# Princeton University

# Ph501

# Electrodynamics

# Problem Set 6

Kirk T. McDonald

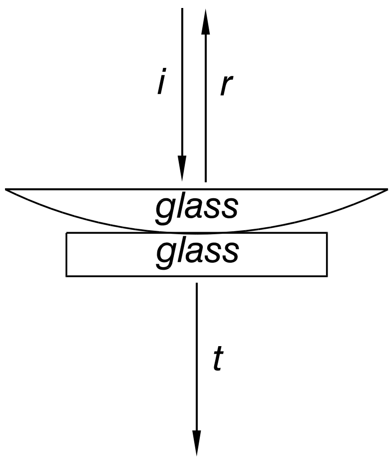
(2001)

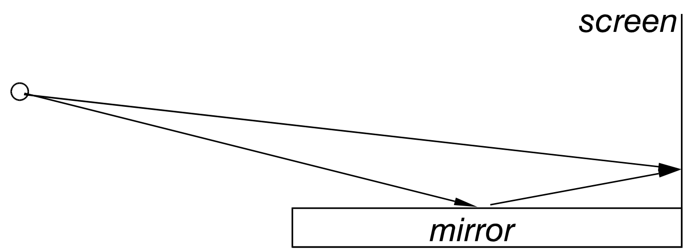
kirkmcd@princeton.edu

http://physics.princeton.edu/~mcdonald/examples/

1. a) The intensity of sunlight at the Earth's orbit is $\approx 1.4 \times 10^6$ erg/s/cm$^2$. What size chunk of earth ($\rho \approx 5$ g/cm$^3$) could be levitated without orbiting, but at the radius of the Earth's orbit?

   b) **Newton's Rings**. Explain briefly whether a dark or bright spot appears at the center, when viewing the reflected and transmitted fringe patterns in the apparatus sketched on the left. Ignore multiple reflections inside the glass in both parts b) and c).




   c) **Lloyd's Mirror**. Explain whether a dark or bright spot appears at the base of the screen in the apparatus sketched on the right.

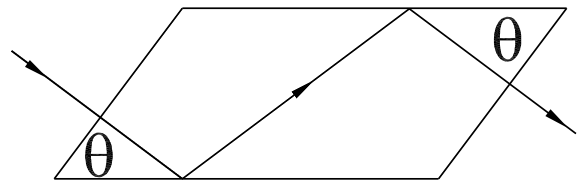
2. Carry out the derivation of Fresnel's equations by matching the fields at the dielectric boundary, as discussed on p. 143 of the Notes. Deduce the four ratios:

$$\frac{E_{0r}}{E_{0i}}, \qquad \frac{E_{0t}}{E_{0i}} \tag{1}$$

for $\mathbf{E}_i$ polarized parallel and perpendicular to the plane of incidence.

The derivation of Fresnel's equations from Maxwell's equation was first performed by Helmholtz.

3. **Fresnel's Rhomb.**




Linearly polarized light can be converted to circularly polarized light, and *vice versa*, with Fresnel's rhomb: a piece of glass cut in the shape of a rhombic prism.

If the glass has index of refraction $n = 1.5$, show that the angle $\theta$ must be $\approx 50.2^\circ$ or $53.3^\circ$.

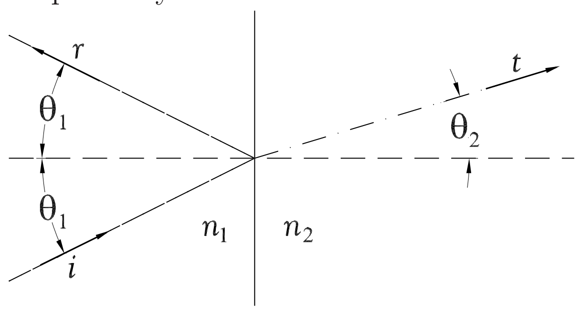
The effect is based on the phase change of totally internally reflected light. Hint: write

$$E_{0r} = E_{0i} e^{-i\phi}, \tag{2}$$

with $\phi_\perp (\phi_\parallel)$ for $\mathbf{E} \perp (\parallel)$ to the plane of incidence, and show that

$$\tan\left(\frac{\phi_\perp - \phi_\parallel}{2}\right) = -\frac{\cos\theta_i \sqrt{\sin^2\theta_i - 1/n^2}}{\sin^2\theta_i}. \tag{3}$$

4. (a) **More amplitude analysis.** On pp. 141-142 of the Notes,
http://physics.princeton.edu/~mcdonald/examples/ph501/ph501lecture12.pdf
we considered reflection and transmission at a dielectric boundary, using the amplitudes $i$, $r$ and $t$, which are proportional to the electric fields of the incident, reflected, and transmitted waves, respectively.




We found that if $i = 1$, then

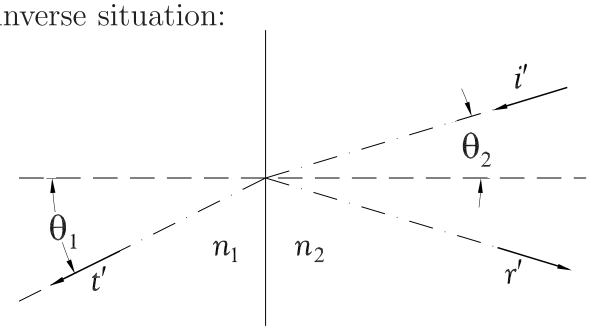
$$|r|^2 = \frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)}, \qquad \text{for } \mathbf{E} \perp \text{ to the plane of incidence,} \tag{4}$$

and

$$|r|^2 = \frac{\tan^2(\theta_1 - \theta_2)}{\tan^2(\theta_1 + \theta_2)}, \qquad \text{for } \mathbf{E} \parallel \text{ to the plane of incidence.} \tag{5}$$

Can we deduce relations for the phases of $r$ and $t$, and not merely their amplitudes? See sec. 33-6 of Vol. I of the *Feynman Lectures on Physics*,
http://www.feynmanlectures.caltech.edu/I_33.html

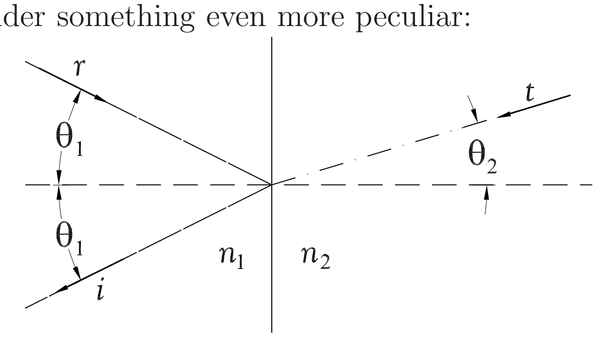
Consider now an inverse situation:




If $i' = 1$, then conservation of energy tells us that

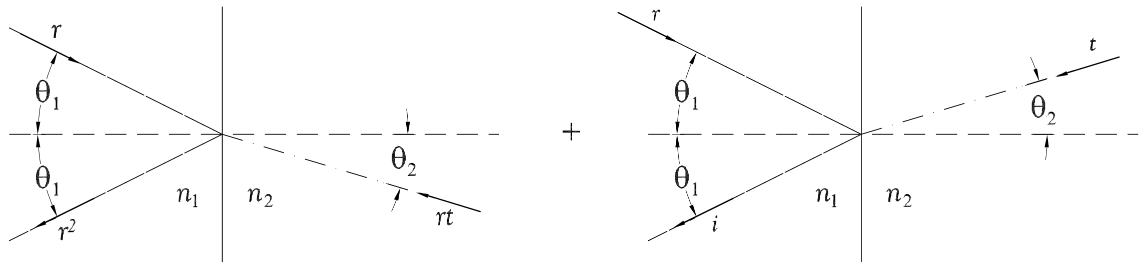
$$|r|^2 + |t|^2 = |r'|^2 + |t'|^2. \tag{6}$$

We can also consider something even more peculiar:



This can be regarded as the "time reversal" of the original situation.

But, we also recognize this as the superposition of two more ordinary configurations:

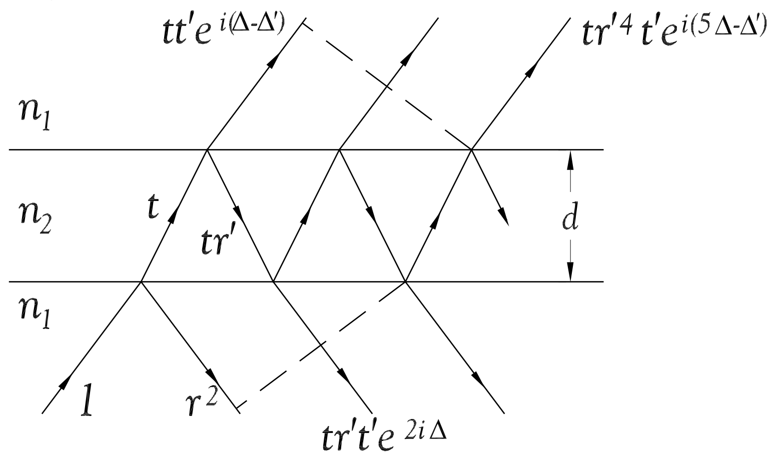



In particular, if a wave of amplitude $r$ were incident from side 1, then the transmitted wave would have amplitude $rt$ in terms of our original definitions.

Show that this implies that $r$ is real, while $t' = t^\star$.

Hint: first deduce that $r' = -r$ and $|t'| = |t|$. Then, multiply the relation $1 = r^2 + tt'$ by its complex conjugate....

(b) **Dielectric Slab.** Consider a plate of thickness $d$ of a dielectric with index of refraction $n_2$, surrounded by a medium of index $n_1 = 1$. A wave of unit amplitude is incident from below. Multiple reflections occur whose interference leads to the reflected and transmitted waves, being the sums of the amplitudes at the dashed wavefronts shown in the figure.




Show that the waves corresponding to a ray and its next higher-order neighbor have a phase difference $2\Delta$ due to the different path lengths they have traveled, where

$$\Delta = \frac{2\pi d \cos\theta_2}{\lambda_2}, \tag{7}$$

while the phase lag for the first transmitted ray compared to the case of no plate is $\Delta - \Delta'$, where

$$\Delta' = \frac{2\pi d \cos\theta_1}{\lambda_1}. \tag{8}$$

Define the total reflected and transmitted **amplitudes** to be $R$ and $T$, respectively.

Sum the partial amplitudes, and use the results of part (a) to show that

$$R = \frac{r(1 - e^{2i\Delta})}{1 - r^2 e^{2i\Delta}}, \qquad \text{and} \qquad T = \frac{(1 - r^2)e^{i(\Delta - \Delta')}}{1 - r^2 e^{2i\Delta}}, \tag{9}$$

which obey energy conservation: $|R|^2 + |T|^2 = 1$.

The ratio of the reflected to transmitted amplitudes is

$$\frac{R}{T} = \frac{r(1 - e^{2i\Delta})}{(1 - r^2)e^{i(\Delta - \Delta')}} = -\frac{2ie^{i\Delta'} r \sin\Delta}{1 - r^2} \tag{10}$$

so there is a phase difference of $\Delta' - \pi/2$ between amplitudes $R$ and $T$. An experiment that would be sensitive to this phase difference could involve a second beam, incident on the beam splitter at angle $\theta_1$, **but from the opposite side of the beam splitter from the original input beam**. Then, the transmitted part of the first beam would interfere with the reflected part of the second beam. If the two input beams are "in phase" at, say, the midplane of the beam splitter, they would have a phase difference of $\Delta'$ at the surface of the splitter onto which the second beam is incident. So, when considering the interference of the two beams, the phase $\Delta'$ found in eq. (10) would drop out, and we should say that **there is an effective phase difference of** $90^\circ$ **between the reflected and transmitted amplitudes in a beam splitter of finite thickness.** This $90^\circ$ phase difference plays an important role when comparing the classical and quantum behavior of a beam splitter:
http://physics.princeton.edu/~mcdonald/examples/bunching.pdf

An argument due to Feynman [Chaps. 31 of Vol. 1 of the *Feynman Lectures on Physics*, http://www.feynmanlectures.caltech.edu/I_31.html; also pp. 282-285, Lecture 23 of the Notes, http://physics.princeton.edu/~mcdonald/examples/ph501/ph501lecture23.pdf ] shows that $R$ is $i$ times a positive number, and hence that $r$ is negative. Then, eqs. (4) and (5) lead to

$$r = -\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}, \qquad \text{for } \mathbf{E} \perp \text{ to the plane of incidence,} \tag{11}$$

and

$$r = -\frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}, \qquad \text{for } \mathbf{E} \parallel \text{ to the plane of incidence.} \tag{12}$$

The expressions (9) for $R$ and $T$ are valid even if $n_1 > n_2$. Then, for large enough $\theta_1$, we expect total internal reflection.

In the Notes, we found for this case that

$$\cos\theta_2 = i\sqrt{(n_1/n_2)^2 \sin^2\theta_1 - 1}, \tag{13}$$

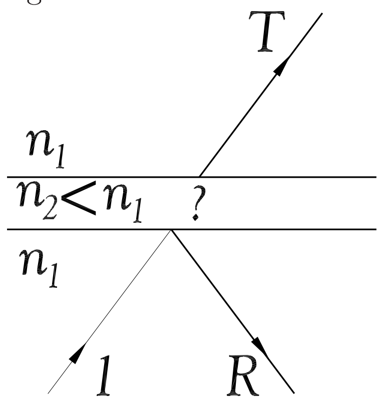
so eq. (7) gives

$$\Delta = \frac{2\pi i d}{\lambda^2}\sqrt{(n_1/n_2)^2 \sin^2\theta_1 - 1} \equiv i\frac{d}{\delta}. \tag{14}$$

Thus,

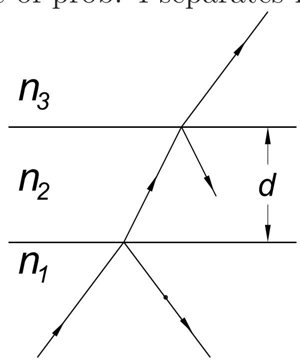
$$T = \frac{(1 - r^2)e^{-d/\delta}e^{-i\Delta'}}{1 - r^2 e^{-2d/\delta}} \to 0 \qquad \text{as} \qquad d \to \infty. \tag{15}$$

But for finite thickness $d$ the transmitted amplitude is nonzero, even though we found no wave motion in medium 2 which traveled normal to the boundaries. This phenomenon is called "tunneling".



5. (a) **Antireflection Coatings**

Suppose the slab of dielectric of prob. 4 separates media of indices $n_1$ and $n_3$.




Show that the reflected and transmitted amplitudes obey

$$R = \frac{r_{12} + r_{23} e^{2i\Delta}}{1 + r_{12} r_{23} e^{2i\Delta}}, \qquad T = \frac{t_{12} t_{23} e^{i\Delta}}{1 + r_{12} r_{23} e^{2i\Delta}}, \tag{16}$$

where $r_{12}$ is the amplitude for a single reflection at boundary 1-2, *etc.*

Consider the special case of normal incidence. Show that if $n_2 = \sqrt{n_1 n_3}$ and $d = \lambda_2/4$, then $R = 0$, which is a prescription for an antireflection lens coating.

Show also that if $d = \lambda_2/2$ the $R$ is independent of $n_2$, and if in addition $n_1 = n_3$ then $R$ vanishes.

(b) **Dielectric Mirrors**

Can we make a good mirror by applying an appropriate dielectric coating on a plate of glass?

Not with only two layers, but consider a multilayer mirror. For example, if a medium 4 exists beyond medium 3, then the reflection at the 2-3 boundary could be described by

$$R_{23} = \frac{r_{23} + r_{34} e^{2i\Delta_3}}{1 + r_{23} r_{34} e^{2i\Delta_3}}, \qquad T_{23} = \frac{t_{23} t_{34} e^{i\Delta_3}}{1 + r_{23} r_{34} e^{2i\Delta_3}}, \tag{17}$$

$$\Rightarrow T = \frac{t_{12} t_{23} t_{34} e^{i(\Delta_2 + \Delta_3)}}{(1 + r_{23} r_{34} e^{2i\Delta_3})(1 + r_{12} R_{23} e^{2i\Delta_2})}, \qquad etc. \tag{18}$$

Then, for a stack of $n$ 2-3 pairs, 1-2-3-2-3- $\cdots$ -3-2-1,

$$T = \frac{t_{12} t_{23} t_{32} t_{23} \cdots t_{32} t_{21} e^{ni(\Delta_2 + \Delta_3)}}{\text{big mess}} = \frac{|t_{12}|^2 \, |t_{23}|^{2n} \, e^{ni(\Delta_2 + \Delta_3)}}{\text{big mess}}, \tag{19}$$

where the big mess is not small if $\Delta_2 = \Delta_3 = \pi/4$, since in that case $1 + r_{23} r_{32} e^{2i\Delta_3} = 1 + r_{23}^2$, *etc.* Since $|t_{23}|^{2n} \to 0$ for large $n$, $T \to 0$ and $R \to 1$.

The prescription given here for a multilayer dielectric mirror works well for only a narrow range of angles of incidence and a narrow range of wavelengths, since we require that $\Delta_2 = \Delta_3 = \pi/4$. Better prescriptions can be given that maintain very high reflectivity over a large range of parameters. See, for example, J.P. Dowling, Science **282**, 1841, (1998).[1]

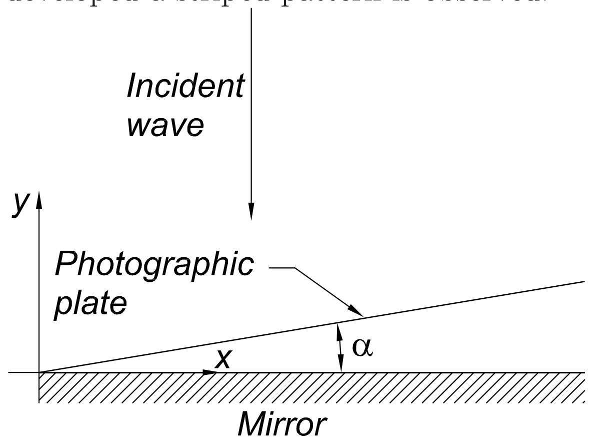
[1] http://physics.princeton.edu/~mcdonald/examples/optics/dowling_science_282_1841_98.pdf

6. In 1890, O. Wiener carried out an experiment that can be said to have photographed electromagnetic waves.[2]

   a) A plane wave is normally incident of a perfectly reflecting mirror. A glass photographic plate is placed on the mirror at a small angle $\alpha$. The polarization of the wave is parallel to the line of intersection of the mirror and the plate.

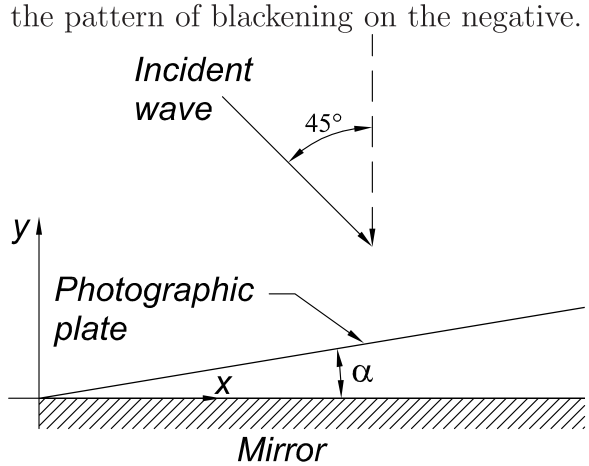
   The photographic emulsion is almost transparent – ignore attenuation and reflection in it and in the glass.

   When the plate is developed a striped pattern is observed.




   Calculate the electromagnetic fields **E** and **B** for $y > 0$, where $y = 0$ is the surface of the mirror. Predict the position and spacing of the dark stripes that appear on the developed "negative" plate.

   b) Repeat the discussion for waves incident at 45°. That is, calculate **E** and **B** for $y > 0$, and predict the pattern of blackening on the negative.




   Distinguish the case of **E** $\perp$ and $\|$ to the plane of incidence.

[2] http://physics.princeton.edu/~mcdonald/examples/EM/wiener_ap_40_203_90.pdf

7. Two airplanes are flying at distance $d$ apart at height $h$ above the ocean whose dielectric constant is $\epsilon$. One plane sends signals to the other. Both airplanes have short vertical antennae. Ignore the curvature of the Earth.

   Show that the ratio of the intensity of the signal reflected off the ocean to that of the direct signal is

$$\frac{d^6}{(d^2+4h^2)^3}\left(\frac{\sqrt{(\epsilon-1)d^2+4\epsilon h^2}-2\epsilon h}{\sqrt{(\epsilon-1)d^2+4\epsilon h^2}+2\epsilon h}\right)^2. \tag{20}$$

   In addition to facts about plane waves you need to know that

   - The intensity of spherical broadcast waves falls off as $1/r^2$. Over small spatial regions (except close to the source) the spherical waves can be considered as plane waves.
   - The amplitude (**E** field) of a broadcast wave varies linearly with the projection perpendicular to the line of sight of the motion (acceleration) of the charges that cause the wave (p. 141 of the Notes).
   - Likewise, the current $I$ excited in a receiving antenna varies as the projection of **E** onto the antenna.
   - The power of the received signal is, of course, $I^2R$, where $R$ is the resistance in the receiving antenna.

8. (a) **Plasma with a dc magnetic field.**

Consider the Earth's ionosphere to be a plasma of uniform density with a static, uniform magnetic field $\mathbf{B}_0$ (the Earth's field) in the $+z$ direction. Discuss the propagation of circularly polarized plane radio waves parallel (or antiparallel) to $\mathbf{B}_0$.

The response of an ionized electron of charge $-e$ and mass $m$ at position $\mathbf{r}$ to the wave of angular frequency $\omega$ is described by

$$m\ddot{\mathbf{r}} + e\frac{\dot{\mathbf{r}}}{c} \times \mathbf{B}_0 = -e\mathbf{E}e^{i(kz-\omega t)}, \tag{21}$$

where for circularly polarized waves the electric field amplitude can be written

$$\mathbf{E}_\pm = E_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}). \tag{22}$$

Show that

$$\mathbf{r}_\pm = -\frac{e\mathbf{E}}{m\omega(\omega \mp \omega_B)}, \tag{23}$$

where

$$\omega_B = \frac{eB_0}{mc}, \tag{24}$$

and that this implies a dielectric constant for the plasma of

$$\epsilon_\pm = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)}, \tag{25}$$

where the plasma frequency $\omega_p$ is given by

$$\omega_p^2 = \frac{4\pi N e^2}{m} \tag{26}$$

for a plasma of number density $N$ per cm$^3$.

Show that for waves of circular polarization $\hat{\mathbf{x}}+i\hat{\mathbf{y}}$ (called left-handed in optics although a "photograph" of the electric field vector would show it to behave like a right-handed screw),

$$v_{\text{group}} = 2v_{\text{phase}} = 2c\frac{\sqrt{\omega\omega_B}}{\omega_p}. \tag{27}$$

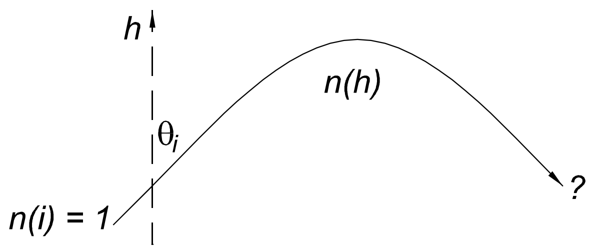
[See pp. 146a-d of the Notes for a discussion of group and phase velocity.]

It turns out that $\omega_B \approx \omega_p \approx 10^7$ Hz in the ionosphere. Estimate the difference in arrival times for signals of $10^5$ and $2 \times 10^5$ Hz originating simultaneously (in lightning flashes) at the opposite side of the Earth. This illustrates the "whistler" or "chirp" effect – as higher frequency waves arrive first.

What is the fate of waves of polarization $\hat{\mathbf{x}} - i\hat{\mathbf{y}}$?

(b) **Reflections and Mirages**

In the ionosphere the density of ionized electrons actually increases with height: the lower atmosphere is screened from the Sun by the upper. Hence, the (frequency-dependent) index of refraction decreases with height, since $n(\omega) \approx \sqrt{1 - (\omega_p/\omega)^2}$, where $\omega_p$ is the plasma frequency.




Suppose at the bottom of the atmosphere, where $n = 1$, a radio wave propagates upwards with angle $\theta_i$ to the vertical.

Use Snell's law to show that the wave is reflected back downwards if the electron density rises until $\omega_p = \omega \cos\theta_i$ at some height.

Mirages are a similar phenomenon in which higher temperatures in the air close to the Earth's surface result in lower density ($N = RT/P$) at lower height, and hence lower index at lower height for optical frequencies ($n \approx \sqrt{1 + \omega_p^2/(\omega_0^2 - \omega^2)}$), so downward going light rays can be reflected upwards near the surface.

9. A plane electromagnetic wave of angular frequency $\omega$ is normally incident on a good conductor that occupies the region $z > 0$. Show that the Poynting vector $\langle \mathbf{S} \rangle$ evaluated at $z$ just greater than zero is equal to the power (per unit area $\perp$ to $z$) lost to Joule heating in the conductor.

10. A plane electromagnetic wave of angular frequency $\omega$ is normally incident on a thin conducting sheet of thickness $a \ll d$, the skin depth. Ignoring reflection, show that the relative transmitted intensity is

$$\mathcal{T} = 1 - \frac{4\pi}{c}\sigma a, \tag{28}$$

where $\sigma$ is the conductivity.

Use an energy argument as in prob. 8.

Extending the argument to show that the relative reflected intensity $\mathcal{R}$ is $(2\pi\sigma a/c)^2$.

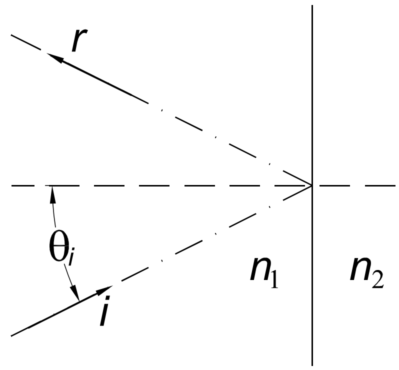
A "trick" derivation is to note that a sheet of unit area and thickness $a$ has resistance

$$R = \frac{l}{\sigma A} = \frac{1}{\sigma a \cdot 1} = \frac{1}{\sigma a} \tag{29}$$

to the induced currents that flow in the plane of the sheet. Hence, power $V^2/R$ if absorbed is a wave of voltage $V$ per unit length crosses the sheet. But, the power carried by a plane wave is $V^2/R_{\rm vac}$ where $R_{\rm vac} = 4\pi/c = 377\ \Omega$. Thus, the fractional power absorbed is $R_{\rm vac}/R = 4\pi\sigma a/c$.

11. **Total Internal Reflection**

A wave of frequency $\omega$ is incident at angle $\theta_i$ on the boundary between dielectrics of indices $n_1 > n_2$. Find the time-averaged Poynting vector of the transmitted wave when $\sin\theta_i > n_2/n_1$, *i.e.*, when total internal reflection occurs.



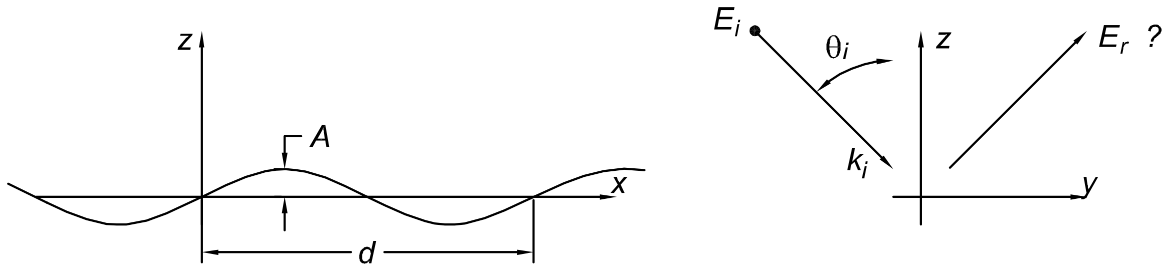

Verify that the transmitted surface wave satisfies the wave equation and Maxwell's equations.

12. **The grating accelerator**

In optics, a reflective grating is a conducting surface with a ripple. For example, consider the surface defined by

$$z = a \sin \frac{2\pi x}{d}. \tag{30}$$

The typical use of such a grating involves an incident electromagnetic wave with wave vector $\mathbf{k}$ in the $x$-$z$ plane, and interference effects lead to a discrete set of reflected waves also with wave vectors in the $x$-$z$ plane.




Consider, instead, an incident plane electromagnetic wave with wave vector in the $y$-$z$ plane and polarization in the $x$ direction:

$$\mathbf{E}_{\rm in} = E_0 \hat{\mathbf{x}} e^{i(k_y y - k_z z - \omega t)}, \tag{31}$$

where $k_y > 0$ and $k_z > 0$. Show that for small ripples ($a \ll d$), this leads to a reflected wave as if $a = 0$, plus two surface waves that are attenuated exponentially with $z$. What is the relation between the grating wavelength $d$ and the optical wavelength $\lambda$ such that the $x$ component of the phase velocity of the surface waves is the speed of light, $c$?

In this case, a charged particle moving with $v_x \approx c$ could extract energy from the wave, which is the principle of the proposed "grating accelerator" [R.B. Palmer, *A Laser-Driven Grating Linac*, Part. Accel. **11**, 81-90 (1980)].[3]

[3] http://physics.princeton.edu/~mcdonald/examples/accel/palmer_pa_11_81_80.pdf

13. A **radiofrequency quadrupole** (RFQ) is a device for focussing beams of charged particles. The electric field in this device can be approximated as that derived from the quasistatic potential

$$\phi(x, y, t) = \frac{E_0}{2d}(y^2 - x^2)\sin\omega t, \tag{32}$$

where $d$ is a length and $\omega$ is the frequency of the field. The magnetic field is ignored in this approximation. While the approximate fields do not satisfy Maxwell's equations, there is little error for $|x|$, $|y| \ll \lambda$, the wavelength of the radiofrequency waves.

Deduce the equations of motion for a particle of charge $e$ and mass $m$ in the radiofrequency quadrupole. Consider solutions of the form

$$x(t) = f(t) + g(t)\sin\omega t \tag{33}$$

where $g \ll f$ and both $f$ and $g$ are slowly varying compared to $\sin\omega t$. The parameters may be assumed to satisfy the conditions that such solutions exist.

Complete the solution for the particular case that

$$\begin{aligned} x(0) &= 0, \qquad \dot{x}(0) = v_0\theta_0, && (34)\\ y(0) &= 0, \qquad \dot{y}(0) = 0, && (35)\\ z(0) &= 0, \qquad \dot{z}(0) = v_0, && (36) \end{aligned}$$

with $\theta_0 \ll 1$. At what distance along the $z$-axis is the first image of the beam 'spot', *i.e.*, where the initially diverging beam is brought back to the $z$-axis?

14. **Pulsar Timing**

The distance from the Earth to a pulsar can be estimated by observing the dispersion of the radio-frequency pulses as they cross the interstellar medium.

a) Suppose the medium is a plasma of $N$ electrons/cm$^3$. What is the index of refraction $n(\omega)$ where $\omega$ is the angular frequency of a wave?

b) The pulsar emits a short pulse that contains a broad range of frequencies. We observe the pulse in a receiver that is tuned to a narrow band $\delta\omega$ about an adjustable central frequency $\omega$. We measure the time difference $\delta t$ between the arrival of two components of the pulse, centered at frequencies $\omega$ and $\omega + \delta\omega$, where $\delta\omega \ll \omega$.

This can be done in a single receiver if the pulsar has a precise pulse rate – as is the case. Pulsars are the most accurately periodic macroscopic phenomenon ever observed!

Which component, $\omega$ or $\omega + \delta\omega$, arrives first, and by how much, as a function of the Earth-pulsar distance $L$?

Use the following representative values to calculate the distance $L$ to pulsar "1913+16": $\omega = 2000$ MHz, $\delta\omega/\omega = 0.01$, $N = 0.04$ electrons/cm$^3$, and $|\delta t| = 0.004$ s.

A pulsar tidbit: Many pulsars occur in binary systems, including one such system where the direction to the Earth lies very close to the plane of the orbit. By observing the small general-relativistic pulse delays that occur when one of the binary partners occults the other, the eccentricity of the orbit can be determined to remarkable accuracy. The data indicate that the orbit has radius $\approx R_{\mathrm{Earth-Sun}}$ and a small eccentricity corresponding to being out of round by less than 1 cm! The orbit is round to one part in $10^{13}$, which makes it the roundest object ever measured – and it's at the other end of the galaxy (F. Camilo, Princeton Ph.D. thesis, $\approx$ 1994).

**Solutions**

1. a) For a black chunk of earth of radius $r$ levitated at distance $R_E = 1.5 \times 10^{13}$ cm in sunlight of intensity $I = 1.4 \times 10^6$ erg/s/cm$^2$,

$$F_G = \frac{GM_S m}{R_E^2} = \frac{4\pi r^3 \rho G M_S}{3R_E^2} = F_{\rm rad} = \pi r^2 \frac{I}{c} . \qquad (37)$$

Hence

$$r = \frac{3IR_E^2}{4c\rho G M_S} = \frac{3 \cdot 1.4 \times 10^6 \cdot (1.5 \times 10^{13})^2}{4 \cdot 3 \times 10^{10} \cdot 5 \cdot 6.7 \times 10^{-8} \cdot 2 \times 10^{33}} \approx 1.2 \times 10^{-5} \text{ cm}, \qquad (38)$$

which is less than a wavelength of light!

b) The center of the reflected spot is dark, being the interference between the reflection off the bottom surface of the lens and the top surface of the glass plate. There is a 180° phase difference between these two cases. (The reflections off the top of the lens and bottom of the glass plate give a "background" intensity that is independent of position.) The center of the transmitted spot is bright, as there is no phase shift during transmission across a dielectric boundary.

c) The fringe pattern is dark at the base of the screen, due to interference of the direct and reflected rays. The latter undergo a 180° phase shift on reflection.

2. We consider plane waves of the form

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \\ \mathbf{H} &= \frac{\mathbf{B}}{\mu} = \frac{n}{\mu}\hat{\mathbf{k}} \times \mathbf{E} = \sqrt{\frac{\epsilon}{\mu}}\hat{\mathbf{k}} \times \mathbf{E}, \end{aligned} \tag{39}$$

where $k = \sqrt{\epsilon\mu}\omega/c = n\omega/c$ in media with dielectric constant $\epsilon$, index $n = \sqrt{\epsilon\mu}$, and permeability $\mu$.

At a boundary between two dielectrics, the perpendicular components of $\mathbf{D}$ and $\mathbf{B}$, and the parallel components of $\mathbf{E}$ and $\mathbf{H}$ are continuous. The incident and reflected waves are in medium 1, and the transmitted wave is in medium 2. The unit normal vector pointing into medium 2 is labeled $\hat{\mathbf{n}}$. Then the boundary conditions are

$$\frac{n_1^2}{\mu_1}(\mathbf{E}_{0i} + \mathbf{E}_{0r}) \cdot \hat{\mathbf{n}} = \frac{n_2^2}{\mu_2}\mathbf{E}_{0t} \cdot \hat{\mathbf{n}}, \tag{40}$$

$$n_1(\hat{\mathbf{k}}_i \times \mathbf{E}_{0i} + \hat{\mathbf{k}}_r \times \mathbf{E}_{0r}) \cdot \hat{\mathbf{n}} = n_2\hat{\mathbf{k}}_t \times \mathbf{E}_{0t} \cdot \hat{\mathbf{n}}, \tag{41}$$

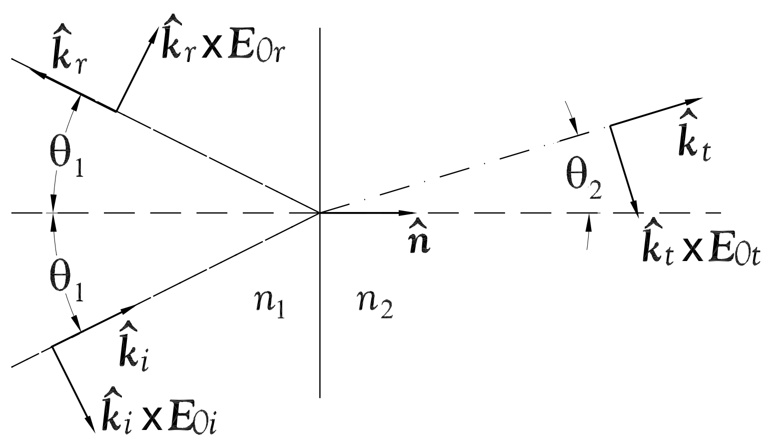
$$(\mathbf{E}_{0i} + \mathbf{E}_{0r}) \times \hat{\mathbf{n}} = \mathbf{E}_{0t} \times \hat{\mathbf{n}}, \tag{42}$$

$$\frac{n_1}{\mu_1}(\hat{\mathbf{k}}_i \times \mathbf{E}_{0i} + \hat{\mathbf{k}}_r \times \mathbf{E}_{0r}) \times \hat{\mathbf{n}} = \frac{n_2}{\mu_2}\hat{\mathbf{k}}_t \times \mathbf{E}_{0t} \times \hat{\mathbf{n}}, \tag{43}$$

Of course, Snell's law tells us that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2. \tag{44}$$

(a) Polarization perpendicular to the plane of incidence (the plane containing $\hat{\mathbf{k}}_i$, $\hat{\mathbf{k}}_r$ and $\hat{\mathbf{k}}_t$).




Relation (40) is satisfied identically. Both relations (41) and (42) yield

$$E_{0i} + E_{0r} = E_{0t}. \tag{45}$$

Relation (43) tells us that

$$\frac{n_1}{\mu_1}(E_{0i} - E_{0r})\cos\theta_1 = \frac{n_2}{\mu_2}E_{0t}\cos\theta_2 = \frac{n_1}{\mu_2}\frac{\sin\theta_1\cos\theta_2}{\sin\theta_2}E_{0t}, \tag{46}$$

and hence,

$$E_{0i} - E_{0r} = \frac{\mu_1}{\mu_2}\frac{\sin\theta_1\cos\theta_2}{\cos\theta_1\sin\theta_2}E_{0t}. \tag{47}$$

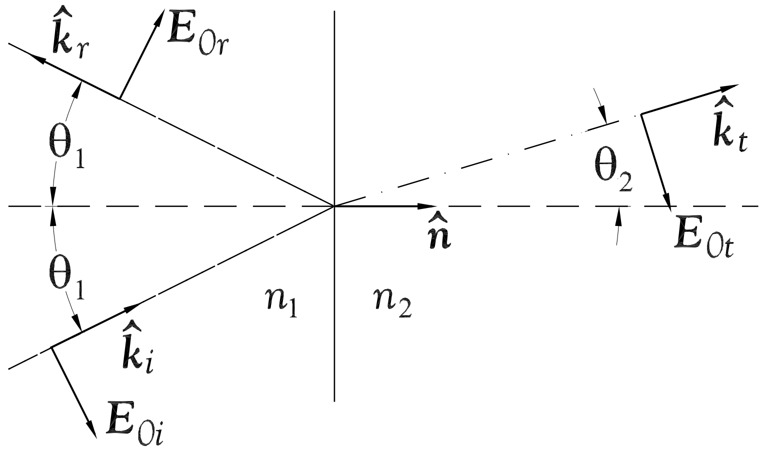
Adding (45) and (47), we find that

$$\frac{E_{0t}}{E_{0i}} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + \frac{\mu_1}{\mu_2}n_2\cos\theta_2} \to \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)} \quad \text{if} \quad \mu_1 = \mu_2 = 1. \tag{48}$$

Then, (45) leads to

$$\frac{E_{0r}}{E_{0i}} = \frac{E_{0t}}{E_{0i}} - 1 = \frac{n_1\cos\theta_1 - \frac{\mu_1}{\mu_2}n_2\cos\theta_2}{n_1\cos\theta_1 + \frac{\mu_1}{\mu_2}n_2\cos\theta_2} \to -\frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)} \quad \text{if} \quad \mu_1 = \mu_2 = 1. \tag{49}$$

(b) Polarization parallel to the plane of incidence.




Relation (40) leads to

$$\frac{n_1^2}{\mu_1}(E_{0i} - E_{0r})\sin\theta_1 = \frac{n_2^2}{\mu_2}E_{0t}\sin\theta_2, \tag{50}$$

which simplifies to

$$E_{0i} - E_{0r} = \frac{\mu_1 n_2}{\mu_2 n_1}E_{0t} \to \frac{\sin\theta_1}{\sin\theta_2}E_{0t} \quad \text{if} \quad \mu_1 = \mu_2 = 1. \tag{51}$$

using Snell's law. Similarly, relation (42) leads to

$$E_{0i} + E_{0r} = \frac{\cos\theta_2}{\cos\theta_1}E_{0t}. \tag{52}$$

Adding (51) and (52), we find that

$$\begin{aligned}\frac{E_{0t}}{E_{0i}} &= \frac{2n_1\cos\theta_1}{n_2\cos\theta_1 + \frac{\mu_1}{\mu_2}n_1\cos\theta_2} \to \frac{2\sin\theta_2\cos\theta_1}{\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2} = \frac{4\sin\theta_2\cos\theta_1}{\sin 2\theta_1 + \sin 2\theta_2} \\ &= \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)} \quad \text{if} \quad \mu_1 = \mu_2 = 1. \end{aligned} \tag{53}$$

Combining this with (52), we have

$$
\begin{aligned}
\frac{E_{0r}}{E_{0i}} &= \frac{\cos\theta_2}{\cos\theta_1}\frac{E_{0t}}{E_{0i}} - 1 = \frac{n_1\cos\theta_2 - \frac{\mu_1}{\mu_2}n_2\cos\theta_1}{n_1\cos\theta_1 + \frac{\mu_1}{\mu_2}n_2\cos\theta_2} \\
&\to \frac{2\sin\theta_2\cos\theta_2}{\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2} - 1 \\
&= \frac{\sin\theta_2\cos\theta_2 - \sin\theta_1\cos\theta_1}{\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2} = -\frac{\sin(\theta_1-\theta_2)\cos(\theta_1+\theta_2)}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)} \\
&= -\frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)} \quad \text{if} \quad \mu_1 = \mu_2 = 1. \qquad (54)
\end{aligned}
$$

(c) Normal Incidence.

Taking the limit of either polarization as $\theta_1 \to 0$ and $\theta_2 \to 0$, we find

$$
\frac{E_{0t}}{E_{0i}} = \frac{2n_1}{n_2 + \frac{\mu_1}{\mu_2}n_1} \to \frac{2n_1}{n_1+n_2} \quad \text{if} \quad \mu_1 = \mu_2 = 1, \qquad (55)
$$

$$
\frac{E_{0r}}{E_{0i}} = \frac{n_1 - \frac{\mu_1}{\mu_2}n_2}{n_1 + \frac{\mu_1}{\mu_2}n_2} \to \frac{n_1-n_2}{n_1+n_2} \quad \text{if} \quad \mu_1 = \mu_2 = 1. \qquad (56)
$$

3. For a linearly polarized wave, $E_\perp$ and $E_\parallel$ are in phase, while for a circularly polarized wave their phase difference is $\Delta\phi = \pm 90°$.

   In Fresnel's rhomb, there are two internal reflections, each of which causes phase changes $\Delta\phi_\perp$ and $\Delta\phi_\parallel$ for light polarized perpendicular and parallel to the plane of incidence, respectively. Hence, if

$$\Delta\phi_\perp - \Delta\phi_\parallel = \pm 45° \tag{57}$$

   at each reflection, we will achieve the desired conversion of linearly into circularly polarized light.

   In case of total internal reflection where media 1 and 2 have indices $n_1 = n$ and $n_2 = 1$, we use Snell's law to write

$$\sin\theta_2 = n\sin\theta_1, \qquad \text{and} \qquad \cos\theta_2 = \sqrt{1 - n^2\sin^2\theta_1} = in\sqrt{\sin^2\theta_1 - 1/n^2}. \tag{58}$$

   Then, eqs. (49) and (54) can be written as

$$\begin{aligned} \left.\frac{E_{0r}}{E_{0i}}\right|_\perp &= -\frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)} = -\frac{\sin\theta_1\cdot in\sqrt{\sin^2\theta_1 - 1/n^2} - \cos\theta_1\cdot n\sin\theta_1}{\sin\theta_1\cdot in\sqrt{\sin^2\theta_1 - 1/n^2} + \cos\theta_1\cdot n\sin\theta_1} \\ &= \frac{\cos\theta_1 - i\sqrt{\sin^2\theta_1 - 1/n^2}}{\cos\theta_1 + i\sqrt{\sin^2\theta_1 - 1/n^2}}, \end{aligned} \tag{59}$$

   and

$$\begin{aligned} \left.\frac{E_{0r}}{E_{0i}}\right|_\parallel &= -\frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)} \\ &= \frac{\cos\theta_1 - i\sqrt{\sin^2\theta_1 - 1/n^2}}{\cos\theta_1 + i\sqrt{\sin^2\theta_1 - 1/n^2}} \cdot \frac{\cos\theta_1\cdot in\sqrt{\sin^2\theta_1 - 1/n^2} - \sin\theta_1\cdot n\sin\theta_1}{\cos\theta_1\cdot in\sqrt{\sin^2\theta_1 - 1/n^2} + \sin\theta_1\cdot n\sin\theta_1} \\ &= \frac{\cos\theta_1 - i\sqrt{\sin^2\theta_1 - 1/n^2}}{\cos\theta_1 + i\sqrt{\sin^2\theta_1 - 1/n^2}} \cdot \frac{\cos\theta_1\left(\cos\theta_1 + i\sqrt{\sin^2\theta_1 - 1/n^2}\right) - 1}{-\cos\theta_1\left(\cos\theta_1 - i\sqrt{\sin^2\theta_1 - 1/n^2}\right) + 1} \\ &= \frac{\cos\theta_1(1 - 1/n^2 - 1) + i\sqrt{\sin^2\theta_1 - 1/n^2}}{-\cos\theta_1(1 - 1/n^2 - 1) + i\sqrt{\sin^2\theta_1 - 1/n^2}} \\ &= -\frac{\cos\theta_1 - in^2\sqrt{\sin^2\theta_1 - 1/n^2}}{\cos\theta_1 + in^2\sqrt{\sin^2\theta_1 - 1/n^2}}. \end{aligned} \tag{60}$$

   Both eqs. (59) and (60) have the form

$$\frac{a - ib}{a + ib} = \frac{a^2 - b^2 - 2iab}{a^2 + b^2} \equiv e^{-i\Delta\phi}, \tag{61}$$

so

$$\tan\Delta\phi = \frac{2ab}{a^2-b^2} = \frac{2(a/b)}{1-(a/b)^2}$$
$$= \tan 2(\Delta\phi/2) = \frac{2\tan(\Delta\phi/2)}{1-\tan^2(\Delta\phi/2)}, \qquad (62)$$

and hence,

$$\tan(\Delta\phi/2) = \frac{a}{b}. \qquad (63)$$

Thus,

$$\tan(\Delta\phi_\perp/2) = \frac{\sqrt{\sin^2\theta_1 - 1/n^2}}{\cos\theta_1}, \qquad \text{and} \qquad \tan(\Delta\phi_\parallel/2) = n^2\frac{\sqrt{\sin^2\theta_1 - 1/n^2}}{\cos\theta_1}, \quad (64)$$

and

$$\tan(\Delta\phi_\perp/2 - \Delta\phi_\parallel/2) = \frac{\tan(\Delta\phi_\perp/2) - \tan(\Delta\phi_\parallel/2)}{1+\tan(\Delta\phi_\perp/2)\tan(\Delta\phi_\parallel/2)}$$
$$= \frac{\sqrt{\sin^2\theta_1 - 1/n^2}}{\cos\theta_1}\cdot\frac{1-n^2}{1+n^2\frac{\sin^2\theta_1-1/n^2}{\cos^2\theta_1}}$$
$$= -\frac{\cos\theta_1\sqrt{\sin^2\theta_1 - 1/n^2}}{\sin^2\theta_1}. \qquad (65)$$

The angle of incidence, $\theta_1$, is the same as angle $\theta$ shown in the figure for Fresnel's rhomb. Thus, condition (57) implies that

$$\frac{\cos\theta\sqrt{\sin^2\theta - 1/n^2}}{\sin^2\theta} = \pm\tan 22.5^\circ = \pm(\sqrt{2}-1), \qquad (66)$$

or

$$(3-2\sqrt{2})\sin^4\theta = (1-\sin^2\theta)(\sin^2\theta - 1/n^2), \qquad (67)$$
$$(4-2\sqrt{2})\sin^4\theta - (1+1/n^2)\sin^2\theta + 1/n^2 = 0, \qquad (68)$$
$$\sin^2\theta = \frac{1+1/n^2 \pm\sqrt{(1+1/n^2)^2 - 4(4-2\sqrt{2})/n^2}}{2(4-2\sqrt{2})}. \qquad (69)$$

For $n = 1.5$, we find $\theta = 50.2^\circ$ and $53.3^\circ$.

4. (a) Since the "time reversed" case is the superposition of the two cases shown below the former, we conclude that

$$i = 1 = r^2 + tt', \tag{70}$$

and

$$0 = rt + tr'. \tag{71}$$

From eq. (71) we learn that

$$r' = -r. \tag{72}$$

Combining this with conservation of energy, eq. (6), we find that

$$|t'| = |t|\,. \tag{73}$$

Given these relations, we can now write

$$r = r_0 e^{i\alpha}, \qquad t = t_0 e^{i\beta}, \qquad \text{and} \qquad t' = t_0 e^{i\gamma}, \tag{74}$$

where $r_0$ and $t_0$ are real and obey

$$r_0^2 + t_0^2 = 1. \tag{75}$$

Inserting relations (74) into eq. (70), we find

$$1 = r_0^2 e^{2i\alpha} + t_0^2 e^{i(\beta+\gamma)}. \tag{76}$$

Multiplying eq.(76) by its complex conjugate yields

$$1 = r_0^4 + t_0^4 + 2r_0^2 t_0^2 \cos(2\alpha - \beta - \gamma). \tag{77}$$

In view of relation (75), we must have

$$2\alpha = \beta + \gamma. \tag{78}$$

Then, eq. (76) can be rewritten as

$$1 = (r_0^2 + t_0^2)e^{2i\alpha} = e^{2i\alpha}. \tag{79}$$

Hence,

$$\alpha = n\pi, \tag{80}$$

where $n$ is an integer, from which we conclude that $r$ (and $r'$) is real. Further,

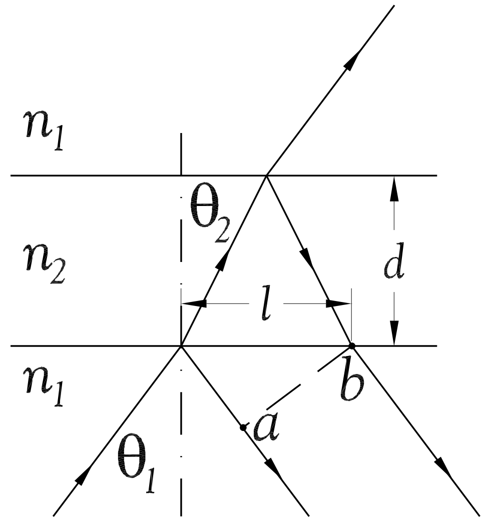
$$\beta + \gamma = 2n\pi, \tag{81}$$

which implies that

$$t' = t^\star, \tag{82}$$

according to the definitions (74).

(b) The phase difference $2\Delta$ is that between points $a$ and $b$ in the figure




Namely,

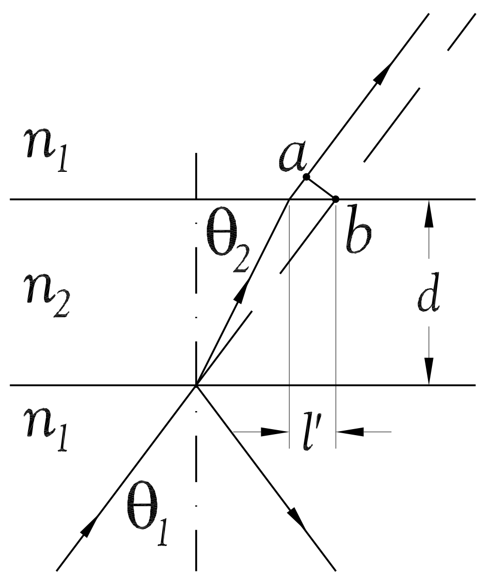
$$2\Delta = \frac{2\pi \cdot 2d}{\lambda_2 \cos\theta_2} - \frac{2\pi l \sin\theta_1}{\lambda_1}, \tag{83}$$

where $l = 2d\tan\theta_2$. The wavelengths are related by $n_1\lambda_1 = n_2\lambda_2$, so eq. (83) becomes

$$2\Delta = \frac{2\pi \cdot 2d}{\lambda_2 \cos\theta_2}\left(1 - \frac{n_1 \sin\theta_1 \tan\theta_2 \cos\theta_2}{n_2}\right) = \frac{4\pi d\cos\theta_2}{\lambda_2}, \tag{84}$$

using Snell's law.

The phase lag for the first transmitted ray is that between points $a$ and $b$ in the figure below:




We note that $l' = d(\tan\theta_1 - \tan\theta_2)$, so that

$$\phi = \frac{2\pi l' \sin\theta_1}{\lambda_1} + \frac{2\pi d}{\lambda_2\cos\theta_2} - \frac{2\pi d}{\lambda_1\cos\theta_1}$$

$$\begin{aligned} &= \frac{2\pi d}{\lambda_2 \cos\theta_2} - \frac{2\pi d \sin\theta_1 \tan\theta_2}{\lambda_1} - \frac{2\pi d}{\lambda_1 \cos\theta_1} + \frac{2\pi d \sin\theta_1 \tan\theta_1}{\lambda_1} \\ &= \Delta - \Delta', \end{aligned} \tag{85}$$

with

$$\Delta' = \frac{2\pi d \cos\theta_1}{\lambda_1}. \tag{86}$$

The total reflected amplitude is

$$\begin{aligned} R &= r + tr't'e^{2i\Delta} + tr'^3t'e^{4i\Delta} + ... = r - |t|^2\, re^{2i\Delta} \sum_{n=0} (r^2 e^{2i\Delta})^2 \\ &= r - \frac{(1-r^2)re^{2i\Delta}}{1-r^2e^{2i\Delta}} = \frac{r(1-e^{2i\Delta})}{1-r^2e^{2i\Delta}}, \end{aligned} \tag{87}$$

using relations (72) and (82).

Similarly, the total transmitted amplitude is

$$T = tt'e^{i(\Delta-\Delta')} \sum_{n=0} (r^2 e^{2i\Delta})^2 = \frac{(1-r^2)e^{i(\Delta-\Delta')}}{1-r^2e^{2i\Delta}}. \tag{88}$$

For a discussion of "tunneling" via the Poynting vector, see D. Mugani, Opt. Comm. **175**, 309 (2000).[4]

[4] http://physics.princeton.edu/~mcdonald/examples/EM/mugnai_oc_175_309_00.pdf

5. (a) The reflected amplitude for the 3-layer medium is

$$\begin{aligned} R &= r_{12} + t_{12}r_{23}t_{21}e^{2i\Delta} + t_{12}r_{23}r_{21}r_{23}t_{21}e^{4i\Delta} + ... \\ &= r_{12} + t_{12}t_{21}r_{23}e^{2i\Delta}\sum_{j=0}\left(r_{21}r_{23}e^{2i\Delta}\right)^j \\ &= r_{12} + t_{12}t_{21}\frac{r_{23}e^{2i\Delta}}{1 - r_{21}r_{23}e^{2i\Delta}}\,. \end{aligned} \tag{89}$$

From prob. 4, $r_{21} = -r_{12}$, and $t_{12}t_{21} = |t_{12}|^2 = 1 - r_{12}^2$, so eq. (89) becomes

$$\begin{aligned} R &= r_{12} + (1 - r_{12}^2)\frac{r_{23}e^{2i\Delta}}{1 + r_{12}r_{23}e^{2i\Delta}} \\ &= \frac{r_{12} + r_{23}e^{2i\Delta}}{1 + r_{12}r_{23}e^{2i\Delta}}\,. \end{aligned} \tag{90}$$

Similarly,

$$\begin{aligned} T &= t_{12}t_{23}e^{i\Delta} + t_{12}e^{3i\Delta} + t_{12}r_{23}r_{21}r_{23}r_{21}t_{23}e^{5i\Delta} + ... \\ &= t_{12}t_{23}e^{i\Delta} + t_{12}r_{23}r_{21}t_{23}e^{3i\Delta}\sum_{j=0}\left(r_{21}r_{23}e^{2i\Delta}\right)^j \\ &= t_{12}t_{23}e^{i\Delta} + t_{12}t_{23}\frac{r_{21}r_{23}e^{3i\Delta}}{1 - r_{21}r_{23}e^{2i\Delta}} \\ &= t_{12}t_{23}e^{i\Delta}\left(1 - \frac{r_{12}r_{23}e^{2i\Delta}}{1 + r_{12}r_{23}e^{2i\Delta}}\right) \\ &= \frac{t_{12}t_{23}e^{i\Delta}}{1 + r_{12}r_{23}e^{2i\Delta}}\,. \end{aligned} \tag{91}$$

At normal incidence the reflected amplitude at the $a$-$b$ boundary is

$$r_{ab} = \frac{n_a - n_b}{n_a + n_b}\,. \tag{92}$$

If $d = \lambda_2/4$, then $\Delta = 2\pi d/\lambda_2 = \pi/2$, and

$$R = \frac{r_{12} - r_{23}}{1 - r_{12}r_{23}}\,, \tag{93}$$

$$r_{12} - r_{23} = \frac{(n_1 - n_2)(n_2 + n_3) - (n_2 - n_3)(n_1 + n_2)}{(n_1 + n_2)(n_2 + n_3)} = \frac{2(n_1n_3 - n_2^2)}{(n_1 + n_2)(n_2 + n_3)}\,, \tag{94}$$

and $R = 0$ if $n_2 = \sqrt{n_1n_3}$.
If $d = \lambda_2/2$, then $\Delta = \pi$, and

$$R = \frac{r_{12} + r_{23}}{1 + r_{12}r_{23}} = \frac{(n_1 - n_2)(n_2 + n_3) + (n_2 - n_3)(n_1 + n_2)}{(n_1 + n_2)(n_2 + n_3) + (n_1 - n_2)(n_2 - n_3)} = \frac{n_1 - n_3}{n_1 + n_3}\,, \tag{95}$$

which is independent of $n_2$, and vanishes if media 1 and 3 are the same.

6. a) The photographic plate intersects the mirror along the line $x = y = 0$. The incident wave of frequency $\omega$ and polarization along the $z$ axis has electric and magnetic fields

$$\mathbf{E}_i = E_0\hat{\mathbf{z}}e^{i(-ky-\omega t)}, \qquad \mathbf{B}_i = -E_0\hat{\mathbf{x}}e^{i(-ky-\omega t)}, \tag{96}$$

where $k = \omega c = 2\pi/\lambda$, $c$ is the speed of light, and $\lambda$ is the wavelength.

The reflected wave has electric field with a 180° phase change so as to satisfy the boundary condition that the tangential electric field vanish at the surface of the mirror. Hence,

$$\mathbf{E}_r = -E_0\hat{\mathbf{z}}e^{i(ky-\omega t)}, \qquad \mathbf{B}_r = -E_0\hat{\mathbf{x}}e^{i(ky-\omega t)}. \tag{97}$$

The total fields are standing waves:

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_r = Re(-2iE_0\hat{\mathbf{z}}\sin kye^{-i\omega t}) = -2E_0\hat{\mathbf{z}}\sin ky \sin\omega t, \tag{98}$$

$$\mathbf{B} = \mathbf{B}_i + \mathbf{B}_r = Re(-2E_0\hat{\mathbf{x}}\cos kye^{-i\omega t}) = -2E_0\hat{\mathbf{x}}\cos ky \cos\omega t. \tag{99}$$

The photographic plated is "exposed" by energy transfer between the electromagnetic fields and the emulsion. Recall that a magnetic field cannot change the energy of a charged particle, while an electric field can. We conclude that it is the electric field (98) whose spatial dependence will determine the pattern of exposure of the photograph. Since the electric field energy depends on $E^2$, the pattern of exposure will actually follow the time average $\langle E^2\rangle \propto \sin^2 ky \propto 1 - \cos 2ky$.

We also recall that the developed photographic "negative" would be black everywhere if it were unexposed. Exposure due to strong electric fields will result in transparent regions on the negative. The blackest stripes on the negative appear where the electric field energy vanishes, *i.e.*, at $y = n\lambda/2$.

For a plate making angle $\alpha$ to the $x$ axis, $y = s\sin\alpha$, where distance $s$ is measured from the edge of the plate in contact with the mirror. Hence, the black stripes on the negative appear at

$$s = \frac{n\lambda}{2\sin\alpha}. \tag{100}$$

If the incident wave had polarization along the $x$ axis, the striping on the negative would be parallel to the $x$ axis with periodicity $\lambda/2$.

b) Now, the incident wave has a 45° angle of incidence, and the plane of incidence is the $x$-$y$ plane.

We first consider the case of polarization perpendicular to the plane of incidence. Then, the incident wave vector is

$$\mathbf{k}_i = \frac{k}{\sqrt{2}}(\hat{\mathbf{x}} - \hat{\mathbf{y}}), \tag{101}$$

and the electromagnetic fields are

$$\mathbf{E}_i = E_0\hat{\mathbf{z}}e^{i(kx/\sqrt{2}-ky/\sqrt{2}-\omega t)}, \tag{102}$$

$$\mathbf{B}_i = \hat{\mathbf{k}}_i \times \mathbf{E}_i = -\frac{E_0}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{y}})e^{i(kx/\sqrt{2}-ky/\sqrt{2}-\omega t)}. \tag{103}$$

The reflected wave has

$$\mathbf{k}_r = \frac{k}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{y}}), \tag{104}$$

and the electromagnetic fields are

$$\mathbf{E}_r = -E_0 \hat{\mathbf{z}} e^{i(kx/\sqrt{2}+ky/\sqrt{2}-\omega t)}, \tag{105}$$

$$\mathbf{B}_r = \hat{\mathbf{k}}_r \times \mathbf{E}_r = -\frac{E_0}{\sqrt{2}}(\hat{\mathbf{x}} - \hat{\mathbf{y}}) e^{i(kx/\sqrt{2}+ky/\sqrt{2}-\omega t)}. \tag{106}$$

The total fields are the waves:

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_i + \mathbf{E}_r = Re\left(-2iE_0\hat{\mathbf{z}}\sin\frac{ky}{\sqrt{2}} e^{i(kx/\sqrt{2}-\omega t)}\right) \\ &= 2E_0\hat{\mathbf{z}}\sin\frac{ky}{\sqrt{2}}\sin(kx/\sqrt{2}-\omega t),\end{aligned} \tag{107}$$

$$\begin{aligned}\mathbf{B} &= \mathbf{B}_i + \mathbf{B}_r = Re\left(-\sqrt{2}E_0\left(\hat{\mathbf{x}}\cos\frac{ky}{\sqrt{2}} + i\hat{\mathbf{y}}\sin\frac{ky}{\sqrt{2}}\right) e^{i(kx/\sqrt{2}-\omega t)}\right) \\ &= -\sqrt{2}E_0\left(\hat{\mathbf{x}}\cos\frac{ky}{\sqrt{2}}\cos(kx/\sqrt{2}-\omega t) + \hat{\mathbf{y}}\sin\frac{ky}{\sqrt{2}}\sin(kx/\sqrt{2}-\omega t)\right).\end{aligned} \tag{108}$$

This is a travelling wave in the $x$ direction, modulated in $y$ by $\sin ky/\sqrt{2}$. Then,

$$\left\langle E^2 \right\rangle \propto \sin^2\frac{ky}{\sqrt{2}} \propto 1 - \cos\sqrt{2}ky, \tag{109}$$

so the dark stripes appear on the plate at positions

$$s = \frac{\sqrt{2}n\lambda}{2\sin\alpha}. \tag{110}$$

For polarization parallel to the plane of incidence,

$$\mathbf{E}_i = \frac{E_0}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{y}}) e^{i(kx/\sqrt{2}-ky/\sqrt{2}-\omega t)}, \tag{111}$$

$$\mathbf{B}_i = \hat{\mathbf{k}}_i \times \mathbf{E}_i = E_0\hat{\mathbf{z}} e^{i(kx/\sqrt{2}-ky/\sqrt{2}-\omega t)}, \tag{112}$$

$$\mathbf{E}_r = -\frac{E_0}{\sqrt{2}}(\hat{\mathbf{x}} - \hat{\mathbf{y}}) e^{i(kx/\sqrt{2}-ky/\sqrt{2}-\omega t)}, \tag{113}$$

$$\mathbf{B}_r = \hat{\mathbf{k}}_r \times \mathbf{E}_r = E_0\hat{\mathbf{z}} e^{i(kx/\sqrt{2}-ky/\sqrt{2}-\omega t)}, \tag{114}$$

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_i + \mathbf{E}_r = Re\left(-\sqrt{2}E_0\left(i\hat{\mathbf{x}}\sin\frac{ky}{\sqrt{2}} - \hat{\mathbf{y}}\cos\frac{ky}{\sqrt{2}}\right) e^{i(kx/\sqrt{2}-\omega t)}\right) \\ &= \sqrt{2}E_0\left(\hat{\mathbf{x}}\sin\frac{ky}{\sqrt{2}}\sin(kx/\sqrt{2}-\omega t) + \hat{\mathbf{y}}\cos\frac{ky}{\sqrt{2}}\cos(kx/\sqrt{2}-\omega t)\right),\end{aligned} \tag{115}$$

$$\begin{aligned}\mathbf{B} &= \mathbf{B}_i + \mathbf{B}_r = Re\left(2E_0\hat{\mathbf{z}}\cos\frac{ky}{\sqrt{2}} e^{i(kx/\sqrt{2}-\omega t)}\right) \\ &= 2E_0\hat{\mathbf{z}}\cos\frac{ky}{\sqrt{2}}\cos(kx/\sqrt{2}-\omega t).\end{aligned} \tag{116}$$

Thus,

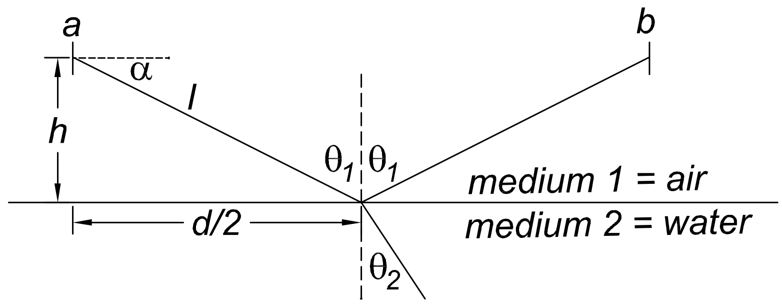
$$\left\langle E^2 \right\rangle \propto \sin^2 \frac{ky}{\sqrt{2}} + \cos^2 \frac{ky}{\sqrt{2}} = 1, \tag{117}$$

so the photographic plate would be uniformly exposed.

7. Since the intensity $I \propto E^2$, and $I \propto 1/r^2$ for a wave that emanates from a localized source, we must have $E \propto 1/r$ for the electric field of the wave. Hence, if we label the field strength of the source at airplane $a$ as $E_a$, the field strength at airplane $b$ is

$$E_b = \frac{E_a}{d}, \tag{118}$$

where $d$ is the distance between the two airplanes, taken to be at the same height $h$.




As shown in the figure, the reflected wave makes angle of incident $\theta_1$ with respect to the surface of the ocean. The reflected angle is, of course, also $\theta_1$, while the transmitted angle $\theta_2$ obeys Snell's law,

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 = \frac{\sin\theta_1}{\sqrt{\epsilon}} = \frac{d/2l}{\sqrt{\epsilon}} = \frac{d}{\sqrt{\epsilon(d^2+4h^2)}}, \tag{119}$$

in the approximation that index $n_1 = 1$ for air. For later use, we note that

$$\tan\theta_1 = \frac{d}{2h}, \tag{120}$$

and

$$\tan\theta_2 = \frac{\sin\theta_2}{\sqrt{1-\sin^2\theta_2}} = \frac{d}{\sqrt{(\epsilon-1)d^2+4\epsilon h^2}}. \tag{121}$$

The amplitude of the wave emitted at airplane $a$ which is reflected by the ocean is smaller than that of the direct wave by the factor $\cos\alpha = \sin\theta_1$.

Likewise, the amplitude of the currents excited in the antenna on airplane $b$ by a wave that makes angle $\alpha$ is smaller by the factor $\cos\alpha$ than that due to the direct wave.

Since the antenna on airplane $a$ is vertical, the polarization of the emitted wave is in a vertical plane, which is also the plane of incidence of the wave with the ocean. Upon reflection, the wave suffers a loss of amplitude described by the ratio

$$\begin{aligned} \left.\frac{E_r}{E_i}\right|_{\parallel} &= -\frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)} = \frac{\tan\theta_2-\tan\theta_1}{\tan\theta_2+\tan\theta_1}\cdot\frac{1-\tan\theta_1\tan\theta_2}{1+\tan\theta_1\tan\theta_2} \\ &= \frac{\sqrt{(\epsilon-1)d^2+4\epsilon h^2}-2\epsilon h}{\sqrt{(\epsilon-1)d^2+4\epsilon h^2}+2\epsilon h}, \end{aligned} \tag{122}$$

according to the Fresnel equation (54), and eqs. (120)-(121).

Also, the path length of the reflected wave between airplanes $a$ and $b$ is $2l$, so the amplitude of the reflected wave has fallen off by factor $1/2l$.

Altogether, the excitation in antenna $b$ due to the reflected wave is proportional to

$$E_R = E_a \cdot \cos\alpha \cdot \frac{1}{2l} \cdot \left.\frac{E_r}{E_i}\right|_{\parallel} \cdot \cos\alpha, \tag{123}$$

while that due to the direct wave is proportional to

$$E_D = E_a \cdot \frac{1}{d}. \tag{124}$$

Therefore, the ratio of the intensity of the reflected to the direct signal is

$$\begin{aligned} \frac{I_R}{I_D} &= \frac{E_R^2}{E_D^2} = \frac{d^2}{4l^2} \sin^4\theta_1 \frac{\tan^2(\theta_1 - \theta_2)}{\tan^2(\theta_1 + \theta_2)} \\ &= \frac{d^6}{(d^2 + 4h^2)^3} \left( \frac{\sqrt{(\epsilon - 1)d^2 + 4\epsilon h^2} - 2\epsilon h}{\sqrt{(\epsilon - 1)d^2 + 4\epsilon h^2} + 2\epsilon h} \right)^2 . \end{aligned} \tag{125}$$

8. (a) The equation of motion of an ionized electron of chage $-e$ and mass $m$ in a circularly polarized plane wave is

$$m\ddot{\mathbf{r}} + e\frac{\dot{\mathbf{r}}}{c} \times B_0\hat{\mathbf{z}} = -eE_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{i(kz-\omega t)}. \tag{126}$$

We seek solutions of a similar form:

$$\mathbf{r}_\pm = r_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})e^{i(kz-\omega t)}. \tag{127}$$

Inserting eq. (127) into (126) we find

$$-m\omega^2 r_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) - i\frac{e\omega B_0}{c} r_0(-\hat{\mathbf{y}} \pm i\hat{\mathbf{x}}) = -eE_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}), \tag{128}$$

$$r_0\left[-m\omega^2 \pm \frac{e\omega B_0}{c}\right](\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) = -eE_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}), \tag{129}$$

and hence,

$$r_0 = \frac{eE_0}{m\omega(\omega \mp \omega_B)}, \tag{130}$$

where

$$\omega_B = \frac{eB_0}{mc}. \tag{131}$$

Note that for propagation antiparallel to the direction of the magnetic field, $\omega_B$ is a negative number.

Since $\mathbf{r}_\pm$ measures the separation of electrons from positive ions, the resulting polarization density is

$$\mathbf{P}_\pm = -Ne\mathbf{r}_\pm = -\frac{Ne^2}{m\omega(\omega \mp \omega_B)}\mathbf{E} \equiv \chi_\pm \mathbf{E}, \tag{132}$$

and the dielectric "constant" is

$$\epsilon_\pm = 1 + 4\pi\chi_\pm = 1 - \frac{4\pi Ne^2}{m\omega(\omega \mp \omega_B)} = 1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_B)}, \tag{133}$$

where the square of the plasma frequency is given by

$$\omega_p^2 = \frac{4\pi Ne^2}{m}. \tag{134}$$

For radio waves with $\omega \ll \omega_B \approx \omega_p$,

$$\epsilon_\pm \approx \pm\frac{\omega_p^2}{\omega\omega_B}. \tag{135}$$

The phase velocity of the plane waves is related by

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{c}{n} = \frac{c}{\sqrt{\epsilon}}. \tag{136}$$

Comparing eqs. (135) and (136) we see that the phase velocity is imaginary for waves with polarization $\hat{\mathbf{x}} - i\,\hat{\mathbf{y}}$, which means that these waves are attenuated rapidly. Only the waves with polarization $\hat{\mathbf{x}}+i\,\hat{\mathbf{y}}$ propagate in the ionosphere, and their phase velocity is

$$v_{\rm phase} = c\frac{\sqrt{\omega\omega_B}}{\omega_p}\,. \tag{137}$$

*For propagation opposite to the direction of the magnetic field, the situation is reversed, and only wave with polarization $\hat{\mathbf{x}} - i\,\hat{\mathbf{y}}$ survive.* For the surviving waves, the wave vector is related to frequency by

$$k = \frac{\omega}{c}\sqrt{\epsilon_+} = \frac{\omega_p}{c}\sqrt{\frac{\omega}{\omega_B}}\,, \tag{138}$$

and so the group velocity is given by

$$v_{\rm group} = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = 2c\frac{\sqrt{\omega\omega_B}}{\omega_p} = 2v_{\rm phase}\,. \tag{139}$$

For waves with $\omega \approx 10^5$ Hz and $\omega_B \approx \omega_p \approx 10^7$ Hz, $v_{\rm phase} \approx c/10$.

The difference in arrival times for pulses centered on frequencies $\omega_1 = 10^5$ and $\omega_2 = 2\times 10^5$ Hz from the opposite side of the Earth ($d = 2\times 10^9$ cm, which used to be the definition of a centimeter) is

$$\begin{aligned}
\Delta t &= \frac{d}{v_{g,1}} - \frac{d}{v_{g,2}} = \frac{d}{2c}\frac{\omega_p}{\sqrt{\omega_2\omega_B}}\left(\sqrt{\frac{\omega_2}{\omega_1}} - 1\right) \\
&= \frac{2\times 10^9}{2\cdot 3\times 10^{10}}\frac{10^7}{\sqrt{2\times 10^5\cdot 10^7}}\left(\sqrt{2}-1\right) \\
&\approx 0.1 \text{ s.}
\end{aligned} \tag{140}$$

(b) For the radio wave to be reflected back downwards, there must be some height $h$ such that $\theta(h) = 90^\circ$. According to Snell,

$$n_i \sin\theta_i = n(h)\sin\theta(h), \tag{141}$$

so with $n_i = 1$, $\sin\theta(h) = 1$, and $n(h) = \sqrt{1-(\omega_p(h)/\omega)^2}$, we need

$$\omega_p(h) = \omega\cos\theta_i. \tag{142}$$

9. Inside the good conductor the plane wave has the form

$$\mathbf{E} = \mathbf{E}_0 e^{-\beta z} e^{i(\beta z - \omega t)}, \qquad \beta = \frac{\sqrt{2\pi\sigma\mu\omega}}{c}, \qquad \mathbf{H} = \sqrt{\frac{2\pi\sigma}{\mu\omega}}(1+i)\hat{\mathbf{z}} \times \mathbf{E}, \tag{143}$$

where $\sigma$ is the conductivity and $\mu$ is the permeability. Hence, the time-averaged Poynting vector is

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi} Re(\mathbf{E} \times \mathbf{H}^\star) = \frac{c}{8\pi}\sqrt{\frac{2\pi\sigma}{\mu\omega}} \left|\mathbf{E}_0\right|^2 e^{-2\beta z}\hat{\mathbf{z}}. \tag{144}$$

The power lost per unit area to Joule heating is

$$\begin{aligned} \langle P \rangle &= \int_0^\infty \langle \mathbf{J} \cdot \mathbf{E} \rangle \; dz = \frac{1}{2}\int_0^\infty Re(\sigma \mathbf{E} \cdot \mathbf{E}^\star) \; dz = \frac{\sigma \left|\mathbf{E}_0\right|^2}{2} \int_0^\infty e^{-2\beta z} \; dz = \frac{\sigma \left|\mathbf{E}_0\right|^2}{4\beta} \\ &= \frac{c\sigma \left|\mathbf{E}_0\right|^2}{4\sqrt{2\pi\sigma\mu\omega}} = \frac{c}{8\pi}\sqrt{\frac{2\pi\sigma}{\mu\omega}} \left|\mathbf{E}_0\right|^2 = \langle \mathbf{S}(z=0) \rangle . \end{aligned} \tag{145}$$

10. Ignoring reflections, the incident power is either absorbed or transmitted, so recalling prob. 8 we can write

$$\begin{aligned}\langle \mathbf{S}_{\rm out}\rangle &= \langle \mathbf{S}_{\rm in}\rangle - \text{Joule heating} = \langle \mathbf{S}_{\rm in}\rangle - \int_0^a \langle \mathbf{J}\cdot\mathbf{E}\rangle\ dz = \frac{c\,|\mathbf{E}_0|^2}{8\pi} - \frac{\sigma\,|\mathbf{E}_0|^2}{2}\int_0^a e^{-2z/d}\ dz \\ &\approx \frac{c\,|\mathbf{E}_0|^2}{8\pi} - \frac{\sigma a\,|\mathbf{E}_0|^2}{2}, \end{aligned} \tag{146}$$

where the approximation holds since $a \ll d$. The relative transmitted intensity is

$$\mathcal{T} = \frac{\langle \mathbf{S}_{\rm out}\rangle}{\langle \mathbf{S}_{\rm in}\rangle} = 1 - \frac{4\pi\sigma a}{c}\,. \tag{147}$$

To analyze the reflected intensity, we first consider the reflected and transmitted amplitudes. In particular, we focus on the magnetic field **H**, since its transverse component is continuous across a metallic boundary. Furthermore, since the thickness of the sheet is much less than the skin depth, the magnitude $H$ is essentially unchanged from one side of the sheet to the other. That is,

$$H_i + H_r \approx H_t, \tag{148}$$

where $i$, $r$ and $t$ indicate incident, reflected and transmitted, respectively. We can deduce $H_t$ from the transmitted intensity ratio $\mathcal{T} = |H_t|^2 / |H_i|^2$,

$$|H_t| = |H_i|\sqrt{\mathcal{T}} \approx |H_i|\left(1 - \frac{2\pi\sigma a}{c}\right)\,. \tag{149}$$

Thus, from eq. (148)

$$H_r \approx -\frac{2\pi\sigma a}{c} H_i, \tag{150}$$

and

$$\mathcal{R} = \frac{|H_r|^2}{|H_i|^2} \approx \left(\frac{2\pi\sigma a}{c}\right)^2. \tag{151}$$

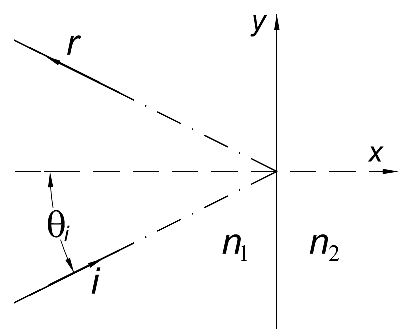
11. According to Snell's law, the angle $\theta_2$ of the transmitted wave obeys the formal relation

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1. \tag{152}$$

For the stated conditions, $\sin\theta_2 > 1$. Then,

$$\cos\theta_2 = \sqrt{1-\sin^2\theta_2} = i\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1}. \tag{153}$$

We take the $x$ axis to be normal to the 1-2 boundary, and the $y$ axis along the boundary in plane of incidence.




Formally, the transmitted wave vector $\mathbf{k}_t$ has components

$$k_x = \frac{n_2\omega}{c}\cos\theta_2 = i\frac{\omega}{c}\sqrt{n_1^2\sin^2\theta_1 - n_2^2} \equiv i\beta, \qquad k_y = \frac{n_2\omega}{c}\sin\theta_2 = \frac{n_1\omega}{c}\sin\theta_1. \tag{154}$$

The space-time dependence of the transmitted wave is therefore

$$e^{i(\mathbf{k}_t\cdot\mathbf{r}-\omega t)} = e^{-\beta x}e^{i(k_y y-\omega t)}, \tag{155}$$

which describes a surface wave that propagates in the $+y$ direction at phase velocity $c/(n_1\sin\theta_1) < c$, and whose amplitude is significant only for $x \lesssim 1/\beta$.

For incident electric field perpendicular to the plane of incidence, the Fresnel equation (48) and eq. (155) tell us that

$$\mathbf{E}_t = E_t\hat{\mathbf{z}} = \frac{2\sin\theta_2\cos\theta_1}{\sin(\theta_1+\theta_2)}E_{0i}\hat{\mathbf{z}}e^{-\beta x}e^{i(k_y y-\omega t)}. \tag{156}$$

We can find the magnetic field via Faraday's equation:

$$\boldsymbol{\nabla}\times\mathbf{E}_t = -\frac{1}{c}\frac{\partial\mathbf{B}_t}{\partial t} = i\frac{\omega}{c}\mathbf{B}_t, \tag{157}$$

so that

$$\mathbf{B}_t = -i\frac{c}{\omega}\hat{\mathbf{x}}\frac{\partial E_z}{\partial y} + i\frac{c}{\omega}\hat{\mathbf{y}}\frac{\partial E_z}{\partial x} = \left(n_1\sin\theta_1\hat{\mathbf{x}} - i\beta\frac{c}{\omega}\hat{\mathbf{y}}\right)E_t. \tag{158}$$

The electric field (156) satisfies $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ since $E_z$ does not depend on $z$. Similarly, the magnetic field (158) satisfies $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, and hence Maxwell's equations are satisfied. The wave equation is also obeyed by eq. (156) since

$$\nabla^2 E_t = (\beta^2 - k_y^2)E_t = \frac{n_2^2\omega^2}{c^2}E_t = \frac{n_2^2}{c^2}\frac{\partial^2 E_t}{\partial t^2}\,. \tag{159}$$

The time-averaged Poynting vector of the transmitted wave is

$$\langle \mathbf{S}_t \rangle = \frac{c}{8\pi} Re(\mathbf{E}_t \times \mathbf{B}_t^\star) = \frac{c}{8\pi} \frac{4\sin^2\theta_2 \cos^2\theta_1}{|\sin(\theta_1+\theta_2)|^2} |E_{0i}|^2 e^{-2\beta x} n_1 \sin\theta_1 \ \hat{\mathbf{y}}. \tag{160}$$

Now,

$$\begin{aligned}
|\sin(\theta_1+\theta_2)|^2 &= |\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2|^2 \\
&= \sin^2\theta_1 |\cos\theta_2|^2 + \cos^2\theta_1 \sin^2\theta_2 \\
&= \sin^2\theta_1(\sin^2\theta_2 - 1) + (1-\sin^2\theta_1)\sin^2\theta_2 \\
&= \sin^2\theta_2 - \sin^2\theta_1 \\
&= \sin^2\theta_2 \left(1 - \frac{n_2^2}{n_1^2}\right),
\end{aligned} \tag{161}$$

so that

$$\langle \mathbf{S}_t \rangle = \frac{c}{2\pi} \frac{n_1}{1-\frac{n_2^2}{n_1^2}} \sin\theta_1 \cos^2\theta_1 |E_{0i}|^2 e^{-2\beta x} \ \hat{\mathbf{y}}. \tag{162}$$

As expected, the Poynting vector is parallel to the $y$ axis, so no energy is transmitted into medium 2. However, the formal result (162) is that a non-negligible energy flows in the thin layer near the bounding surface of medium 2.

For the case of the electric field parallel to the plane of incidence, the Fresnel equation (53) is

$$\frac{E_{0t}(x=0)}{E_{0i}} = \frac{2\sin\theta_2\cos\theta_1}{\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2} \tag{163}$$

However, the transmitted electric field cannot be only in the $x$ direction, as this would not satisfy $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$. There must be a $y$ component as well, as we found for the magnetic field when the electric field was perpendicular to the plane of incidence. Hence, we expect that

$$\mathbf{E}_t = \left(n_1 \sin\theta_1 \hat{\mathbf{x}} - i\beta\frac{c}{\omega}\hat{\mathbf{y}}\right) A e^{-\beta x} e^{i(k_y y - \omega t)}, \tag{164}$$

where $A$ is chosen to satisfy eq. (163) at $x = 0$. That is,

$$\begin{aligned}
(2n_1^2\sin^2\theta_1 - n_2^2)\,|A|^2 &= \frac{4\sin^2\theta_2\cos^2\theta_1}{|\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2|^2} |E_{0i}|^2 \\
&= \frac{4\frac{n_1^2}{n_2^2}\sin^2\theta_1\cos^2\theta_1}{\sin^2\theta_1(1-\sin^2\theta_1) + \frac{n_1^2}{n_2^2}\sin^2\theta_1(\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1)} |E_{0i}|^2 \\
&= \frac{4\cos^2\theta_1}{(1-\frac{n_2^2}{n_1^2})((1+\frac{n_1^2}{n_2^2})\sin^2\theta_1 - 1)} |E_{0i}|^2 .
\end{aligned} \tag{165}$$

Faraday's law (157) gives us the magnetic field as

$$\mathbf{B}_t = i\frac{c}{\omega}\left(\frac{\partial E_{tx}}{\partial y} - \frac{\partial E_{ty}}{\partial x}\right) = -n_2^2 A e^{-\beta x} e^{i(k_y y - \omega t)} \hat{\mathbf{z}}. \tag{166}$$

The time-averaged Poynting vector of the transmitted wave is

$$\begin{aligned}\langle \mathbf{S}_t \rangle &= \frac{c}{8\pi} Re(\mathbf{E}_t \times \mathbf{B}_t^\star) = \frac{c}{8\pi} n_1 n_2^2 \sin\theta_1 \left|A\right|^2 e^{-2\beta x} \hat{\mathbf{y}} \\ &= \frac{c}{2\pi} \frac{n_1 \sin\theta_1 \cos^2\theta_1}{\left(1 - \frac{n_2^2}{n_1^2}\right)\left(\left(1 + \frac{n_1^2}{n_2^2}\right)\sin^2\theta_1 - 1\right)} \frac{\left|E_{0i}\right|^2 e^{-2\beta x} \hat{\mathbf{y}}}{\left(2\frac{n_1^2}{n_2^2}\sin^2\theta_1 - 1\right)} . \end{aligned} \tag{167}$$

Perhaps the most interesting feature of the surface wave for the case of polarization in the plane of incidence is that the electric field (164) includes a component along the direction of propagation of the wave, and the wave velocity is less than $c$. A charged particle moving along with this wave would experience a continual force in the direction of motion, and would therefore be accelerated.

A related concept for particle acceleration by surface waves will be explored in prob. 12.

12. The interaction between particle beams and diffraction gratings was first considered by Smith and Purcell, Phys. Rev. **62**, 1069 (1953),[5] who emphasized energy transfer from the particle to free electromagnetic waves. The excitation of surface waves by particles near conducting structures was first discussed by Pierce, J. Appl. Phys. **26**, 627-638 (1955),[6] which led to the extensive topic of wakefields in particle accelerators. The presence of surface waves in the Smith-Purcell effect was noted by di Francia, Nuovo Cim. **16**, 61-77 (1960).[7] A detailed treatment of surface waves near a diffraction grating was given by van den Berg, Appl. Sci. Res. **24**, 261-293 (1971).[8] Here, we construct a solution containing surface waves by starting with only free waves, then adding surface waves to satisfy the boundary condition at the grating surface.

    If the (perfectly) conducting surface were flat, the reflected wave would be

$$\mathbf{E}_{\rm r} = -E_0 \hat{\mathbf{x}} e^{i(k_y y + k_z z - \omega t)}. \tag{168}$$

    However, the sum $\mathbf{E}_{\rm in} + \mathbf{E}_{\rm r}$ does not satisfy the boundary condition that $\mathbf{E}_{\rm total}$ must be perpendicular to the wavy surface (30). Indeed,

$$[\mathbf{E}_{\rm in} + \mathbf{E}_{\rm r}]_{\rm surface} = 2iE_0 \hat{\mathbf{x}} e^{i(k_y y - \omega t)} \sin k_z z \approx 2iak_z E_0 \hat{\mathbf{x}} e^{i(k_y y - \omega t)} \sin k_x x, \tag{169}$$

    where the approximation holds for $a \ll d$, and we have defined $k_x = 2\pi/d$.

    Hence, we require additional fields near the surface to cancel that given by (169). For $z \approx 0$, these fields therefore have the form

$$\mathbf{E} = -ak_z E_0 \hat{\mathbf{x}} e^{i(k_y y - \omega t)} \left( e^{ik_x x} - e^{-ik_x x} \right). \tag{170}$$

    This can be decomposed into two waves $\mathbf{E}_\pm$ given by

$$\mathbf{E}_\pm = \mp ak_z E_0 \hat{\mathbf{x}} e^{i(\pm k_x x + k_y y - \omega t)}. \tag{171}$$

    Away from the surface, we suppose that the $z$ dependence of the additional waves can be described by including a factor $e^{ik'_z z}$. Then, the full form of the additional waves is

$$\mathbf{E}_\pm = \mp ak_z E_0 \hat{\mathbf{x}} e^{i(\pm k_x x + k_y y + k'_z z - \omega t)}. \tag{172}$$

    The constant $k'_z$ is determined on requiring that each of the additional waves satisfy the wave equation,

$$\nabla^2 \mathbf{E}_\pm = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}_\pm}{\partial t^2}. \tag{173}$$

    This leads to the dispersion relation

$$k_x^2 + k_y^2 + k_z'^2 = \frac{\omega^2}{c^2}. \tag{174}$$

---

[5] http://physics.princeton.edu/~mcdonald/examples/accel/smith_pr_92_1069_53.pdf
[6] http://physics.princeton.edu/~mcdonald/examples/accel/pierce_jap_26_627_55.pdf
[7] http://physics.princeton.edu/~mcdonald/examples/accel/toraldo_di_francia_nc_16_61_60.pdf
[8] http://physics.princeton.edu/~mcdonald/examples/accel/vandenberg_asr_24_261_71.pdf

The component $k_y$ of the incident wave vector can be written in terms of the angle of incidence $\theta_{\rm in}$ and the wavelength $\lambda$ as

$$k_y = \frac{2\pi}{\lambda} \sin\theta_{\rm in}. \tag{175}$$

Combining (174) and (175), we have

$$k'_z = 2\pi i \sqrt{\frac{1}{d^2} - \left(\frac{\cos\theta_{\rm in}}{\lambda}\right)^2}. \tag{176}$$

For short wavelengths, $k'_z$ is real and positive, so the reflected wave (168) is accompanied by two additional plane waves with direction cosines $(k_x, k_y, k'_z)$. But for long enough wavelengths, $k'_z$ is imaginary, and the additional waves are exponentially attenuated in $z$.

When surface waves are present, consider the fields along the line $y = 0$, $z = \pi/2k_z$. Here, the incident plus reflected fields vanish (see the first form of (169)), and the surface waves are

$$\mathbf{E}_\pm = \mp a k_z e^{-\pi|k'_z|/2k_z} E_0 \hat{\mathbf{x}} e^{i(\pm k_x x - \omega t)}. \tag{177}$$

The phase velocity of these waves is

$$v_p = \frac{\omega}{k_x} = \frac{d}{\lambda} c. \tag{178}$$

When $d = \lambda$, the phase velocity is $c$, and $k'_z = i k_y$ according to (176). The surface waves are then,

$$\mathbf{E}_\pm = \mp \frac{2\pi a \cos\theta_{\rm in}}{d} e^{-(\pi/2)\tan\theta_{\rm in}} E_0 \hat{\mathbf{x}} e^{i(\pm k_x x - \omega t)}. \tag{179}$$

A relativistic charged particle that moves in, say, the $+x$ direction remains in phase with the wave $\mathbf{E}_+$, and can extract energy from that wave for phases near $\pi$. On average, the particle's energy is not affected by the counterpropagating wave $\mathbf{E}_-$. In principle, significant particle acceleration can be achieved via this technique. For a small angle of incidence, and with $a/d = 1/2\pi$, the accelerating field strength is equal to that of the incident wave.

13. This problem was abstracted from T.P. Wangler, *Strong focusing and the radiofrequency quadrupole accelerator*, Am. J. Phys. **64**, 177 (1996).[9]

The electric field in the RFQ can be obtained from the potential via $\mathbf{E} = -\boldsymbol{\nabla}\phi$, so

$$E_x = \frac{x}{d} E_0 \sin\omega t, \tag{180}$$

$$E_y = -\frac{y}{d} E_0 \sin\omega t. \tag{181}$$

The equations of motion are

$$\ddot{x} = \frac{x}{d}\frac{eE_0}{m} \sin\omega t, \tag{182}$$

$$\ddot{y} = -\frac{y}{d}\frac{eE_0}{m} \sin\omega t, \tag{183}$$

$$\ddot{z} = 0. \tag{184}$$

Then,

$$z(t) = z_0 + v_{0z}t = v_0 t \tag{185}$$

for the particular case specified.

For the $x$ motion, we consider the form (33),

$$\dot{x} = \dot{f} + \dot{g}\sin\omega t + \omega g \cos\omega t, \tag{186}$$

$$\ddot{x} = \ddot{f} + \ddot{g}\sin\omega t + 2\omega\dot{g}\cos\omega t - \omega^2 g \sin\omega t. \tag{187}$$

The $x$ equation of motion now yields

$$\ddot{f} + 2\omega\dot{g}\cos\omega t = \left[-\ddot{g} + \omega^2 g + \frac{f + g\sin\omega t}{d}\frac{eE_0}{m}\right]\sin\omega t. \tag{188}$$

Since $g$ is both small and slowly varying by hypothesis, we neglect the terms involving $\dot{g}$ and $\ddot{g}$, leaving

$$\ddot{f} \approx \left[\omega^2 g + \frac{f}{d}\frac{eE_0}{m}\right]\sin\omega t + \frac{g}{d}\frac{eE_0}{m}\sin^2\omega t. \tag{189}$$

In this, the coefficent of the rapidly varying term $\sin\omega t$ should vanish, and $\ddot{f}$ should be the average of the term in $\sin^2\omega t$. The first condition tells us that

$$g = -\frac{eE_0}{m\omega^2 d} f, \tag{190}$$

which combines with the (averaged) second condtion to give a differential equation for $f$:

$$\ddot{f} = -\frac{1}{2}\left(\frac{eE_0}{m\omega d}\right)^2 f. \tag{191}$$

[9] http://physics.princeton.edu/~mcdonald/examples/accel/wangler_ajp_64_177_96.pdf

Thus,

$$f \approx A\cos\Omega t + B\sin\Omega t, \qquad \text{where} \qquad \Omega = \frac{eE_0}{\sqrt{2}m\omega d}. \tag{192}$$

Together we have

$$x(t) \approx (A\cos\Omega t + B\sin\Omega t)\left(1 - \frac{eE_0}{m\omega^2 d}\sin\omega t\right). \tag{193}$$

The particular initial conditions (34)-(36) are satisifed by

$$x(t) \approx \frac{v_0\theta_0}{\Omega}\sin\Omega t\left(1 - \frac{eE_0}{m\omega^2 d}\sin\omega t\right). \tag{194}$$

For this to be consistent we must have that

$$\frac{eE_0}{m\omega^2 d} \ll 1. \tag{195}$$

Then, the beam returns to the $z$-axis at time $t = \pi/\Omega$, corresponding to distance $z = \pi v_0/\Omega$.

The argument is similar for the $y$ motion. The opposite sign of the electric field leads to

$$g = +\frac{eE_0}{m\omega^2 d}f, \tag{196}$$

and so

$$y(t) \approx (C\cos\Omega t + D\sin\Omega t)\left(1 + \frac{eE_0}{m\omega^2 d}\sin\omega t\right). \tag{197}$$

The particular initial conditions (34)-(36), however, require that both $C$ and $D$ vanish.

Experts will recognize that the dimensionless quantity

$$\eta \equiv \frac{eE_0}{m\omega c}, \tag{198}$$

where $c$ is the speed of light, is a useful invariant of the field. In terms of this invariant the condition of validity of the solution is

$$\eta\frac{\lambda}{2\pi d} \ll 1. \tag{199}$$

If $d$ is a characteristic aperture of the RFQ, we earlier required that $\lambda \gg d$ so the quasistatic approximation to the fields would be valid. Hence, the invariant field strength $\eta$ cannot be too large in the RFQ.

The physical meaning of the invariant $\eta$ is that it is the ratio of the energy gain over distance $\lambda/2\pi$ to the electron rest energy $mc^2$:

$$\eta = \frac{eE_0}{m\omega c} = \frac{eE_0\lambda/2\pi}{mc^2}. \tag{200}$$

Thus, the RFQ should not impart relativistic transverse motion to the particles if it is to function as described above.

14. a) We ignore the interstellar magnetic field (see prob. 7 for a discussion of the effect of such fields), so the usual analysis of waves incident on free electrons applies:

$$m\ddot{\mathbf{r}} = e\mathbf{E}e^{i(kz-\omega t)}, \tag{201}$$

$$\mathbf{r} = -\frac{e\mathbf{E}}{m\omega^2}\,, \tag{202}$$

$$\mathbf{p} = Ne\mathbf{r} = -N\frac{e^2\mathbf{E}}{m\omega^2} = \chi\mathbf{E}, \tag{203}$$

$$\epsilon = 1 + 4\pi\chi = 1 - \frac{4\pi Ne^2}{m\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\,, \tag{204}$$

where the plasma frequency $\omega_p$ is given by

$$\omega_p^2 = \frac{4\pi Ne^2}{m} = \frac{4\pi Ne^2c^2}{mc^2} = 4\pi Nr_0c^2, \tag{205}$$

where $r_0 = e^2/mc^2 = 2.8 \times 10^{-13}$ cm is the classical electron radius. Then,

$$n(\omega) = \sqrt{\epsilon} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}\,. \tag{206}$$

b) The propagation time for a wave of frequency $\omega$ over distance $L$ is

$$t(\omega) = \frac{n(\omega)L}{c}\,, \tag{207}$$

so the propagation-time difference for frequencies separated by $\delta\omega$ is

$$\delta t = \frac{dt}{d\omega}\delta\omega = \frac{L}{cn}\frac{\omega_p^2}{\omega^2}\frac{\delta\omega}{\omega}. \tag{208}$$

The higher frequency takes longer to arrive.

For the example that $\omega = 2000$ MHz, $\delta\omega/\omega = 0.01$, $N = 0.04$ electrons/cm$^3$, and $|\delta t| = 0.004$ s, we find that

$$\omega_p^2 = 4\pi Nr_0c^2 = 4\pi \cdot 0.04 \cdot 2.8 \times 10^{-13} \cdot (3 \times 10^{10})^2 \approx 1.3 \times 10^8, \tag{209}$$

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \sqrt{1 - \frac{1.3 \times 10^8}{4 \times 10^{18}}} \approx 1, \tag{210}$$

and the distance to the pulsar is

$$L = cn\delta t\frac{\omega^2}{\omega_p^2}\frac{\omega}{\delta\omega} \approx 3 \times 10^{10} \cdot 1 \cdot 0.004 \cdot \frac{4 \times 10^{18}}{1.3 \times 10^8} \cdot 100 \approx 3.7 \times 10^{20} \text{ cm} \approx 400 \text{ light years.} \tag{211}$$