$$q(t) = -q_0 \frac{\ln(b/r(t))}{\ln(b/a)}. \tag{101}$$

The current is

$$I(t) = -\dot{q}(t) = -\frac{q_0}{\ln(b/a)} \frac{v(t)}{r(t)}. \tag{102}$$

To calculate the dynamical quantities $r(t)$ and $v(t)$, we must return to the full problem of the ions in a chamber with voltage $V$. The electric field in the chamber is only slightly perturbed by the presence of the ions, and so is given by

$$E(r) = \frac{V}{r \ln(b/a)}. \tag{103}$$

According to (5), the positive ions have velocity

$$v(r) = \frac{\mu V}{r \ln(b/a)},$$
(104)

which integrates to give

$$r^2(t) = a^2 + \frac{2\mu V}{\ln(b/a)} t.$$
(105)

Inserting (104-105) in (102), we find

$$I(t) = -\frac{q_0}{2t_0 \ln(b/a)} \frac{1}{1 + t/t_0},$$
(106)

where

$$t_0 = \frac{a^2 \ln(b/a)}{2\mu V}.$$
(107)

The idealized current pulse has a very sharp rise, and falls off rapidly over characteristic time $t_0$, which is about 20 nsec in typical straw tube chambers.

### $I(t)$ **via Reciprocity**

Referring to the prescription in the statement of the problem, we first solve case C, in which the inner electrode is at unit potential and the outer electrode is grounded. We quickly find that

$$V_C(r) = \frac{\ln(b/r)}{\ln(b/a)}.$$
(108)

According to (11), the current off the inner electrode is therefore,

$$I(t) = -q_0 \frac{dV_C}{dr} v(r) = -\frac{q_0}{\ln(b/a)} \frac{v(t)}{r(t)},$$
(109)

as previously found in (102). We again solve for $v$ and $r(t)$ as in (103-105), which corresponds to the use of case A, to obtain the solution (106-107).

### **The Charge Distribution $q(z)$ on the Cathode**

The more detailed question as to the longitudinal charge distribution on the cathode can be solved by the reciprocation method if we conceptually divide the cathode cylinder into a ring of length $dz$ at position $z_1$ plus two cylinders that extends to $z = \pm l/2$ where $l$ is the length of the cylinder. We label the ring as electrode 1, and calculate the charge $\Delta Q_1 = q(z)dz$ induced on this ring when the positive ion charge $q_0$ is at position $(r_0, 0, z_0)$ in cylindrical coordinates $(r, \theta, z)$.

According to the prescription (10) given in the statement of the problem,

$$\Delta Q_1 = -q_0 V_C(r_0, 0, z_0),$$
(110)

where case C now consists of a cylinder of radius $b$ grounded except for the ring at position $z_1$ at unit potential, and a grounded cylinder at radius $a$. For $z$ not close to

the ends of the cylinder, the end surfaces $z = \pm l/2$ may be approximated as at ground potential.

This problem is very similar to that discussed in sec. 5.36 of W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (Mcgraw-Hill, New York, 1968).

Laplace's equation, $\nabla^2 \phi_C(\mathbf{r}) = 0$ holds for the potential in the region $a < r < b$. The problem has azimuthal symmetry, so $\phi_C$ will be independent of $\theta$. Since the planes $z = \pm l/2$ are grounded, the longitudinal functions in the Fourier series expansion,

$$\phi_C = \sum_n R_n(r) Z_n(z), \tag{111}$$

must have the form $Z_n = \sin 2n\pi z/l$. The equation for the radial functions $R_n(r)$ follows from Laplace's equation as

$$\frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} - \left(\frac{2n\pi}{l}\right)^2 R_n = 0. \tag{112}$$

The solutions of this are the modified Bessel functions of order zero, $I_0(2n\pi r/l)$ and $K_0(2n\pi r/l)$. Both of these are finite on the interval $a < r < b$, so the expansion (111) will include them both.

The boundary condition that $\phi_C(a, \theta, z) = 0$ is satisfied by the expansion

$$\phi_C = \sum_n A_n \frac{\frac{I_0(2n\pi r/l)}{I_0(2n\pi a/l)} - \frac{K_0(2n\pi r/l)}{K_0(2n\pi a/l)}}{\frac{I_0(2n\pi b/l)}{I_0(2n\pi a/l)} - \frac{K_0(2n\pi b/l)}{K_0(2n\pi a/l)}} \sin\frac{2n\pi z}{l}, \tag{113}$$

where the form of the denominator is chosen to simplify the evaluation of the boundary condition at $r = b$. Here, $\phi_C = 0$, except of an interval $dz$ long about $z$ where it is unity. Hence, the Fourier coefficients are

$$A_n = \frac{2}{l} \sin\frac{2n\pi z_1}{l}dz. \tag{114}$$

In sum, the charge distribution q(z) on the cathode at radius $b$ due to positive charge $q_0$ at $(r_0, 0, z_0)$ follows from (110) and (112-113) as

$$q(z) = -\frac{2q_0}{l} \sum_n \frac{\frac{I_0(2n\pi r_0/l)}{I_0(2n\pi a/l)} - \frac{K_0(2n\pi r_0/l)}{K_0(2n\pi a/l)}}{\frac{I_0(2n\pi b/l)}{I_0(2n\pi a/l)} - \frac{K_0(2n\pi b/l)}{K_0(2n\pi a/l)}} \sin\frac{2n\pi z}{l} \sin\frac{2n\pi z_0}{l}. \tag{115}$$

A numerical evaluation of (115) is illustrated in Fig. 1. As is to be expected, the induced charge distribution on the cathode has characteristic width of order $b - r_0$, the distance of the positive charge from the cathode.
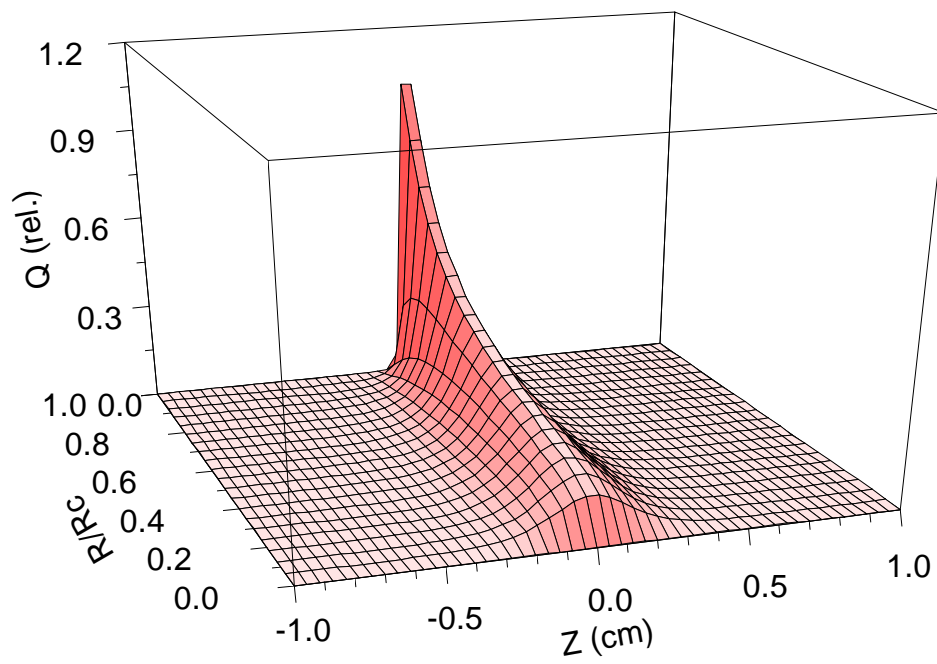
Figure 1:   The induced charge distribution (115) on the cathode of a straw
tube chamber of radius $R_C = 0.25$ cm due to positive ion charge at radius $R$.

5. This problem is posed on p. 363 of *The Mathematical Theory of Electricity and Magnetism*, 5th ed., by James Jeans.

We will evaluate the resistance $R$ via Ohm's Law, $R = V/I$, by calculating the current $I$ that flows when a potential difference $V$ is established between the two contacts.

For a thin disk, the current flow is 2-dimensional. Since $\mathbf{J} = \sigma \mathbf{E}$, where $\mathbf{J}$ is the current density and $\mathbf{E}$ is the electric field, the electric field is 2-dimensional also. And, since $\mathbf{E} = -\nabla \phi$, where $\phi$ is the electric potential, the potential is 2-dimensional as well.

The form of the 2-dimensional potential is well approximated (for distances more than $\delta/2$ from the centers of the contacts) by considering a cylinder of radius $a$, rather than the disk, with a line charge density $\lambda$ that passes through the center of one contact, and line charge $-\lambda$ that passes through the center of the other contact.

The electric field from the wire of charge density $\lambda$ has magnitude

$$E_1 = \frac{2\lambda}{r_1}, \tag{116}$$

according to Gauss' law, where $r_1$ is the distance from the wire to the observer. The corresponding electric potential is

$$\phi_1 = 2\lambda \ln \frac{r_1}{r_0}, \tag{117}$$

where $r_0$ is a constant of integration. The potential due to the wire with charge density $-\lambda$ is similarly

$$\phi_2 = -2\lambda \ln \frac{r_2}{r_0}, \tag{118}$$

where $r_2$ is the distance from the observer to wire 2. The potential at an arbitrary point is then given by

$$\phi = \phi_1 + \phi_2 = 2\lambda \ln \frac{r_1}{r_2}. \tag{119}$$

The total potential difference between the two line charges is formally divergent. To make physical sense, we can suppose that expression (119) holds only for $r_1$ and $r_2$ greater than $\delta/2$, the half width of the electrical contacts, and the potential is essentially constant for smaller values of $r_1$ and $r_2$. That is, we approximate the contacts of width $\delta$ by perfectly conducting wires of radii $\delta/2$, as shown in the figure below. Then, the potential of contact 2 is estimated from eq. (119) by setting $r_1 = d - \delta/2$ and $r_2 = \delta/2$

$$\phi(\text{contact } 2) = 2\lambda \ln \frac{d - \delta/2}{\delta/2} \approx 2\lambda \ln \frac{2d}{\delta}. \tag{120}$$

The potential at the surface of contact 1 is just the negative of eq. (120), so the potential difference is

$$V \approx 4\lambda \ln \frac{2d}{\delta}. \tag{121}$$
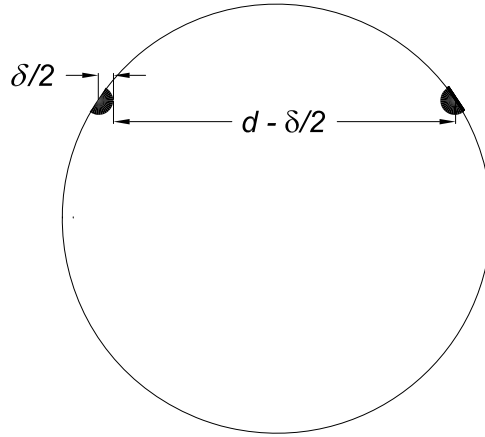
We note that the current and the electric field must be tangential to the edge of the disk. We recall that the equipotentials of eq. (119) are circles, and that the corresponding electric field lines are also circles which, of course, pass through the line charges. Hence, the boundary condition on the electric field at the edge of the disk is indeed satisfied.

To complete the solution, we must calculate the current $I$ that is flowing. For this, we can integrate the current density $\mathbf{J}$ across any surface between the two contacts. For convenience, consider a cylindrical surface of radius $r$ centered on one of the contacts, such that $\delta/2 < r \ll d$. Since $r \ll d$, this surface is essentially an equipotential, and the electric field is essentially that due to the nearby charge density $\lambda$. Namely, the electric field is normal to this surface, with magnitude

$$E = \frac{2\lambda}{r}. \tag{122}$$

The current density across this surface is given by $J = \sigma E$. Restricting the problem to a disk of thickness $t$, the relevant area of the surface is $\pi r t$, so the total current is

$$I = \pi r t \cdot \sigma \cdot \frac{2\lambda}{r} = 2\pi\sigma\lambda t, \tag{123}$$

which is independent of the choice of $r$.

Finally, the resistance is found by combining eqs. (121) and (123):

$$R = \frac{V}{I} \approx \frac{4\lambda \ln 2d/\delta}{2\pi\sigma\lambda t} = \frac{2}{\pi\sigma t} \ln \frac{2d}{\delta}, \tag{124}$$

independent of the radius $a$ of the disk.

6. The series expansion approach is unsuccessful in treating the full problem of a "checker-board" array of two phases if those phases meet in sharp corners as shown above. However, an analytic form for the electric potential of a two-phase (and also a four-phase) checkerboard can be obtained using conformal mapping of certain elliptic functions; see R.V. Craster and Yu.V. Obnosov, *Checkerboard composites with separated phases*, J. Math. Phys. **42**, 5379 (2001).[10] If the regions of one phase are completely surrounded by the other phase, rather lengthy series expansions for the potential can be given; see Bao Ke-Da, Jörger Axell and Göran Grimvall, *Electrical conduction in checker-board geometries*, Phys. Rev. B **41**, 4330 (1990).[11] The present problem is based on M. Söderberg and G. Grimvall, *Current distribution for a two-phase material with chequer-board geometry*, J. Phys. C: Solid State Phys. **16**, 1085 (1983),[12] and Joseph B. Keller, *Effective conductivity of periodic composites composed of two very unequal conductors*, J. Math. Phys. **28**, 2516 (1987).[13]

In the steady state, the electric field obeys $\nabla \times \mathbf{E} = 0$, so that $\mathbf{E}$ can be deduced from a scalar potential $\phi$ via $\mathbf{E} = -\nabla\phi$. The steady current density obeys $\nabla \cdot \mathbf{J} = 0$, and is related to the electric field by Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$. Hence, within regions of uniform conductivity, $\nabla \cdot \mathbf{E} = 0$ and $\nabla^2\phi = 0$. Thus, we seek solutions to Laplace's equations in the four regions of uniform conductivity, subject to the stated boundary conditions at the outer radius, as well as the matching conditions that $\phi$, $E_\parallel$, and $j_\perp$ are continuous at the boundaries between the regions.

We analyze this two-dimensional problem in a cylindrical coordinate system $(r, \theta)$ with origin at the corner between the phases and $\theta = 0$ along the radius vector that bisects the region whose potential is unity at $r = a$. The four regions of uniform conductivity are labeled $I$, $II$, $III$ and $IV$ as shown below.
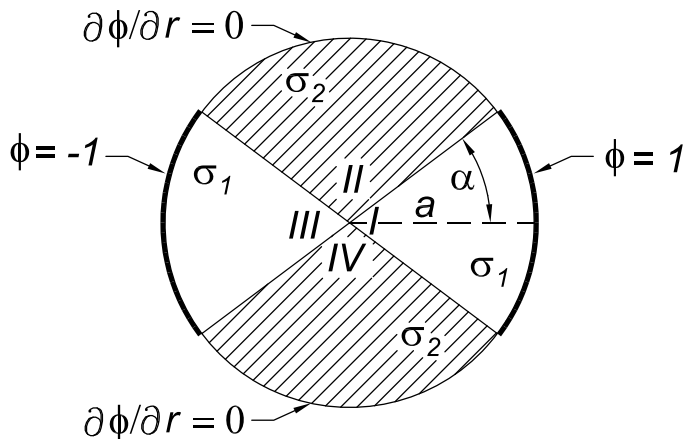


Since $\mathbf{J}_\perp = J_r = \sigma E_r = -\sigma\partial\phi/\partial r$ at the outer boundary, the boundary conditions at $r = a$ can be written

$$\phi_I(r = a) = 1, \tag{125}$$

[10]http://kirkmcd.princeton.edu/examples/EM/craster_prsla_456_2741_00.pdf

[11]http://kirkmcd.princeton.edu/examples/EM/ke-da_prb_41_4330_90.pdf

[12]http://kirkmcd.princeton.edu/examples/EM/soderberg_jpc_16_1085_83.pdf

[13]http://kirkmcd.princeton.edu/examples/EM/keller_jmp_28_2516_87.pdf

$$\frac{\partial \phi_{II}(r=a)}{\partial r} = \frac{\partial \phi_{IV}(r=a)}{\partial r} = 0, \tag{126}$$

$$\phi_{III}(r=a) = -1. \tag{127}$$

Likewise, the condition that $J_\perp = J_\theta = \sigma E_\theta = -(\sigma/r)\partial\phi/\partial\theta$ is continuous at the boundaries between the regions can be written

$$\sigma_1 \frac{\partial \phi_I(\theta = \alpha)}{\partial \theta} = \sigma_2 \frac{\partial \phi_{II}(\theta = \alpha)}{\partial \theta}, \tag{128}$$

$$\sigma_1 \frac{\partial \phi_{III}(\theta = \pi - \alpha)}{\partial \theta} = \sigma_2 \frac{\partial \phi_{II}(\theta = \pi - \alpha)}{\partial \theta}, \tag{129}$$

$$etc.$$

From the symmetry of the problem we see that

$$\phi(-\theta) = \phi(\theta), \tag{130}$$

$$\phi(\pi - \theta) = -\phi(\theta), \tag{131}$$

and in particular $\phi(r = 0) = 0 = \phi(\theta = \pm\pi/2)$.

We recall that two-dimensional solutions to Laplace's equations in cylindrical coordinates involve sums of products of $r^{\pm k}$ and $e^{\pm ik\theta}$, where $k$ is the separation constant that in general can take on a sequence of values. Since the potential is zero at the origin, the radial function is only $r^k$. The symmetry condition (130) suggests that the angular functions for region $I$ be written as $\cos k\theta$, while the symmetry condition (131) suggests that we use $\sin k(\pi/2 - |\theta|)$ in regions $II$ and $IV$ and $\cos k(\pi - \theta)$ in region $III$. That is, we consider the series expansions

$$\phi_I = \sum A_k r^k \cos k\theta, \tag{132}$$

$$\phi_{II} = \phi_{IV} = \sum B_k r^k \sin k\left(\frac{\pi}{2} - |\theta|\right), \tag{133}$$

$$\phi_{III} = -\sum A_k r^k \cos k(\pi - \theta). \tag{134}$$

The potential must be continuous at the boundaries between the regions, which requires

$$A_k \cos k\alpha = B_k \sin k\left(\frac{\pi}{2} - \alpha\right). \tag{135}$$

The normal component of the current density is also continuous across these boundaries, so eq. (128) tells us that

$$\sigma_1 A_k \sin k\alpha = \sigma_2 B_k \cos k\left(\frac{\pi}{2} - \alpha\right). \tag{136}$$

On dividing eq. (136) by eq. (135) we find that

$$\tan k\alpha = \frac{\sigma_2}{\sigma_1} \cot k\left(\frac{\pi}{2} - \alpha\right). \tag{137}$$

There is an infinite set of solutions to this transcendental equation. When $\sigma_2/\sigma_1 \ll 1$ we expect that only the first term in the expansions (132)-(133) will be important, and in this case we expect that both $k\alpha$ and $k(\pi/2 - \alpha)$ are small. Then eq. (137) can be approximated as

$$k\alpha \approx \frac{\sigma_2/\sigma_1}{k(\frac{\pi}{2} - \alpha)}, \tag{138}$$

and hence

$$k^2 \approx \frac{\sigma_2/\sigma_1}{\alpha(\frac{\pi}{2} - \alpha)} \ll 1. \tag{139}$$

Equation (135) also tells us that for small $k\alpha$,

$$A_k \approx B_k k \left(\frac{\pi}{2} - \alpha\right). \tag{140}$$

Since we now approximate $\phi_I$ by the single term $A_k r^k \cos k\theta \approx A_k r^k$, the boundary condition (125) at $r = a$ implies that

$$A_k \approx \frac{1}{a^k}, \tag{141}$$

and eq. (140) then gives

$$B_k \approx \frac{1}{ka^k(\frac{\pi}{2} - \alpha)} \gg A_k. \tag{142}$$

The boundary condition (126) now becomes

$$0 = kB_k a^{k-1} \sin k \left(\frac{\pi}{2} - \theta\right) \approx \frac{k(\frac{\pi}{2} - \theta)}{a(\frac{\pi}{2} - \alpha)}, \tag{143}$$

which is approximately satisfied for small $k$.

So we accept the first terms of eqs. (132)-(134) as our solution, with $k$, $A_k$ and $B_k$ given by eqs. (139), (141) and (142).

In region $I$ the electric field is given by

$$E_r = -\frac{\partial \phi_I}{\partial r} \approx -k\frac{r^{k-1}}{a^k} \cos k\theta \approx -k\frac{r^{k-1}}{a^k}, \tag{144}$$

$$E_\theta = -\frac{1}{r}\frac{\partial \phi_I}{\partial \theta} \approx k\frac{r^{k-1}}{a^k} \sin k\theta \approx k^2\theta\frac{r^{k-1}}{a^k}. \tag{145}$$

Thus, in region $I$, $E_\theta/E_r \approx k\theta \ll 1$, so the electric field, and the current density, is nearly radial. In region $II$ the electric field is given by

$$E_r = -\frac{\partial \phi_{II}}{\partial r} \approx -k\frac{r^{k-1}}{ka^k(\frac{\pi}{2} - \alpha)} \sin k \left(\frac{\pi}{2} - \theta\right) \approx -k\frac{r^{k-1}}{a^k}\frac{\frac{\pi}{2} - \theta}{\frac{\pi}{2} - \alpha}, \tag{146}$$

$$E_\theta = -\frac{1}{r}\frac{\partial \phi_{II}}{\partial \theta} \approx k\frac{r^{k-1}}{ka^k(\frac{\pi}{2} - \alpha)} \cos k \left(\frac{\pi}{2} - \theta\right) \approx \frac{r^{k-1}}{a^k(\frac{\pi}{2} - \alpha)}. \tag{147}$$

Thus, in region $II$, $E_r/E_\theta \approx k(\pi/2 - \theta) \ll 1$, so the electric field, and the current density, is almost purely azimuthal.

The current density $\mathbf{J}$ follows the lines of the electric field $\mathbf{E}$, and therefore behaves as sketched below:
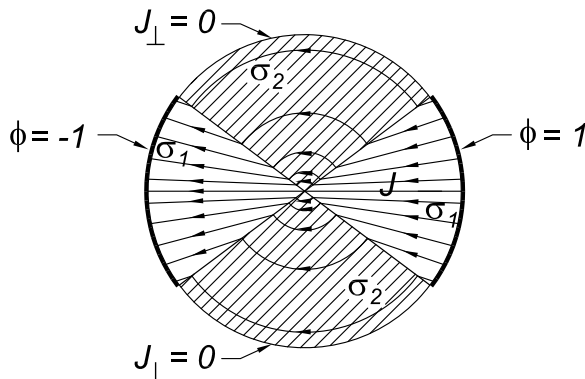


The total current can be evaluated by integrating the current density at $r = a$ in region $I$:

$$I = 2a \int_0^\alpha J_r d\theta = 2a\sigma_1 \int_0^\alpha E_r(r = a)d\theta \approx -2k\sigma_1 \int_0^\alpha d\theta = -2k\sigma_1\alpha = -2\sqrt{\frac{\sigma_1\sigma_2\alpha}{\frac{\pi}{2} - \alpha}}.$$

(148)

In the present problem the total potential difference $\Delta\phi$ is -2, so the effective conductivity is

$$\sigma = \frac{I}{\Delta\phi} = \sqrt{\frac{\sigma_1\sigma_2\alpha}{\frac{\pi}{2} - \alpha}}.$$

(149)

For a square checkerboard, $\alpha = \pi/4$, and the effective conductivity is $\sigma = \sqrt{\sigma_1\sigma_2}$. It turns out that this result is independent of the ratio $\sigma_2/\sigma_1$, and holds not only for the corner region studied here but for the entire checkerboard array; see Joseph B. Keller, *A Theorem on the Conductivity of a Composite Medium*, J. Math. Phys. **5**, 548 (1964).[14]

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[14]http://kirkmcd.princeton.edu/examples/EM/keller_jmp_5_548_64.pdf

7. Taking the wire that carries current $I$ to be along the $z$ axis, the magnetic field at distance $r$ from the wire is

$$\mathbf{B} = \frac{2I}{cr}\hat{\boldsymbol{\theta}}. \tag{150}$$

The force on an element $d\mathbf{l}$ of the loop that carries current $I'$ is

$$d\mathbf{F} = \frac{I'}{c}d\mathbf{l} \times \mathbf{B}. \tag{151}$$

The torque about the axis of the loop due to that force element is

$$d\mathbf{N} = s\hat{\mathbf{n}} \times d\mathbf{F} , \tag{152}$$

where $s$ is the distance from the axis to the element and $\hat{\mathbf{n}}$ is the unit vector from the axis to the element. Combining (150-152), the total torque on the loop is

$$\mathbf{N} = \frac{2II'}{c} \oint \frac{s\hat{\mathbf{n}} \times (d\mathbf{l} \times \hat{\boldsymbol{\theta}})}{r} . \tag{153}$$

The two sides of length $2b$ will have equal and opposite contributions which therefore cancel, leaving only the contributions from the two sides of length $2a$. For the side nearest the wire, $s = b$, and $d\mathbf{l} = \hat{\mathbf{z}}dl$, since the currents flow in the same direction, so that $d\mathbf{l} \times \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}}_{\text{near}}dl$ (parallel currents attract),

$$s\hat{\mathbf{n}} \times (d\mathbf{l} \times \hat{\boldsymbol{\theta}}) = -b\hat{\mathbf{n}} \times \hat{\mathbf{r}}_{\text{near}}dl = -b\sin\alpha\hat{\mathbf{z}}dl = -\frac{bd\sin\theta}{r_{\text{near}}}\hat{\mathbf{z}}dl, \tag{154}$$

using the law of sines for the triangle shown.
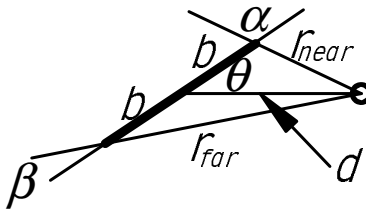


The distance $r$ from the wire to this side of the loop is

$$r_{\text{near}} = \sqrt{b^2 + d^2 - 2bd\cos\theta}, \tag{155}$$

so that the contribution of this side of the loop to the torque is

$$\mathbf{N}_{\text{near}} = -\frac{4abdII'}{c}\frac{\sin\theta}{b^2 + d^2 - 2bd\cos\theta}\hat{\mathbf{z}}. \tag{156}$$

For the side furthest the wire, again $s = b$, but $d\mathbf{l} = -\hat{\mathbf{z}}dl$, so $d\mathbf{l} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}_{\text{far}}dl$. Then,

$$s\hat{\mathbf{n}} \times (d\mathbf{l} \times \hat{\boldsymbol{\theta}}) = b\hat{\mathbf{n}} \times \hat{\mathbf{r}}_{\text{far}}dl = -b\sin\beta\hat{\mathbf{z}}dl = -\frac{bd\sin\theta}{r_{\text{far}}}\hat{\mathbf{z}}dl. \tag{157}$$

and

$$r_{\text{far}} = \sqrt{b^2 + d^2 + 2bd\cos\theta},\tag{158}$$

so that the contribution of the far side of the loop to the torque is

$$\mathbf{N}_{\text{far}} = -\frac{4abdII'}{c}\frac{\sin\theta}{b^2 + d^2 + 2bd\cos\theta}\hat{\mathbf{z}}.\tag{159}$$

The total torque on the loop is then,

$$\mathbf{N} = \mathbf{N}_{\text{near}} + \mathbf{N}_{\text{far}} = -\frac{8abdII'}{c}\frac{(b^2 + d^2)\sin\theta}{b^4 + d^4 - 2b^2d^2\cos 2\theta}\hat{\mathbf{z}},\tag{160}$$

where the identity $2\cos^2\theta = \cos 2\theta + 1$ has been used. This torque is down the axis, that is, it acts to decrease $\theta$ and bring the loop into the plane of the wire and the axis.

8. a) The magnetic field on the axis of the loop centered at $z = b$ is obtained from the Biot-Savart law:

$$\mathbf{B}_1 = \frac{I}{c} \oint \frac{d\mathbf{l} \times \mathbf{r}}{r^3} = \frac{2\pi I a^2}{c(a^2 + (z-b)^2)^{\frac{3}{2}}} \hat{\mathbf{z}}, \qquad (161)$$

and that due to the other loop is

$$\mathbf{B}_2 = \frac{2\pi I a^2}{c(a^2 + (z+b)^2)^{\frac{3}{2}}} \hat{\mathbf{z}}. \qquad (162)$$

The total field is then the sum of the (161) and (162). This is unchanged under $z \to -z$, so all odd derivatives with respect to $z$ automatically vanish at the origin. We can choose the separation $b$ to cancel any desired even derivative at the origin.

We first accumulate a catalog of derivatives (some of which are needed in prob. 6):

$$\frac{cB_z(z)}{2\pi I a^2} = \frac{1}{(a^2 + (z-b)^2)^{\frac{3}{2}}} + \frac{1}{(a^2 + (z+b)^2)^{\frac{3}{2}}}, \qquad (163)$$

$$\frac{cB_z'(z)}{2\pi I a^2} = -\frac{3(z-b)}{(a^2 + (z-b)^2)^{\frac{5}{2}}} - \frac{3(z+b)}{(a^2 + (z+b)^2)^{\frac{5}{2}}}, \qquad (164)$$

$$\frac{cB_z''(z)}{2\pi I a^2} = \frac{12(z-b)^2 - 3a^2}{(a^2 + (z-b)^2)^{\frac{7}{2}}} + \frac{12(z+b)^2 - 3a^2}{(a^2 + (z+b)^2)^{\frac{7}{2}}}. \qquad (165)$$

$$\frac{cB_z'''(z)}{2\pi I a^2} = -\frac{90(z-b)^3 - 15a^2(z-b)}{(a^2 + (z-b)^2)^{\frac{9}{2}}} - \frac{90(z+b)^3 - 15a^2(z+b)}{(a^2 + (z+b)^2)^{\frac{9}{2}}}, \qquad (166)$$

$$\frac{cB_z''''(z)}{2\pi I a^2} = \frac{540(z-b)^4 - 450a^2(z-b)^2 + 15a^4}{(a^2 + (z-b)^2)^{\frac{11}{2}}}$$
$$+ \frac{540(z+b)^4 - 450a^2(z+b)^2 + 15a^4}{(a^2 + (z+b)^2)^{\frac{11}{2}}}, \qquad (167)$$

The second derivative of $B_z$ with respect to $z$ at the origin is proportional to

$$\frac{4b^2 - a^2}{(a^2 + b^2)^{\frac{7}{2}}}, \qquad (168)$$

which vanishes when $a = 2b$. That is, the separation of a pair of Helmholtz coil is equal to their radius.

b) In a Helmholtz coil pair, the field at the origin is proportional to $I/a$ according to (163), so the 4rth derivative at the origin is proportional to $I/a^5$. If we add a second Helmholtz pair with current $I'$ and radius $a' = 2/a$, the combined 4th derivative at the origin is proportional to $I/a^5 + 32I'/a^5$. Hence, the current $I' = -I/32$ will cancel the 4rth derivative at the origin. The field at the origin is then proportional to $I/a - (I/32)(2/a) = 15I/16$. That is, the central field has been reduced by 1/16.

c) Far outside the coils, the leading behavior of the magnetic field is due to the dipole moment, $2\pi I a^2/c$, of the coils. If a second Helmholtz pair is placed at $a' = 2a$ to cancel the dipole moment of the first pair, we need $I' = -I/4$. Since the central field of a

Helmholtz pair varies as $I/a$, the combined central field will be $I/a - (I/4)(1/2a) = 7I/8a$, *i.e.*, $1/8$ of the central field is lost to insure that the field far outside the coils is extremely weak.

To characterize the field well outside a set of Helmholtz coils in more detail, we use the variable $u = 1/z$, and expand about $u = 0$. From (163), and using $a = 2b$,

$$\frac{cB_z^{(0)}(u)}{2\pi I a^2} = \frac{u^3}{(1 - au + 5a^2u^2/4)^{\frac{3}{2}}} + \frac{u^3}{(1 + au + 5a^2u^2/4)^{\frac{3}{2}}}. \tag{169}$$

Using the Taylor expansion,

$$\frac{1}{(1 + \epsilon)^{\frac{3}{2}}} = 1 - \frac{3}{2}\epsilon + \frac{15}{8}\epsilon^2 - \frac{105}{48}\epsilon^3 + ...., \tag{170}$$

we find that

$$B_z^{(0)}(u) = \frac{4\pi I a^2}{c}\left(u^3 + \frac{75}{16}a^3u^6 + ...\right). \tag{171}$$

The leading term, $u^3 = 1/z^3$, is due to the dipole moment of the pair, which, of course, is proportional to $Ia^2$. The next nonvanishing term, $u^6$, is due to the hexadecupole moment, which is proportional to $Ia^5$. The brevity of this derivation hides that fact that the Helmholtz condition, $a = 2b$, served to cancel the octupole moment. The quadrupole moment vanishes due to the symmetry of the coil pair.

9. Since the divergence of the magnetic field vanishes, the proposed expansions (15) obey

$$\mathbf{\nabla} \cdot \mathbf{B} = \frac{1}{r}\frac{\partial B_r}{\partial r} + \frac{\partial B_z}{\partial z} = \sum_n \left[ (n+1)b_n r^{n-1} + a_n^{(1)} r^n \right] = 0, \tag{172}$$

where $a^{(m)}(z) \equiv d^m a / dz^m$. For this to be true at all $r$, the coefficients of $r^n$ must separately vanish for all $n$. Hence,

$$b_0 = 0, \tag{173}$$

$$b_n = -\frac{a_{n-1}^{(1)}}{n+1}. \tag{174}$$

Since the curl of the field vanishes,

$$(\mathbf{\nabla} \times \mathbf{B})_\theta = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \sum_n \left( b_n^{(1)} r^n - n a_n r^{n-1} \right) = 0 , \tag{175}$$

Again, the coefficient of $r^n$ must vanish for all $n$, so that

$$b_n^{(1)} = (n+1)a_{n+1}. \tag{176}$$

Using (176) in (174), we find

$$b_n = -\frac{b_{n-2}^{(2)}}{(n+1)(n+3)}. \tag{177}$$

Since $b_0$ vanishes, $b_{2n}$ vanishes for all $n$, and from (176), $a_{2n+1}$ vanishes for all $n$. Then, using (177) in (176), we find

$$a_{2n} = -\frac{a_{2n-2}^{(2)}}{4n^2}. \tag{178}$$

Repeatedly applying this to itself gives

$$a_{2n} = (-1)^n \frac{a_0^{(2n)}}{2^{2n}(n!)^2}. \tag{179}$$

Inserting this in (174), we get

$$b_{2n+1} = (-1)^{n+1} \frac{a_0^{(2n+1)}}{2^{2n+1}(n+1)(n!)^2}. \tag{180}$$

Combining (179-180) with (15), we arrive at the desired forms (16-17) for the fields.

The axial field of a pair of Helmholtz coils has the form

$$a_0(z) = B_0 + B_4 z^4 + ... \tag{181}$$

The first four derivatives are

$$a_0^{(1)} = 4B_4 z^3, \qquad a_0^{(2)} = 12B_4 z^2, \qquad a_0^{(3)} = 24B_4 z, \qquad a_0^{(4)} = 24B_4. \tag{182}$$

From (179-180), the other non-vanishing functions through fourth order are

$$a_2 = -3B_4 z^2, \qquad a_4 = \frac{3B_4}{8}, \qquad b_1 = -2B_4 z^3, \qquad b_3 = \frac{3B_4}{2} z \, . \qquad (183)$$

The fields, correct to fourth order, are

$$B_z = B_0 + B_4 z^4 - 3B_4 r^2 z^2 + \frac{3B_4}{8} r^4 + \dots \, , \qquad (184)$$

$$B_r = -2B_4 r z^3 + \frac{3B_4}{2} r^3 z + \dots \qquad (185)$$

The constants $B_0$ and $B_4$ are obtained from the catalog of derivatives in prob. 5, using

$$B_z(0, z) = B_0 + B_4 z^4 = B_z(0, z) + \frac{B_z''''(0, z) z^4}{4!} + \dots \qquad (186)$$

From (163),

$$B_0 = \frac{4\pi I a^2}{c(a^2 + b^2)^{\frac{3}{2}}} = \frac{32\sqrt{5}\pi I}{25 ca} \, , \qquad (187)$$

and from (167),

$$B_4 = \frac{5\pi I a^2 (a^4 - 30 a^2 b^2 + 36 b^4)}{2c(a^2 + b^2)^{\frac{11}{2}}} = -\frac{4864\sqrt{5}\pi I}{3125 ca^5} = -\frac{152}{125 a^4} B_0 \, , \qquad (188)$$

using $b = a/2$. Since $B_4 < 0$, the axial field $B_z$ decreases as we move away from the origin, as is to be expected.

These results are overly detailed for some purposes. If one is interested only in the leading behavior at small $r$, then (184-185) simplify to

$$B_z(r, z) \approx B_z(0, z), \qquad B_r(r, z) \approx -\frac{r}{2} \frac{\partial B_z(0, z)}{\partial z} \, . \qquad (189)$$

The result for $B_r$ also follows quickly from $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, according to eq. (172),

$$B_r(r, z) = -\int_0^r r \frac{\partial B_z(r, z)}{\partial z} dr \approx -\int_0^r r \frac{\partial B_z(0, z)}{\partial z} dr = -\frac{r}{2} \frac{\partial B_z(0, z)}{\partial z} \, . \qquad (190)$$

It is also instructive that the approximation (190) can be deduced quickly from the integral form of Gauss' law (without the need to recall the form of $\boldsymbol{\nabla} \cdot \mathbf{B}$ in cylindrical coordinates). Consider a Gaussian pillbox of radius $r$ and thickness $dz$ centered on $(r = 0, z)$. Then,

$$\begin{aligned} 0 &= \int \mathbf{B} \cdot d\mathbf{S} \approx \pi r^2 [B_z(0, z + dz) - B_z(0, z)] + 2\pi r \, dz \, B_r(r, z) \\ &\approx \pi r^2 \, dz \, \frac{\partial B_z(0, z)}{\partial z} + 2\pi r \, dz \, B_r(r, z) \, , \end{aligned} \qquad (191)$$

which again implies eqs. (189).

10. We first seek a solution in rectangular coordinates, and expect that separation of variables will apply. Thus, we consider the form

$$B_x = f(x)g(y)\cos kz, \tag{192}$$
$$B_x = F(x)G(y)\sin kz, \tag{193}$$
$$B_z = A(x)B(y)C(z). \tag{194}$$

Then
$$\nabla \cdot \mathbf{B} = 0 = f'g\cos kz + FG'\sin kz + ABC', \tag{195}$$

where the $'$ indicates differentiation of a function with respect to its argument. Equation (195) can be integrated to give

$$ABC = -\frac{f'g}{k}\sin kz + \frac{FG'}{k}\cos kz. \tag{196}$$

The $z$ component of $\nabla \times \mathbf{B} = 0$ tells us that

$$\frac{\partial B_x}{\partial y} = fg'\cos kz = \frac{\partial B_y}{\partial x} = F'G\sin kz, \tag{197}$$

which implies that $g$ and $F$ are constant, say 1. Likewise,

$$\frac{\partial B_x}{\partial z} = -fk\sin kz = \frac{\partial B_z}{\partial x} = A'BC = -\frac{f''}{k}\sin kz, \tag{198}$$

using (196-197). Thus, $f'' - k^2 f = 0$, so

$$f = f_1 e^{kx} + f_2 e^{-kx}. \tag{199}$$

Finally,
$$\frac{\partial B_y}{\partial z} = Gk\cos kz = \frac{\partial B_z}{\partial y} = AB'C = \frac{G''}{k}\sin kz, \tag{200}$$

so
$$G = G_1 e^{ky} + G_2 e^{-ky}. \tag{201}$$

The "boundary conditions" $f(0) = B_0 = G(0)$ are satisfied by

$$f = B_0 \cosh kx, \qquad G = B_0 \cosh ky, \tag{202}$$

which together with (196) leads to the solution

$$B_x = B_0 \cosh kx \cos kz, \tag{203}$$
$$B_y = B_0 \cosh ky \sin kz, \tag{204}$$
$$B_z = -B_0 \sinh kx \sin kz + B_0 \sinh ky \cos kz, \tag{205}$$

This satisfies the last "boundary condition" that $B_z(0, 0, z) = 0$.

However, this solution does not have helical symmetry.

Suppose instead, we look for a solution in cylindrical coordinates $(r, \theta, z)$. We again expect separation of variables, but we seek to enforce the helical symmetry that the field at $z + \delta$ be the same as the field at $z$, but rotated by angle $k\delta$. This symmetry implies that the argument $kz$ should be replaced by $kz - \theta$, and that the field has no other $\theta$ dependence.

We begin constructing our solution with the hypothesis that

$$B_r = F(r)\cos(kz - \theta), \tag{206}$$

$$B_\theta = G(r)\sin(kz - \theta). \tag{207}$$

To satisfy the condition (24) on the $z$ axis, we first transform this to rectangular components,

$$B_z = F(r)\cos(kz - \theta)\cos\theta + G(r)\sin(kz - \theta)\sin\theta, \tag{208}$$

$$B_y = -F(r)\cos(kz - \theta)\sin\theta + G(r)\sin(kz - \theta)\cos\theta, \tag{209}$$

from which we learn that the "boundary conditions" on $F$ and $G$ are

$$F(0) = G(0) = B_0. \tag{210}$$

A suitable form for $B_z$ can be obtained from $(\nabla \times \mathbf{B})_r = 0$:

$$\frac{1}{r}\frac{\partial B_z}{\partial \theta} = \frac{\partial B_\theta}{\partial z} = kG\cos(kz - \theta), \tag{211}$$

so

$$B_z = -krG\sin(kz - \theta), \tag{212}$$

which vanishes on the $z$ axis as desired.

From either $(\nabla \times \mathbf{B})_\theta = 0$ or $(\nabla \times \mathbf{B})_z = 0$ we find that

$$F = \frac{d(rG)}{dr}. \tag{213}$$

Then, $\nabla \cdot \mathbf{B} = 0$ leads to

$$(kr)^2\frac{d^2(krG)}{d(kr)^2} + kr\frac{d(krG)}{d(kr)} - [1 + (kr)^2](krG) = 0. \tag{214}$$

This is the differential equation for the modified Bessel function of order 1. See, for example, M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, D.C., 1964), sec. 9.6. Hence,

$$G = C\frac{I_1(kr)}{kr} = \frac{C}{2}\left[1 + \frac{(kr)^2}{8} + \cdots\right], \tag{215}$$

$$F = C\frac{dI_1}{d(kr)} = C\left(I_0 - \frac{I_1}{kr}\right) = \frac{C}{2}\left[1 + \frac{3(kr)^2}{8} + \cdots\right]. \tag{216}$$

The "boundary conditions" (210) require that $C = 2B_0$, so our second solution is

$$B_r = 2B_0 \left( I_0(kr) - \frac{I_1(kr)}{kr} \right) \cos(kz - \theta), \tag{217}$$

$$B_\theta = 2B_0 \frac{I_1}{kr} \sin(kz - \theta), \tag{218}$$

$$B_z = -2B_0 I_1 \sin(kz - \theta), \tag{219}$$

which is the form discussed by Blewett and Chasman.

For a realization of the axial field pattern (24), we consider a wire that carries current I and is wound in the form of a helix of radius $a$ and period $\lambda = 2\pi/k$. A suitable equation of this helix is

$$x_1 = a \sin kz, \qquad y_1 = -a \cos kz. \tag{220}$$

The magnetic field due to this winding has a nonzero $z$ component along the axis, which is not desired. Therefore, we also consider a second helical winding,

$$x_2 = -a \sin kz, \qquad y_2 = a \cos kz, \tag{221}$$

which is offset from the first by half a period and which carries current $-I$. The combined magnetic field from the two helices has no component along their common axis.

The unit vector $\hat{\mathbf{l}}_{1,2}$ that is tangent to helix 1(2) at a point

$$\mathbf{r}'_{1,2} = (x'_{1,2}, y'_{1,2}, z') = (\pm a \sin kz', \mp a \cos kz', z') \tag{222}$$

has components

$$\hat{\mathbf{l}}_{1,2} = \frac{(\pm 2\pi a \cos kz', \pm 2\pi a \sin kz', \lambda)}{\sqrt{\lambda^2 + (2\pi a)^2}}, \tag{223}$$

and the element $d\mathbf{l}'_{1,2}$ of arc length along the helix is related by

$$d\mathbf{l}'_{1,2} = \hat{\mathbf{l}}'_{1,2} dz' \frac{\sqrt{\lambda^2 + (2\pi a)^2}}{\lambda} = dz'(\pm ka \cos kz', \pm ka \sin kz', 1). \tag{224}$$

The magnetic field $\mathbf{B}$ at a point $\mathbf{r} = (0, 0, z)$ on the axis is given by

$$
\begin{aligned}
\mathbf{B}(0,0,z) &= \frac{I}{c} \int_1 \frac{d\mathbf{l}'_1 \times (\mathbf{r}'_1 - \mathbf{r})}{|\mathbf{r}'_1 - \mathbf{r}|^3} - \frac{I}{c} \int_2 \frac{d\mathbf{l}'_2 \times (\mathbf{r}'_2 - \mathbf{r})}{|\mathbf{r}'_2 - \mathbf{r}|^3} \\
&= \frac{2Ia}{c} \int_{-\infty}^{\infty} \frac{dz'}{[a^2 + (z' - z)^2]^{3/2}} \left[ \hat{\mathbf{x}}(k(z' - z) \sin kz' + \cos kz') \right. \\
&\qquad\qquad\qquad\qquad \left. + \hat{\mathbf{y}}(-k(z' - z) \cos kz' + \sin kz') \right] \\
&= \frac{2I}{ca} \int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)^{3/2}} \left[ \hat{\mathbf{x}}(kat \sin(kat + kz) + \cos(kat + kz)) \right. \\
&\qquad\qquad\qquad\qquad \left. + \hat{\mathbf{y}}(-kat \cos(kat + kz) + \sin(kat + kz)) \right] \tag{225} \\
&= \frac{4Ik}{c} (\hat{\mathbf{x}} \cos kz + \hat{\mathbf{y}} \sin kz) \left[ \frac{1}{ka} \int_0^{\infty} \frac{\cos kat}{(1 + t^2)^{3/2}} dt + \int_0^{\infty} \frac{t \sin kat}{(1 + t^2)^{3/2}} dt \right],
\end{aligned}
$$

where we made the substitution $z' - z = at$ in going from the second line to the third. Equation 9.6.25 of Abramowitz and Stegun tells us that

$$\int_0^\infty \frac{\cos kat}{(1+t^2)^{3/2}} dt = kaK_1(ka), \tag{226}$$

where $K_1$ also satisfies eq. (214). We integrate the last integral by parts, using

$$u = \sin kat, \quad dv = \frac{t\, dt}{(1+t^2)^{3/2}}, \quad \text{so} \quad du = ka\cos kat\, dt, \quad v = -\frac{1}{\sqrt{1+t^2}}. \tag{227}$$

Thus,

$$\int_0^\infty \frac{t\sin kat}{(1+t^2)^{3/2}} dt = ka \int_0^\infty \frac{\cos kat}{\sqrt{1+t^2}} dt = kaK_0(ka), \tag{228}$$

using 9.6.21 of Abramowitz and Stegun. Hence

$$\mathbf{B}(0,0,z) = \frac{4Ik}{c} \left[ kaK_0(ka) + K_1(ka) \right] (\hat{\mathbf{x}}\cos kz + \hat{\mathbf{y}}\sin kz). \tag{229}$$

Both $K_0(ka)$ and $K_1(ka)$ have magnitudes $\approx 0.5e^{-ka}$ for $ka \approx 1$. That is, the field on the axis of the double helix is exponentially damped in the radius $a$ for a fixed current $I$.

11. We analyze the magnetic field of the solenoid in cylindrical coordinates, $(r, \theta, z)$, with the origin at the center of the solenoid and $z$ axis along that of the solenoid.

First, the field at a point on the axis, $(0, 0, z)$, to a current loop, with current $dI$, centered on and perpendicular to the $z$-axis at $z'$ follows from the Biot-Savart law as

$$\mathbf{B}(0, 0, z) = \frac{dI}{c} \oint \frac{d\mathbf{l} \times \mathbf{r}}{r^3} = \frac{2\pi dI a^2}{c(a^2 + (z - b)^2)^{\frac{3}{2}}} \hat{\mathbf{z}}, \tag{230}$$

For the solenoid,

$$dI = nI \, dz' \ ,$$

where $z'$ runs from $-l/2$ to $l/2$, so the total field on the axis is

$$\begin{aligned} \mathbf{B}(0, 0, z) &= \frac{2\pi nI a^2 \hat{\mathbf{z}}}{c} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{dz'}{(a^2 + (z - z')^2)^{\frac{3}{2}}} \\ &= \frac{2\pi nI a^2 \hat{\mathbf{z}}}{c} \int_{-\frac{l}{2}-z}^{\frac{l}{2}-z} \frac{dz'}{(a^2 + z'^2)^{\frac{3}{2}}} \\ &= \frac{2\pi nI \hat{\mathbf{z}}}{c} \left( \frac{\frac{l}{2} - z}{\sqrt{a^2 + (\frac{l}{2} - z)^2}} + \frac{\frac{l}{2} + z}{\sqrt{a^2 + (\frac{l}{2} + z)^2}} \right) \\ &= \frac{2\pi nI}{c} (\cos\theta_1 + \cos\theta_2) \hat{\mathbf{z}} \ , \end{aligned} \tag{231}$$

where $\theta_1$ is the angle between the $z$ axis and the line joining the observation point, $(0, 0, z)$ to the point $(a, 0, l/2)$ on the end of the solenoid, *etc.*

For $z \ll a \ll l$, we use the next to last line of (231), and the Taylor expansion

$$\frac{1}{\sqrt{1 + \epsilon}} = 1 - \frac{\epsilon}{2} + \frac{3\epsilon^2}{8} - \frac{5\epsilon^3}{16} + ..., \tag{232}$$

to find that near the origin,

$$B_z(0, 0, z) \approx \frac{2\pi nI}{c} \left( 2 - \frac{4a^2}{l^2} - \frac{72a^2 z^2}{l^4} \right) \ . \tag{233}$$

As noted as the end of prob. 6, the radial field near the axis can be obtained from the axial field using (189). Hence,

$$B_r(r, 0, z) \approx \frac{288\pi nI a^2 rz}{cl^4} \ . \tag{234}$$

Near the end of the solenoid at $z = l/2$,

$$\cos\theta_1 = -\sin(\theta_1 - \pi/2) \approx -\frac{z - l/2}{a} \ , \qquad \text{and} \qquad \cos\theta_2 \approx \frac{l}{\sqrt{l^2 + a^2}} \approx 1 - \frac{a^2}{2l^2} \ . \tag{235}$$

Then, from (231)

$$B_z(0,0,z) \approx \frac{2\pi nI}{c} \left( 1 - \frac{a^2}{2l^2} - \frac{z - l/2}{a} \right) . \tag{236}$$

Comparing with (233), we see that the axial field at the end of the solenoid is approximately $1/2$ that at the center. The radial field at the end of the solenoid follows from (189) as

$$B_r \approx \frac{\pi nIr}{ac} . \tag{237}$$

12. a) As discussed on p. 98 of the Notes,[15] a sphere of uniform magnetization density
    **M** has a uniform magnetic field $\mathbf{B} = 8\pi\mathbf{M}/3$ inside. As discussed on p. 93, the **B**
    field associated with magnetization density **M** can be thought of as arising from a
    magnetization current density, $\mathbf{J}_m = c\nabla \times \mathbf{M}$, and a surface current density, $\mathbf{K}_m = c\mathbf{M} \times \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the outward normal from the bounding surface. For uniform
    magnetization, $\mathbf{J}_m = 0$, while, if $\mathbf{M} = M\,\hat{\mathbf{z}}$, then

$$\mathbf{K}_m = cM \sin\theta\,\hat{\boldsymbol{\phi}}. \quad = \frac{3cB \sin\theta\,\hat{\boldsymbol{\phi}}}{8\pi} \tag{238}$$

Since this is to be produced by windings on the surface of the sphere, with the same
current flowing through each turn of the winding, the density of windings must be
proportional to $\sin\theta$ (or to $\cos\theta$ for $\theta$ measured with respect to the "equatorial" plane
(colatitude).

On the other hand, if the surface current is due to a surface charge density $\sigma(\theta)$ that
rotates as a rigid body with angular velocity $\omega$, then the surface current density is
$\omega a \sin\theta\sigma(\theta)$, which varies as $\sin\theta$ for uniform $\sigma$. That is, a rotating, uniform spherical
shell of charge produces the desired uniform magnetic field inside the shell.

b) If the sphere has radius $a$ and surface charge density $\sigma$ and rotates with angular
velocity $\omega$, its magnetic moment will be

$$\begin{aligned}
\mu &= \frac{1}{c}\int \text{Area}\,dI = \frac{1}{c}\int \text{Area}\frac{dQ}{T} = \frac{1}{c}\int_0^\pi \pi a^2 \sin^2\theta \cdot \frac{\sigma 2\pi a \sin\theta a\,d\theta}{2\pi/\omega}\\
&= \frac{\pi\sigma\omega a^4}{c}\int_0^\pi \sin^3\theta\,d\theta = \frac{4\pi\sigma\omega a^4}{3c} = \frac{Q\omega a^2}{3c} = \frac{Qav_{\text{eq}}}{3c},
\end{aligned} \tag{239}$$

where $Q = 4\pi\sigma a^2$ is the total charge on the sphere, and $v_{\text{eq}} = a\omega$ is the equatorial
velocity of the rotating shell.

*In case of a solid sphere with uniform chaged density, $\mu = Q\omega a^2/5c = Qav_{\text{eq}}/5c$.*

*These results were first given by W. Voigt, ELektronenhypothese und Theorie des
Magnetismus, Ann. d. Phys. **9**, 115 (1902),*
http://kirkmcd.princeton.edu/examples/EM/voigt_ap_9_115_02.pdf
*See his eqs. 9-10, where $\mu$ is the magnetic moment, $\mathsf{M} = 2ma^2/3$ is the moment of
inertia for a spherical shell, $r = \omega$, but for Voigt, $\omega = c$.*

For the classical model of the electron with magnetic moment $\mu = e\hbar/2mc$, the velocity
at the equator is

$$v_{\text{eq}} = \omega a = \frac{3\mu c}{Qa} = 3\frac{e\hbar}{2mc}c\frac{1}{e}\frac{mc^2}{e^2} = \frac{3}{2}\frac{\hbar c}{e^2}c = \frac{3}{2\alpha}c. \tag{240}$$

Since the fine structure constant $\alpha = e\hbar/c^2$, is $\approx 1/137$, this velocity is more than two
hundred times the speed of light!

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[15]http://kirkmcd.princeton.edu/examples/ph501/ph501lecture8.pdf

Even if we suppose the radius of the electron is the (reduced Compton wavelength $\lambda = \hbar/mc$, the velocity at the equator is $3c/2$ for a spherical shell (and $5c/2$ for a solid sphere).

The magnetic field of the rotating shell of charge, with magnetic moment $\boldsymbol{\mu} = \mu\,\hat{\mathbf{z}}$ is

$$\mathbf{B} = \begin{cases} \frac{2\boldsymbol{\mu}}{a^3} = \frac{2\mu\,\hat{\mathbf{z}}}{a^3 c} & (r < a) \\ \frac{3(\boldsymbol{\mu}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \boldsymbol{\mu}}{r^3} = \frac{2\mu\cos\theta\,\hat{\mathbf{r}} - \mu\sin\theta\,\hat{\boldsymbol{\theta}}}{cr^3} & (r > a). \end{cases} \tag{241}$$

The magnetic field energy is

$$\begin{aligned} U_{\mathrm{M}} &= \int \frac{B^2}{8\pi}\,d\mathrm{Vol} = \frac{1}{8\pi}\frac{4\mu^2}{a^6 c^2}4\pi a^3 + \frac{\mu^3}{4}\int_{-1}^{1} d\cos\theta \int_{a}^{\infty} r^2\,dr\,\frac{4\cos^2\theta + \sin^2\theta}{r^6 c^2} \\ &= \frac{2\mu^2}{a^3 c^2} + \frac{\mu^3}{3a^3 c^2} = \frac{8Q^2 v_{\mathrm{eq}}^2}{3ac^2} = \frac{8v_{\mathrm{eq}}^2}{3c^2}\,U_{\mathrm{E}}, \end{aligned} \tag{242}$$

where the electrostatic field energy is $U_{\mathrm{E}} = Q^2/a$. Perhaps surprisingly, the magnetic field energy is larger than the electric field energy as $v_{\mathrm{eq}} \to c$, as is needed for a classical model of an electron to reproduce its observed magnetic moment $e\hbar/2mc$.

Indeed, the evidence from scattering experiments that an electron is "pointlike" (meaning not larger than its Compton wavelength) requires a classical model of the electron magnetic moment to involve an equatorial velocity faster than light, such that classical models of the electron are unsatisfactory.

13. a) If $n$ is the number density of alignable electrons, each of magnetic moment $\mu = e\hbar/2mc$, then the resulting bulk magnetic field strength is

$$B = 4\pi M = 4\pi n\mu = \frac{2\pi ne\hbar}{mc} = 2\pi n\frac{e\hbar}{m^2c^3}mc^2 = \frac{2\pi nmc^2}{B_{\text{crit}}}, \qquad (243)$$

where $B_{\text{crit}} = m^2c^3/e\hbar = 4.4 \times 10^{13}$ Gauss is the so-called QED critical field strength. Since $mc^2 = 0.511$ MeV $= 8.2 \times 10^{-7}$ erg we have

$$n = \frac{2 \times 10^4 \cdot 4.4 \times 10^{13}}{2\pi \cdot 8.2 \times 10^{-7}} = 1.7 \times 10^{23}/\text{cm}^3. \qquad (244)$$

Iron has atomic weight $A = 56$ and mass density 8 g/cm$^3$, so its number density is

$$n_{\text{atom}} = \frac{6 \times 10^{23} \cdot 8}{56} = 8.6 \times 10^{22}/\text{cm}^2. \qquad (245)$$

Thus, two electrons per iron atom participate in its bulk magnetization.

b) The interaction energy $U$ of two magnetic dipoles $\mathbf{m}_1$ and $\mathbf{m}_2$ of equal magnitude $m$ separated by distance $\mathbf{r}_2 - \mathbf{r}_1 = d\hat{\mathbf{z}}$ can be calculated by supposing that, say, dipole 2 is held fixed while dipole 1 is brought into place from a large distance. The force on dipole 1 due to dipole 2 is then

$$\mathbf{F}_1 = \boldsymbol{\nabla}_1(\mathbf{m}_1 \cdot \mathbf{B}_2), \qquad (246)$$

which can be integrated to give the interaction energy

$$U = -\int \mathbf{F}_1 \cdot d\mathbf{r}_1 = -\mathbf{m}_1 \cdot \mathbf{B}_2 = -\mathbf{m}_2 \cdot \mathbf{B}_1, \qquad (247)$$

where the second form follows from a similar argument in which dipole 1 was held fixed while dipole 2 is moved into place. The field of a dipole is

$$\mathbf{B}_1 = \frac{3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1}{r^3}, \qquad (248)$$

so

$$U = -\frac{3(\mathbf{m}_1 \cdot \hat{\mathbf{z}})(\mathbf{m}_2 \cdot \hat{\mathbf{z}}) - \mathbf{m}_1 \cdot \mathbf{m}_2}{d^3}. \qquad (249)$$

If dipole 1(2) makes angle $\theta_{1(2)}$ to the $z$ axis, and both lie in, say, the $x$-$z$ plane, then

$$U(\theta_1, \theta_2) = -\frac{m^2}{d^3}[3\cos\theta_1 \cos\theta_2 - \cos(\theta_1 - \theta_2)]. \qquad (250)$$

This is a minimum for $\theta_1 = \theta_2 = 0$ or $\pi$, and

$$U_{\text{min}} = -\frac{2m^2}{d^3}. \qquad (251)$$

In the absence of an external field, the dipoles are aligned, and both are either parallel or antiparallel to the $z$ axis.

We now add a uniform external magnetic field $\mathbf{B}$ that makes angle $\theta$ to the $z$ axis in the $x$-$z$ plane. The interaction energy of the system is then,

$$U(\theta_1, \theta_2, \theta) = -\frac{m^2}{d^3}[3\cos\theta_1\cos\theta_2 - \cos(\theta_1 - \theta_2)] - mB[\cos(\theta - \theta_1) + \cos(\theta - \theta_2)]. \quad (252)$$

First, consider when $\mathbf{B}$ is at right angles to the line of centers of the dipoles, $\theta = \pi/2$. Then,

$$U(\theta_1, \theta_2) = -\frac{m^2}{d^3}[3\cos\theta_1\cos\theta_2 - \cos(\theta_1 - \theta_2)] - mB[\sin\theta_1 + \sin\theta_2]. \quad (253)$$

If the dipoles remain in their original orientation, say $\theta_1 = \theta_2 = 0$, then the interaction energy $U_0$ is still given by (251). Suppose the two dipoles rotate together towards $\mathbf{B}$. Then,

$$\begin{aligned} U(\theta_1 = \theta_2) &= -\frac{m^2}{d^3}[3\cos^2\theta_1 - 1] - 2mB\sin\theta_1 \\ &= \frac{m^2}{d^3}\left[3\sin^2\theta_1 - 2 - \frac{2d^3 B}{m}\sin\theta_1\right]. \end{aligned} \quad (254)$$

The dipoles will rotate from angle 0 to $\pi/2$ provided that $U(\theta_1 = \theta_2)$ decreases monotonically along this path. Since (254) is a quadratic function of $\sin\theta_1$, this requires that the minimum occur for $\sin\theta_1 \geq 1$. The critical condition is then

$$B = \frac{3m}{d^3}, \quad (255)$$

above which field strength the dipoles always align with the transverse $\mathbf{B}$. Thus, (255) is the saturation magnetic field.

We note that for $B < 3m/d^3$, the energy minimum occurs at $\sin\theta_1 = \sin\theta_2 = d^3 B/3m < 1$. From (253), it can be verified that this is the absolute minimum for all $(\theta_1, \theta_2)$. So the equilibrium configuration falls short of full alignment with $\mathbf{B}$ until the field is larger than the saturation value (255).

Finally, we consider the case where the dipoles originally have $\theta_1 = \theta_2 = 0$, and external field $\mathbf{B} = -B\hat{\mathbf{z}}$ is applied. We first suppose that the dipoles rotate together, $\theta_1 = \theta_2$, so from (252) the energy of an intermediate state is

$$U(\theta_1 = \theta_2) = -\frac{m^2}{d^3}\left(3\cos^2\theta_1 - 1 - \frac{2d^3 B}{m}\cos\theta_1\right). \quad (256)$$

This form is concave downward in $\cos\theta_1$, so the alignment can occur so long as the maximum occurs at $\cos\theta_1 \geq 1$. Thus, on this path the critical condition is again (255).

We might wonder whether the alternative path, $\theta_2 = -\theta_1$ leads to a lower critical field. From (252),

$$U(\theta_1 = -\theta_2) = -\frac{m^2}{d^3}\left(\cos^2\theta_1 + 1 - \frac{2d^3 B}{m}\cos\theta_1\right). \quad (257)$$

This is also concave downward in $\cos\theta_1$, so requiring the maximum to occur at $\cos\theta_1 = 1$ leads to the critical condition

$$B = \frac{m}{d^3}. \tag{258}$$

In our model of a permanent magnet as consisting of only 2 magnetic dipoles, we find that it is favorable for the transition from one ferromagnetic (aligned dipoles) state to another at $180°$ due to application of an external field to pass through an antiferromagnetic state (anti-aligned dipoles). We leave it to statistical mechanics to decide whether this can occur in a system of a large number of magnetic dipoles.

14. The technique of mapping two dimensional magnetostatic fields with conducting paper is NOT based on the analogy between electrostatics and magnetostatics that was mentioned on p. 97 of the Notes. Rather, we start from (32-33). The 2-dimensional magnetic field derived from the vector potential $A_z\hat{\mathbf{z}}$ is

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y}, -\frac{\partial A_z}{\partial x}, 0\right). \tag{259}$$

With the identification $A_z = k\phi$ we can write

$$\mathbf{E} = -\boldsymbol{\nabla}\phi = -\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) = -\frac{1}{k}\left(\frac{\partial A_z}{\partial x}, \frac{\partial A_z}{\partial y}\right) = \frac{1}{k}(B_y, B_x) = \frac{\mathbf{B} \times \hat{\mathbf{z}}}{k}. \tag{260}$$

Hence, $\mathbf{B}$ is perpendicular to $\mathbf{E}$ and therefore parallel to equipotentials of $\phi$.

When current $I$ is feed into a region of conducting paint, it spreads out on the paper as described by current density $\mathbf{J}$. Then, current conservation and Ohm's law allow us to write

$$I\hat{\mathbf{z}} = \oint \mathbf{J} \times d\mathbf{l} = \oint \frac{\mathbf{E}}{\sigma} \times d\mathbf{l} = \frac{1}{k\sigma}\oint(\mathbf{B} \times \hat{\mathbf{z}}) \times d\mathbf{l} = \frac{\hat{\mathbf{z}}}{k\sigma}\oint \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi I_B}{ck\sigma}\hat{\mathbf{z}}, \tag{261}$$

where $I_B$ is the current needed to produce magnetic field $\mathbf{B}$. Thus, if we set

$$k = \frac{4\pi I_B}{c\sigma I}, \tag{262}$$

we can extract numerical values of $B$ from the potential distribution $\phi$ on the paper. To be more precise, note that

$$\Delta\phi = -\int \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{k}\int(\mathbf{B} \times \hat{\mathbf{z}}) \cdot d\mathbf{l} = \frac{1}{k}\int \mathbf{B} \cdot (d\mathbf{l} \times \hat{\mathbf{z}}) = \frac{\Phi_B}{k}, \tag{263}$$

where $\Phi_B$ is the magnetic flux per unit length in $z$ that passes between the end points of the integration.

As to the boundary conditions, first consider the patches of conducting paint. The electric field is perpendicular to the boundaries of these, and hence the model $\mathbf{B}$ is parallel to them. This is the magnetic boundary condition at a perfect conductor.

As mentioned in the statement of the problem, one can use a patch of conducting paint to simulate a surface on which the magnetic field is known to lie, thereby reducing the extent of the model.

The electric field is parallel to the edge of the conducting paper, and zero outside it, since $E = J/\sigma$. Hence, the model magnetic field is perpendicular to the edge of the paper. Outside the paper, (260) and the vanishing of $E$ would imply that $B$ vanishes also. This awkwardness can be avoided by supposing that we were actually modeling the field $\mathbf{H}$ rather than $\mathbf{B}$, and the paper corresponds to a region of permeability $\mu = 1$, where $\mathbf{B} = \mathbf{H}$. Then, if we suppose that the region outside the paper has very high permeability, the continuity of $B_\perp$ at the boundary implies that $H$ is extremely

small outside the paper, which restores consistency of the actual electrical boundary conditions with a class of magnetic boundary conditions.

Thus, we conclude that the conducting paper technique is particularly well suited for mapping 2-dimensional magnetic fields in situations bounded by a high permeability material, such as iron. Of course, the actual fields must not be so high that the iron saturates and the effective permeability drops to near unity.

Use of conducting paper to study two-dimensional patterns of electric field lines seems to haven been first made by G. Kirchhoff, *Ueber den Durchgang einem elektrischen Stromes durch einie Ebene, insbesondere durch eine kreissfórmige.* Ann. d Phys. **64**, 497 (1845), `http://kirkmcd.princeton.edu/examples/EM/kirchhoff_apc_64_497_45.pdf`
See also B. Liebmann, *Electrical Analógues*, Brit. J. Appl. Phys. **4**, 193 (1953),
`http://kirkmcd.princeton.edu/examples/EM/liebmann_bjap_4_193_53.pdf`
and J.H.O. Harries, *The Rubber Membrane and Resistance Paper Analogies*, Proc. IRE **44**, 236 (1956), `http://kirkmcd.princeton.edu/examples/EM/harries_pire_44_236_56.pdf`