Princeton University

Ph501
Electrodynamics
Problem Set 9

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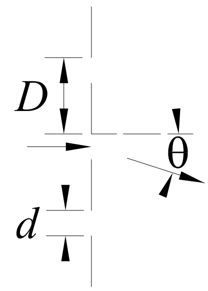
http://kirkmcd.princeton.edu/examples/

1. **Multiple-Slit Diffraction Pattern**

A flat, perfectly absorbing screen has N infinite slits, each of width d, separated by distance D.



Light of wavelength λ is normally incident on the screen.

Show that for Fraunhofer diffraction, the angular distribution of the transmitted intensity, far from the screen, is,

$$I(\theta) = I_0 \left(\frac{\sin(ud)}{ud}\right)^2 \left(\frac{\sin(NuD)}{N\sin(uD)}\right)^2, \qquad \text{where} \qquad u = \frac{\pi}{\lambda}\sin\theta. \tag{1}$$

Sketch this for $N = 4$ and $D = 2d$.

If this "grating" is used to resolve spectral lines of different λ, show that the "resolving power" is $\lambda/\Delta\lambda = ND/\lambda$ according to Rayleigh's criterion.

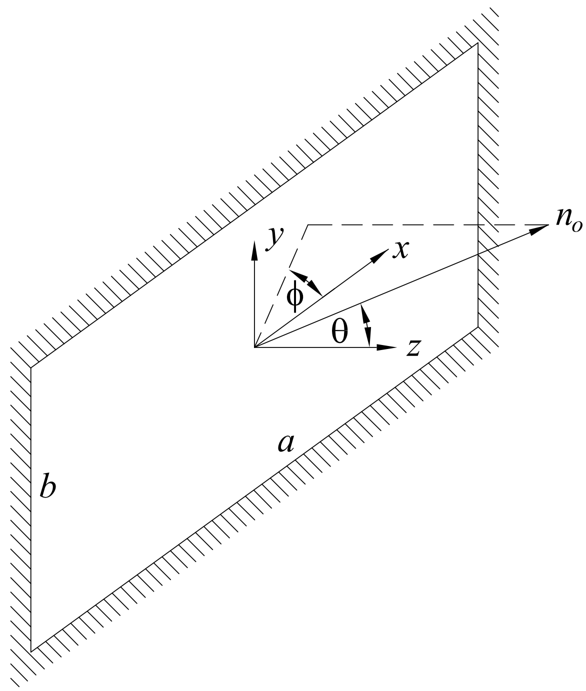
The greatest resolving power involves use of the largest principal maximum visible in the diffraction pattern.

Show that the angle between a principal maximum and the nearest minimum of $I(\theta)$ is $\Delta\theta \approx \lambda/ND$.

2. **Rectangular Aperture**

Show that the Fraunhofer diffraction pattern for plane waves of wavelength λ normally incident on a rectangular aperture of size $a \times b$ in a perfectly absorbing screen is,

$$I(\theta) = I_0 \left(\frac{\sin u}{u}\right)^2 \left(\frac{\sin v}{v}\right)^2, \qquad \text{where} \qquad u = \frac{\pi a}{\lambda}\sin\theta\cos\phi, \qquad v = \frac{\pi b}{\lambda}\sin\theta\sin\phi. \tag{2}$$



3. **Circular Aperture**

A plane wave $\psi = A\,e^{i(kz-\omega t)}$ is incident on an opaque screen at $z = 0$ with a circular aperture of radius a centered on the origin.

Expand the Fraunhofer diffraction integral in a power series, and evaluate it term by term to show that,

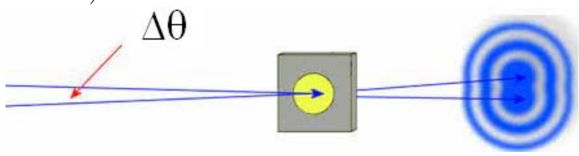
$$\psi \propto \sum_n (-1)^n \frac{(u/2)^{2n}}{n!(n+1)!}, \qquad \text{where} \qquad u = ka\sin\theta. \tag{3}$$

This turns out to be,

$$\psi \propto \frac{J_1(u)}{u}, \tag{4}$$

where J_1 is the ordinary Bessel function of order 1.

Given that the first zero of $J_1(u)$ is at $u = 3.832$, what is the minimum angle between distant sources than can be resolved, when viewed through the circular aperture (which might also contain a lens)?



Ans: $\Delta\theta_{\min} = 1.22\lambda/d$ *according to Rayleigh's criterion, where* $d = 2a$ *is the diameter of the aperture.*

This problem was first solved by Airy, before Bessel functions were well known: G.B. Airy, *On the Diffraction of an Object-glass with Circular Aperture*, Trans. Camb. Phil. Soc. **5**. 283 (1834), http://kirkmcd.princeton.edu/examples/optics/airy_tcps_5-3_283_34.pdf

4. **Partially Opaque Disk**

Suppose a circular disk of radius a absorbs only a fraction η of the amplitude of incident radiation.

Consider Fraunhofer diffraction of normally incident plane waves of wave number k.

Calculate the cross sections $\sigma_{\rm scattering}$, $\sigma_{\rm absorption}$, $\sigma_{\rm total}$ as well as the relative scattering amplitude $f(\theta)$.

Show that,

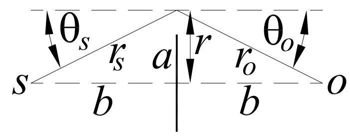
$$\sigma_{\rm total} = \frac{4\pi}{k} {\rm Im} f(0) = 2\pi a^2 \eta, \tag{5}$$

so that the optical theorem holds here.

Hint: Consider Babinet's principle. When calculating $\sigma_{\rm absorption}$ note that the absorbed intensity is that which is not transmitted.

5. **Fresnel Diffraction by an Opaque Circular Disk**

A source s and observer o both lie on the axis of an opaque circular disk of radius a.



Qualitatively, do you expect the observed intensity to increase or decrease as the observer moves slightly off axis?

To be more quantitative, suppose both s and o are at the same distance b from the center of the disk. Include the obliquity factor $(\cos\theta_s + \cos\theta_0/2$ in the Fresnel diffraction integral. Show by appropriate manipulation of the integral into a power series that

$$I(b) \approx \frac{I_0}{4} \frac{b^2}{b^2 + a^2}, \tag{6}$$

where I_0 is the intensity at the edge of the disk.

6. **Time-Reversed Diffraction**

In the usual formulation of the Kirchhoff diffraction integral, a scalar field with harmonic time dependence at frequency ω is deduced at the interior of a charge-free volume from knowledge of the field (or its normal derivative) on the bounding surface. In particular, the field is propagated forwards in time from the boundary to the desired observation point.

Construct a time-reversed version of the Kirchhoff integral in which the knowledge of the field on the boundary is propagated backwards in time into the interior of the volume.

Consider the example of an optical focus at the origin for a system with the z axis as the optic axis. In the far field beyond the focus a Gaussian beam has cone angle $\theta_0 \equiv \sqrt{2}\sigma_\theta$, and the x component of the electric field in a spherical coordinate system is given approximately by,

$$E_x(r, \theta, \phi, t) = E(r) e^{i(kr-\omega t)} e^{-\theta^2/\theta_0^2}, \tag{7}$$

where $k = \omega/c$ and c is the speed of light. Deduce the field near the focus.

Since the Kirchhoff diffraction formalism requires the volume to be charge free, the time-reversed technique is not applicable to cases where the source of the field is inside the volume. Nonetheless, apply the time-reversed diffraction integral to the example of an oscillating dipole at the origin.

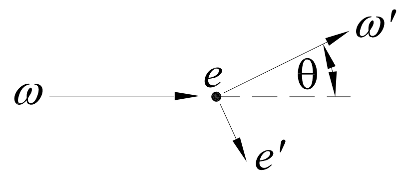
7. **Compton Effect**

In 1905, Einstein suggested that a light wave (in vacuum) of angular frequency ω and direction $\hat{\mathbf{k}}$ can be thought of (in some situations) as consisting of quanta of energy $E = h\nu = \hbar\omega$ and momentum $\mathbf{P} = \hbar\mathbf{k} = \hbar\omega\,\hat{\mathbf{k}}/c$, where c is the speed of light in vacuum, and h is Planck's constant. Then, the energy-momentum 4-vector of such a quantum is,

$$(E, \mathbf{P}c) = (\hbar\omega, \hbar\mathbf{k}c) = \hbar(\omega, \mathbf{k}c) = \hbar\omega(1, \hat{\mathbf{k}})m \tag{8}$$

where $(\omega, \mathbf{k}c) = \omega(1, \hat{\mathbf{k}})$ is the wave 4-vector.

A striking consequence of the light-quantum hypothesis was demonstrated by Compton in 1923, in the scatter of very short wavelength light by "free" electrons at rest.



Suppose a single quantum of light with 4-vector $\hbar(\omega, \mathbf{k}c)$ strikes an electron of rest mass m that is initially at rest. Apply energy and momentum conservation to the 4-vectors involved to show that if the quantum (photon) scatters by angle θ, it emerges with angular frequency,

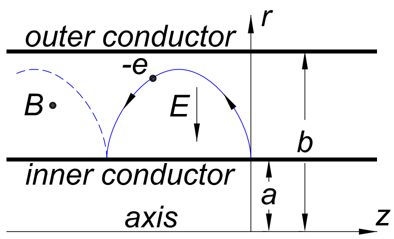
$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)} < \omega. \tag{9}$$

The result that $\omega' < \omega$ is not expected in a classical analysis of light scattering. The observation of this effect by Compton convinced most physicists (including Niels Bohr!) to take the light-quantum hypothesis seriously.

8. A coaxial cable has inner conductor of radius a, outer conductor of radius b, and vacuum between. A constant voltage V is maintained between the conductors, and steady current I flows on the inner conductor (with current $-I$ on the outer conductor). Electrons leave the inner conductor with negligible velocity (due to thermionic emission) and are attracted to the outer conductor. Show that the electrons cannot reach the outer conductor if,

$$I > \frac{cV}{2\ln b/a}\sqrt{1 + \frac{2mc^2}{eV}}, \tag{10}$$

in Gaussian units, where $e > 0$ and m are the charge and mass of the electron and c is the speed of light. The fields due to the electrons in the vacuum can be ignored.

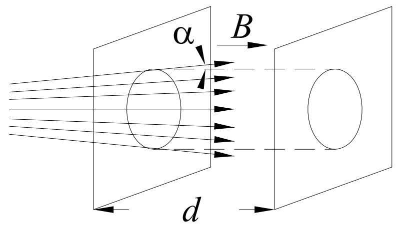


A relativistic analysis adds the term 1 in eq. (10), which is negligible compared to mc^2/eV *in most practical cases.*

The problem can be solved in the lab frame, or by transforming to a moving frame in which one of **E** *or* **B** *is zero.*

9. **Magnetic Lens**

A beam of particles of electric charge e and momentum $|\mathbf{P}| = P$ passes through a circular aperture of radius a. The beam diverges slightly, with maximum angle $\alpha \ll 1$ with respect to the beam axis. To have the beam pass through a second aperture of radius a, at distance d from the first, a uniform, axial magnetic field is applied over that distance.



Show that the minimum field strength to accomplish this is,

$$B_{\min} = \frac{\pi P c}{ed}\left(1 + \frac{2\alpha d}{\pi^2 a}\right). \tag{11}$$

If B is increased above $B_{\min}$, what is the value, $B_{\max}$, for which the beam no longer entirely passes through the second aperture?

There are additional ranges of larger B for which the beam does pass completely through the second aperture.

10. **Fields of a Uniformly Moving Charge**

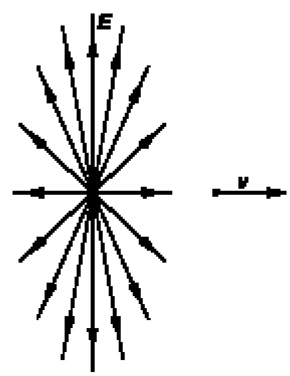
Obtain expressions for the electric and magnetic fields of a charge e moving with uniform velocity $\mathbf{v}$ via a Lorentz transformation of the static electric field of the charge.

Show that,

$$\mathbf{E} = \frac{e\mathbf{R}}{\gamma^2 R^3 (1 - \beta^2 \sin^2\theta)^{3/2}}, \qquad \mathbf{B} = \boldsymbol{\beta} \times \mathbf{E}, \tag{12}$$

where $\boldsymbol{\beta} = \mathbf{v}/c$, $\gamma = 1/\sqrt{1-\beta^2}$, $\mathbf{R}$ is its present position of the charge, and θ is the angle between $\mathbf{v}$ and $\mathbf{R}$.

Thus, $\mathbf{E}$ is radial with respect to the present position of the charge. The magnitude $E(\theta)$ is minimal for $\theta = 0$ and $180°$, and maximal for $\theta = 90°$. Lines of $\mathbf{E}$ are "squeezed" towards the plane $\theta = 90°$.

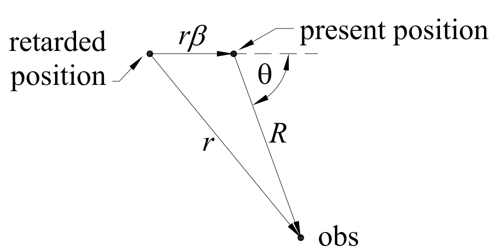


$E(\theta = 0) = e/\gamma^2 R^2 < e/R^2$. How can this be consistent with the transformation $E_\| = E_\|$?

Show also that,

$$\mathbf{E} = \frac{e(\mathbf{r} - r\boldsymbol{\beta})}{\gamma^2 (r - \mathbf{r}\cdot\boldsymbol{\beta})^3}, \qquad \mathbf{B} = \hat{\mathbf{r}} \times \mathbf{E}, \tag{13}$$

where $\mathbf{r}$ is the retarded position of the charge.



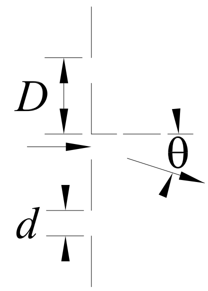
11. Two electrons with equal velocity $\mathbf{v}$ move side by side, separated by distance a. Midway between them is an infinite sheet of surface charge density σ. What must this density be such that the two electrons maintain constant separation a?

    Solve this problem both in the rest frame of the sheet and in the rest frame of the electrons.

Solutions

1. **Multiple-Slit Diffraction Pattern**

A flat, perfectly absorbing screen has N infinite slits, each of width d, separated by distance D.



Light of wavelength $\lambda = 2\pi/k$ is normally incident on the screen. Observations are made at angle θ, at a distance from the screen large compared to λ.

The Fraunhofer diffraction approximation for the transmitted amplitude of a wave incident on a set of N slits in an opaque screen is, from pp. 203-204, Lecture 17 of the Notes, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture17.pdf,

$$\begin{aligned}\psi(\theta) &\propto \int e^{ik\mathbf{x}\cdot(\hat{\mathbf{n}}_s-\hat{\mathbf{n}}_o)}\,dx = \sum_{n=0}^{N-1}\int_{x_{n,\rm lower}}^{x_{n,\rm upper}} e^{ikx\sin\theta}\,dx = \sum_{n=0}^{N-1} e^{iknD\sin\theta}\int_{-d/2}^{d/2} e^{ikx\sin\theta}\,dx \\ &= \frac{e^{ikd\sin\theta/2}-e^{-ikd\sin\theta/2}}{ik\sin\theta}\sum_{n=0}^{N-1} e^{iknD\sin\theta} = \frac{2}{k\sin\theta}\sin\frac{kd\sin\theta}{2}\,\frac{1-e^{ikND\sin\theta}}{1-e^{ikD\sin\theta}} \\ &= d\frac{\sin(ud)}{ud}\,e^{i(N-1)kD\sin\theta/2}\frac{\sin(NkD\sin\theta/2)}{\sin(kD\sin\theta/2)} = d\,e^{i(N-1)uD}\frac{\sin(ud)}{ud}\frac{\sin(NuD)}{\sin(uD)}\,, \qquad (14)\end{aligned}$$

where $u = (\pi/\lambda)\sin\theta = (k/2)\sin\theta$.

The transmitted intensity is,

$$I_N(\theta) = |\psi|^2 \propto d^2\left(\frac{\sin(ud)}{ud}\right)^2\left(\frac{\sin(NuD)}{\sin(uD)}\right)^2, \qquad (15)$$

such that $I_N(0) \equiv I_0 \propto N^2d^2$, and hence,

$$I_N(\theta) = I_0\left(\frac{\sin(ud)}{ud}\right)^2\left(\frac{\sin(NuD)}{N\sin(uD)}\right)^2. \qquad (16)$$

For $N = 1$, we find the single-slit (Fraunhofer) diffraction pattern,

$$I_1(\theta) = I_0\left(\frac{\sin(ud)}{ud}\right)^2. \qquad (17)$$

and hence we can write,

$$I_N(\theta) = I_1(\theta)\left(\frac{\sin(NuD)}{N\sin(uD)}\right)^2. \qquad (18)$$

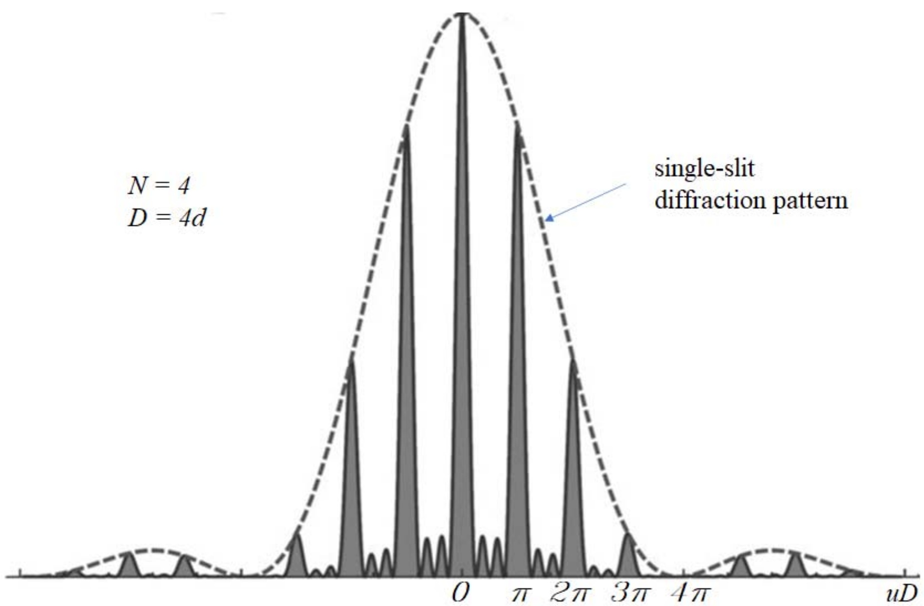
The factor $\sin(NuD)$ vanishes for $uD = l\pi/N$, while $\sin(uD)$ vanishes when $uD = m\pi$, for integers l and m. Whenever $l/N = m$, the ratio $\sin(NuD)/N\sin(uD)$ is just 1, and the diffraction pattern is a maximum (called a principal maximum) rather than a zero.

Between adjacent principal maxima, there are at $N - 1$ zeroes of the pattern, and hence $N - 2$ secondary maxima between adjacent principal maxima.

The rapidly varying function $[\sin(NuD)/N\sin(uD)]^2$ is modulated by the slowly varying single-slit diffraction pattern (17).

Note that if $D = kNd$, then for $uD = m\pi$, where k is a positive integer, we have $ud = m\pi/kN$, and the single-slit pattern has a zero when m is an integer multiple of kN. In this case, the principal maximum of order m is "missing".

The 4-slit diffraction pattern, with $D = 4d$ looks like:



The principal maxima of orders $4k$ are missing.

Show that the angle between a principal maximum and the nearest minimum of $I(\theta)$ is $\Delta\theta \approx \lambda/ND$.

If this "grating" is used to resolve spectral lines of wavelengths λ and $\lambda' = \lambda + \Delta\lambda$, Rayleigh's criterion is that the largest visible maximum for λ' is at the minimum next to the largest visible principal maximum in the pattern for λ.

Recalling that $u = (\pi/\lambda)\sin\theta$, the order $m_{\rm max}$ of the largest visible principal maximum is related by

$$\sin\theta = \frac{\lambda u}{\pi} = \frac{\lambda m_{\rm max}}{D} \approx 1, \qquad m_{\rm max} = \frac{D}{\lambda}. \tag{19}$$

The minimum next to the largest visible principal maximum occurs in the pattern for

λ at,

$$uD = m_{\max}\pi + \frac{\pi}{N}, \qquad \sin\theta = \frac{u\lambda}{\pi} = \frac{\lambda}{D}\left(m_{\max} + \frac{1}{N}\right) = \frac{m_{\max}\lambda}{D}\left(1 + \frac{\lambda}{ND}\right), \quad (20)$$

while the largest visible principal maximum in the pattern for λ' is at,

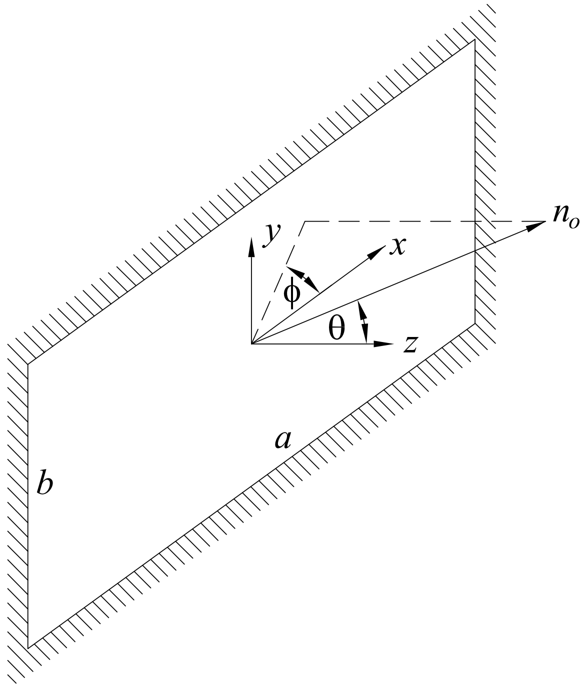
$$m_{\max}\pi = u'D = \frac{\pi D}{\lambda'}\sin\theta', \qquad \sin\theta' = m_{\max}\frac{\lambda'}{D} = m_{\max}\frac{\lambda + \Delta\lambda}{D} = \frac{m_{\max}\lambda}{D}\left(1 + \frac{\Delta\lambda}{\lambda}\right). (21)$$

When angles θ and θ' are the same,

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda}{ND}, \quad (22)$$

which is the resolving power of the grating.

2. **Rectangular Aperture**



The Fraunhofer diffraction approximation for the transmitted amplitude of a wave normally incident on a rectangular aperture of size $a \times b$ in an opaque screen in the plane $z = 0$ is, noting that for a point in the aperture, $\mathbf{x} = x\,\hat{\mathbf{x}} = y\,\hat{\mathbf{y}}$, $\hat{\mathbf{n}}_s = \hat{\mathbf{z}}$ and $\hat{\mathbf{n}}_o = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$,

$$\psi(\theta,\phi) \propto \int e^{ik\mathbf{x}\cdot(\hat{\mathbf{n}}_s - \hat{\mathbf{n}}_o)}\, d\text{Area} = \int_{-a/2}^{a/2} dx\, e^{-ikx\sin\theta\cos\phi} \int_{-b/2}^{b/2} dy\, e^{-iky\sin\theta\sin\phi}$$

$$= \frac{-e^{ika\sin\theta\cos\phi/2} + e^{-ikd\sin\theta\cos\phi/2}}{-ik\sin\theta\cos\phi}\, \frac{-e^{ikb\sin\theta\sin\phi/2} + e^{-ikd\sin\theta\sin\phi/2}}{-ik\sin\theta\sin\phi}$$

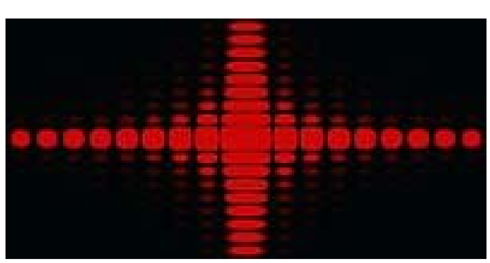
$$= \frac{2}{k\sin\theta\cos\phi}\sin\frac{ka\sin\theta\cos\phi}{2}\,\frac{2}{k\sin\theta\sin\phi}\sin\frac{kb\sin\theta\sin\phi}{2} = ab\frac{\sin u}{u}\frac{\sin v}{v}, \qquad (23)$$

where $u = (\pi/\lambda)a\sin\theta\cos\phi = (ka/2)\sin\theta\cos\phi$ and
$v = (\pi/\lambda)b\sin\theta\sin\phi = (kb/2)\sin\theta\sin\phi$.

The intensity is the square of the diffraction amplitude,

$$I(\theta,\phi) = I_0 \left(\frac{\sin u}{u}\right)^2 \left(\frac{\sin v}{v}\right)^2, \qquad (24)$$

where $I_0 = I(0,0)$.



3. **Circular Aperture**

The Fraunhofer diffraction approximation for the transmitted amplitude of a wave normally incident on a circular aperture of radius a in an opaque screen in the plane $z = 0$ is azimuthally symmetric, so if suffices to consider an observe in the x-z plane at angle θ to the z axis. Noting that for a point in the aperture, $\mathbf{x} = r\,\hat{\mathbf{r}} = r\cos\phi\,\hat{\mathbf{x}} + r\sin\phi\,\hat{\mathbf{y}}$, $\hat{\mathbf{n}}_s = \hat{\mathbf{z}}$ and $\hat{\mathbf{n}}_o = \sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{z}}$, we have,

$$\psi(\theta) \propto \int e^{ik\mathbf{x}\cdot(\hat{\mathbf{n}}_s-\hat{\mathbf{n}}_o)}\,d\text{Area} = \int_0^{2\pi} d\phi \int_0^a r\,dr\,e^{-ikr\sin\theta\cos\phi}$$

$$= \sum_{n=0}^{\infty} \frac{(-ik\sin\theta)^n}{n!} \int_0^{2\pi} d\phi\,\cos^n\phi \int_0^a r^{n+1}\,dr = \sum_{n\ \text{even}} \frac{(-ik\sin\theta)^n}{n!} \frac{2\pi(n-1)!}{2^{n-1}(\frac{n}{2})!(\frac{n-2}{2})!} \frac{a^{n+2}}{n+2}$$

$$= \pi a^2 \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+1)!} \left(\frac{ka\sin\theta}{2}\right)^{2m} = 2\pi a^2 \frac{J_1(u)}{u}\,, \qquad (25)$$

using Dwight 858.44, http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf, where $u = ka\sin\theta$, and J_1 is the ordinary Bessel function of order 1.

Noting that $J_1(u)/u \to 1/2$ as $u \to 0$, the transmitted intensity is.

$$I(\theta) = I_0 \left(\frac{2J_1(u)}{u}\right)^2 \qquad (26)$$

By Rayleigh's criterion, two sources of wavelength λ at $\theta_s = 0$ and $\Delta\theta$ can be resolved via their diffraction pattern as seen through a circular aperture if the first minimum of one pattern coincides with the central bright spot of the other.

That is, for the source at, say angle $\theta_s = \Delta\theta$ in the x-z we need the amplitude to be zero when observed at $\theta = 0 = \theta_o$. The Fraunhofer amplitude for this case is, with $\hat{\mathbf{n}}_s = -\sin\theta_s\,\hat{\mathbf{x}} + \cos\theta_s\,\hat{\mathbf{z}}$ and $\hat{\mathbf{n}}_0 = \hat{\mathbf{z}}$,

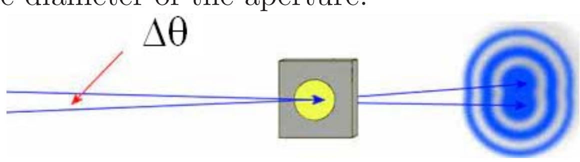
$$\psi(\theta = 0) \propto \int e^{ik\mathbf{x}\cdot(\hat{\mathbf{n}}_s-\hat{\mathbf{n}}_o)}\,d\text{Area} = \int_0^{2\pi} d\phi \int_0^a r\,dr\,e^{-ikr\sin\theta_s\cos\phi} = 2\pi a^2 \frac{J_1(u_s)}{u_s}\,, \qquad (27)$$

as in eq. (25), but now $u_s = ka\sin\theta_s \approx ka\Delta\theta$.

The first zero of $J_1(u)$ is at $u = 3.832$, so the amplitude (27) is zero for $3.823 = u_s \approx 2\pi a\Delta\theta/\lambda$, and hence the two sources can be resolved if,

$$\Delta\theta = \frac{3.823}{\pi}\frac{\lambda}{2a} = 1.22\frac{\lambda}{d}\,, \qquad (28)$$

where $d = 2a$ is the diameter of the aperture.



4. **Partially Opaque Disk**

This problem is based on sec. 8.8, p. 293 of M. Schwartz, *Principles of Electrodynamics* (McGraw-Hill, 1972), http://kirkmcd.princeton.edu/examples/EM/schwartz_72.pdf

The diffraction/scattering of a plane wave normally incident on a disk of radius a that absorbs only a fraction η of the incident amplitude is the complement the case of an opaque screen with a circular aperture of radius a that transmits only fraction η of the incident amplitude.

The complementary problem is the same as Prob. 3 above, but with the diffraction amplitude multiplied by the transmission factor η.

As indicated on p. 207, Lecture 17 of the Notes, we should include the normalization factors in the diffraction amplitude:

$$\psi_1 = \eta A \frac{ka^2}{i} \frac{e^{i(kr-\omega t)}}{r} \frac{J_1(ka\sin\theta)}{ka\sin\theta}\,, \tag{29}$$

where A is the amplitude of the incident plane wave, $\psi_{\rm in} = A\,e^{i(kz-\omega t)}$.

Then, by Babinet's principle the outgoing amplitude for the case of the partially absorbing disk is,

$$\psi_{\rm out} = \psi_{\rm in} - \psi_1. \tag{30}$$

As on p. 207 of the Notes, we write $\psi_{\rm out} = \psi_{\rm in} + \psi_{\rm scat}$, such that,

$$\psi_{\rm scat} = -\psi_1 = i\eta ka^2 A \frac{e^{i(kr-\omega t)}}{r} \frac{J_1(ka\sin\theta)}{ka\sin\theta}\,. \tag{31}$$

The relative scattering amplitude is,

$$f(\theta) = i\eta ka^2 \frac{J_1(ka\sin\theta)}{ka\sin\theta}\,, \qquad f(0) = \frac{i\eta ka^2}{2}\,, \tag{32}$$

recalling that $J_1(u)/u \to 1/2$ as $u \to 0$.

The scattering cross section is,[1]

$$\sigma_{\rm scat} = \int \frac{d\sigma}{d\Omega}\,d\Omega = 2\pi \int_0^\pi |f(\theta)|^2\,d\theta = \eta^2 \pi a^2\,, \tag{33}$$

recalling the factoid at the top of p. 208 of the Notes.

We also consider the absorption cross section, which is just the area of the absorber if it is completely opaque. The partially opaque disk absorbs fraction η of the wave amplitude incident on it, so the transmitted amplitude is $1-\eta$ of the incident amplitude. The fraction of the intensity absorbed is 1 minus the fraction of the transmitted intensity, $1-(1-\eta)^2 = 2\eta - \eta^2$. Hence, the absorption cross section is,

$$\sigma_{\rm abs} = (2\eta - \eta^2)\pi a^2\,. \tag{34}$$

The total cross section is,

$$\sigma_{\rm tot} = \sigma_{\rm scat} + \sigma_{\rm abs} = 2\eta\pi a^2 = \frac{4\pi}{k}{\rm Im} f(0), \tag{35}$$

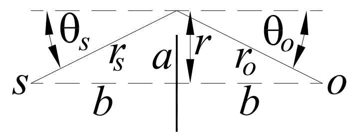
recalling eq. (32). This result is called the optical theorem.

[1]The cross section (33) is sometimes called the elastic cross section.

5. **Fresnel Diffraction by an Opaque Circular Disk**

This problem follows sec. 35C, p. 213 of A. Sommerfeld, *Optics* (Academic Press, 1952)

A source s and observer o both lie on the axis of an opaque circular disk of radius a, at the same distances b from the disk.



The Fresnel diffraction integral, including the obliquity factor and normalization, is

$$\begin{aligned}\psi(a) &= \frac{kA}{2\pi i}\int \frac{e^{ik(r_s+r_0)}}{r_s r_o}\frac{\cos\theta_s + \cos\theta_0}{2}\,d\text{Area} = \int_a^\infty r\,dr,\frac{e^{2ik\sqrt{r^2+b^2}}}{r^2+b^2}\frac{b}{\sqrt{r^2+b^2}} \\ &= \frac{bkA}{i}\int_{\sqrt{a^2+b^2}}^\infty dx\,\frac{e^{2ikx}}{x^2} = \frac{bkA}{i}\left.\frac{e^{2ikx}}{2ikx^2}\right|_{\sqrt{a^2+b^2}}^\infty - \frac{bkA}{i}\int_{\sqrt{a^2+b^2}}^\infty dx\,\frac{-3e^{2ikx}}{ikx^3} \\ &= \frac{bA}{2}\frac{e^{2ik\sqrt{a^2+b^2}}}{a^2+b^2} + \mathcal{O}\left(\frac{1}{(a^2+b^2)^{3/2}}\right), \end{aligned} \tag{36}$$

using $x = \sqrt{r^2+b^2}$, $dx = r\,dr/x$, and noting that integrating the last integral in the second line of eq. (36) by parts over and over again leads to terms of order $1/(a^2+b^2)^{3/2}$ and higher.

The intensity observed at distance a is,

$$I(a) = |\psi(a)|^2 \approx \frac{b^2|A|^2}{4(a^2+b^2)^2} = \frac{I_0}{4}\frac{b^2}{a^2+b^2}, \tag{37}$$

where $I_0 = |A|^2/(a^2+b^2)$ is the intensity at the edge of the disk.

The intensity observed on the axis is nonzero, although the observer is in the nominal "shadow" of the opaque disk.

This result was first predicted by Fresnel in 1818,
http://kirkmcd.princeton.edu/examples/optics/fresnel_acp_11_337_19.pdf
It is counterintuitive, and Poisson immediately insisted that it must be wrong. However, Arago soon demonstrated experimentally that the (Arago/Fresnel/Poisson) bright spot exists, http://kirkmcd.princeton.edu/examples/optics/arago_acp_11_5_19.pdf
This was a "turning point" for the acceptance of the wave theory of light.

There is a bright spot close to the axis of the disk, while the shadow exists beyond a small radius $\ll a$, extending out to radii of order a.

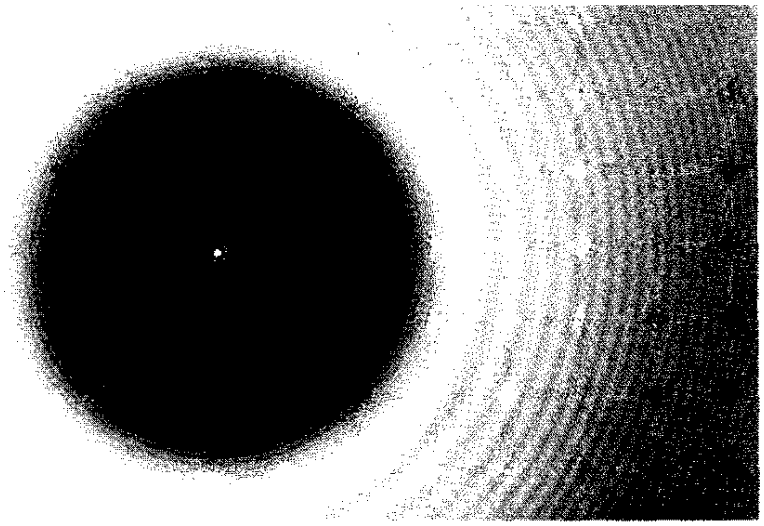


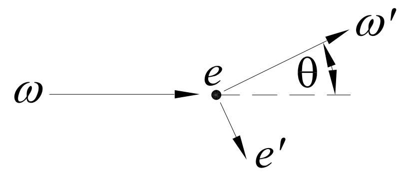
Fig. 1. This photograph shows the appearance of the diffraction pattern due to a penny on a screen which is 20 m from the penny. The source of light is also 20 m from the penny. The Poisson spot is clearly shown in the center of the circular pattern and the distance from this spot to the farthest edge of the photograph is about 5 cm.

From http://kirkmcd.princeton.edu/examples/optics/rinard_ajp_44_70_76.pdf

6. The solution to this problem is at http://kirkmcd.princeton.edu/examples/laserfocus.pdf

7. **Compton Effect**

We write the Compton scattering process of elastic scattering of a photon by an electron as $\omega + e \to \omega' + e'$.



Taking the symbols to represent 4-vectors of the initial and final particles, energy and momentum conservation in the collision can be written as,

$$\omega + e = \omega' + e'. \tag{38}$$

If we only want to deduce the details of the final-state photon, and no of the final-state electron, a useful trick is to rearrange eq. (38) as,

$$e' = \omega + e - \omega', \tag{39}$$

and square this, noting that the square of an energy momentum 4-vector $(E, \mathbf{P}c)$ of a particle of mass m is $E^2 - P^2c^2 = m^2c^4$. Since the rest mass of a photon is zero, and $e^2 = m^2c^4 = e'^2$, squaring eq. (40) gives,

$$e'^2 = \omega^2 + e^2 + \omega'^2 + 2\omega \cdot e - 2\omega' \cdot (\omega + e), \qquad \omega' \cdot (\omega + e) = \omega \cdot e \tag{40}$$

We take the initial photon to have 4-vector $e_\mu = \hbar\omega(1, 0, 0, 1)$, while the 4-vector of the initial electron (at rest) is just $e_\mu = (mc^2, 0, 0, 0)$. Taking the scattering to be in the x=z plane, the 4-vector of the scattered photon is $\omega_\mu = \hbar\omega'(1, \sin\theta, 0, \cos\theta)$. Then the needed 4-vector products are,

$$\omega \cdot e = \hbar\omega mc^2, \qquad \omega' \cdot (\omega + e) = \hbar\omega'(\hbar\omega + mc^2) - \hbar\omega' \cos\theta\, \hbar\omega, \tag{41}$$

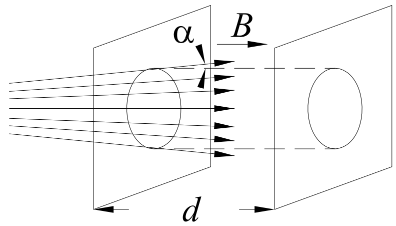
and eq. (40) becomes,

$$\omega = \omega' \left(\frac{\hbar\omega}{mc^2} + 1 - \frac{\hbar\omega}{mc^2} \cos\theta \right), \qquad \omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)}. \tag{42}$$

8. A solution to this problem is at http://kirkmcd.princeton.edu/examples/e_in_coax.pdf.

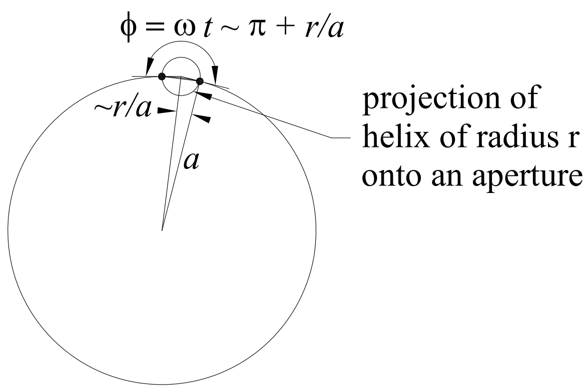
9. **Magnetic Lens**

A beam of relativistic particles of electric charge e and momentum $|\mathbf{P}| = P$ passes through a circular aperture of radius a. The beam diverges slightly, with maximum angle $\alpha \ll 1$ with respect to the beam axis. To have the beam pass through a second aperture of radius a, at distance d from the first, a uniform, axial magnetic field is applied over that distance.



The magnetic field does not change the energy E or magnitude P of the momentum of a charged particle ($dE/dt = \mathbf{F} \cdot \mathbf{v} = e(\mathbf{v}/c \times \mathbf{B}) \cdot \mathbf{v} = 0$), so the direction of $\mathbf{P}$ precesses around the direction of the constant field $\mathbf{B}$. The trajectory of the charged particle (when in the constant field $\mathbf{B}$) is helix, of radius r.

A particle that grazes the edge of the first aperture must precess by at least $\pi + r/a$, but not more than 2π, for $r \ll a$, when it arrives at the second aperture, if it is to pass through that aperture.



For small angles α the axial velocity of a relativistic charged particle is essentially its total velocity, $\approx c$, so the velocity component perpendicular to $\mathbf{B}$ is $v_\perp \approx \alpha c$.

The equation of motion for the precession is,

$$\mathbf{F} = e\frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad F = \frac{\gamma m v_\perp^2}{r} = \frac{e v_\perp B}{c}, \qquad \omega = \frac{v_\perp}{r} = \frac{eB}{\gamma mc} \approx \frac{eB}{P}, \qquad r \approx \frac{\alpha c P}{eB}. \tag{43}$$

The angular velocity ω of the precession (called the cyclotron frequency) is independent of angle α.

The travel time of the particle between the two aperture is $t \approx d/c$, so the minimum B such that the grazing particle passes through the second aperture is related by,

$$\omega t \approx \frac{eB_{\min}}{P}\frac{d}{c} \approx \pi + \frac{r}{a} \approx \pi + \frac{\alpha c P}{aeB_{\min}} \qquad B_{\min} \approx \frac{\pi P c}{ed}\left(1 + \frac{2\alpha d}{\pi^2 a}\right), \tag{44}$$

using the positive root of the quadratic equation for $B_{\rm min}$.

The maximum field strength is roughly twice this (although for $3B_{\rm min} \lesssim B \lesssim 4B_{\rm min}$, *etc.*, the particles also all pass through the second aperture.

10. **Fields of a uniformly moving charge in terms of the present distance**

We consider electric charge e that moves along the x-axis with constant velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$ in the lab (unprimed) frame, is observed at some time, say $t = 0$, when the charge is at the origin, by an observer at $(x_0, y_0, 0)$, at (present) distance from the charge,

$$R = \sqrt{x_0^2 + y_0^2}. \tag{45}$$

We also consider the rest (primed) frame of the charge, where it is at the origin, and whose axes are parallel to those of the lab frame. The Lorentz transformation between these two frames is,

$$\mathbf{x}'_\parallel = \gamma(\mathbf{x}_\parallel - \mathbf{v}t), \qquad \mathbf{x}'_\perp = \mathbf{x}_\perp, \qquad ct' = \gamma(ct - \beta x_\parallel), \qquad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \tag{46}$$

where c is the speed of light in vacuum, and *parallel* means parallel to $\mathbf{v}$, *i.e.*, to the x-axis.

At time $t = 0$, the coordinates of the observer in the rest frame of the charge are $\mathbf{r}' = (\gamma x_0, y_0, 0)$, such that,

$$\begin{aligned} r'^2 = \gamma^2 x_0^2 + y_0^2 = \gamma^2 R^2 + (1-\gamma^2) y_0^2 = \gamma^2 R^2 - \gamma^2\beta^2 y_0^2 = \gamma^2 R^2 \left(1 - \beta^2 \frac{y_0^2}{R^2}\right) \\ = \gamma^2 R^2 (1 - \beta^2 \sin^2\theta), \end{aligned} \tag{47}$$

where θ is the angle between $\mathbf{R}$ and $\mathbf{v}$.

In the rest frame of electric charge e its electromagnetic fields are simply,

$$\mathbf{E}' = \frac{e\mathbf{r}'}{r'^3} = \mathbf{E}'_\parallel + \mathbf{E}'_\perp = \frac{e\mathbf{r}'_\parallel}{r'^3} + \frac{e\mathbf{r}'_\perp}{r'^3}, \qquad \mathbf{B}' = 0\,. \tag{48}$$

where according to eq. (46), $\mathbf{r}'_\parallel = \gamma \mathbf{R}_\parallel$ and $\mathbf{r}'_\perp = \mathbf{R}_\perp$.

The Lorentz transformation of the electromagnetic fields to a frame in which the charge has velocity $\mathbf{v}$ are, p. 221, Lecture 18 of the Notes,

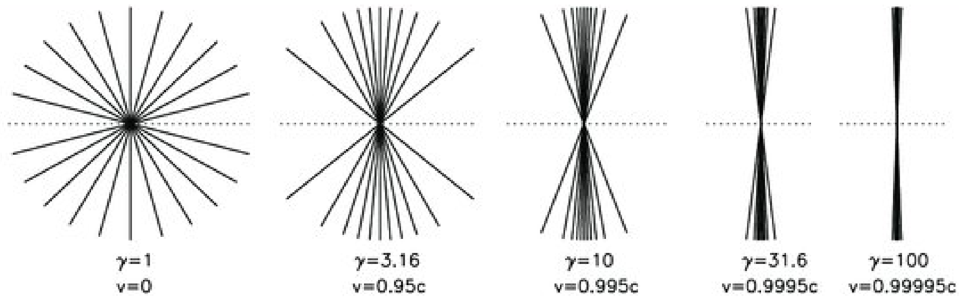
$$\mathbf{E}_\parallel = \mathbf{E}'_\parallel = \frac{e\mathbf{r}'_\parallel}{r'^3} = \frac{\gamma e \mathbf{R}_\parallel}{r'^3}, \qquad \mathbf{E}_\perp = \gamma \mathbf{E}'_\perp = \frac{\gamma e \mathbf{r}'_\perp}{r'^3} = \frac{\gamma e \mathbf{R}_\perp}{r'^3}, \tag{49}$$

$$\mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp = \frac{\gamma e(\mathbf{R}_\parallel + \mathbf{R}_\perp)}{r'^3} = \frac{e\mathbf{R}}{\gamma^2 R^3 (1 - \beta^2 \sin^2\theta)^{3/2}}, \tag{50}$$

$$\mathbf{B}_\parallel = \mathbf{B}'_\parallel = 0, \quad \mathbf{B}_\perp = \gamma(\mathbf{B}'_\perp + \boldsymbol{\beta} \times \mathbf{E}'_\perp) = \boldsymbol{\beta} \times \gamma \mathbf{E}'_\perp = \boldsymbol{\beta} \times \mathbf{E}_\perp = \boldsymbol{\beta} \times \mathbf{E}, \tag{51}$$

$$\mathbf{B} = \boldsymbol{\beta} \times \mathbf{E}, \tag{52}$$

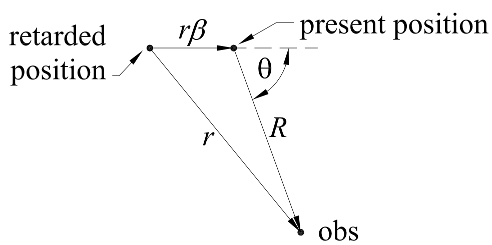
Thus, $\mathbf{E}$ is radial with respect to the present position of the charge. The magnitude $E(\theta)$ is minimal for $\theta = 0$ and $180°$, and maximal for $\theta = 90°$. Lines of $\mathbf{E}$ are "squeezed" towards the plane $\theta = 90°$.



$E(\theta = 0) = e/\gamma^2 R^2 < e/R^2$, which is consistent with eq. (47), in that at $\theta = 0$, the lab frame R is Lorentz contracted from the distance r' in the rest frame, $R = r'/\gamma$.

Fields in terms of the retarded distance

We can also express the fields in terms of the retarded distance between the charge, and the observer, which we write as $\mathbf{r}$. Recall that $r = c(t - t_{\rm ret})$, so the distance $\beta c(t - t_{\rm ret})$ between the present and retarged positions of the charge is just $r\beta$.



From the figure we see that,

$$\mathbf{R} + r\boldsymbol{\beta} + \mathbf{r}, \qquad \mathbf{R} = \mathbf{r} - r\boldsymbol{\beta} = r(\hat{\mathbf{r}} - \boldsymbol{\beta}), \tag{53}$$

$$R^2 = r^2(1 - 2\hat{\mathbf{r}}\cdot\boldsymbol{\beta} + \beta^2), \qquad \mathbf{R}\cdot\boldsymbol{\beta} = r(\hat{\mathbf{r}}\cdot\boldsymbol{\beta} - \beta^2). \tag{54}$$

From eq. (47),[2]

$$\begin{aligned} r'^2 &= \gamma^2 R^2(1 - \beta^2\sin^2\theta) = \gamma^2 R^2(1 - \beta^2 + \beta^2\cos^2\theta) = \gamma^2[R^2(1-\beta^2) + (\mathbf{R}\cdot\boldsymbol{\beta})^2] \\ &= \gamma^2[r^2(1 - 2\hat{\mathbf{r}}\cdot\boldsymbol{\beta} + \beta^2)(1-\beta^2) + r^2((\hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2 - 2(\hat{\mathbf{r}}\cdot\boldsymbol{\beta})\beta^2) + \beta^4)] \\ &= \gamma^2 r^2[1 - 2\hat{\mathbf{r}}\cdot\boldsymbol{\beta} + \beta^2 - \beta^2 + 2(\hat{\mathbf{r}}\cdot\boldsymbol{\beta})\beta^2 - \beta^4 + (\hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2 - 2(\hat{\mathbf{r}}\cdot\boldsymbol{\beta})\beta^2) + \beta^4] \\ &= \gamma^2 r^2[1 - 2\hat{\mathbf{r}}\cdot\boldsymbol{\beta} + (\hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2] = \gamma^2 r^2(1 - \hat{\mathbf{r}}\cdot\boldsymbol{\beta})^2 = [\gamma(r - \mathbf{r}\cdot\boldsymbol{\beta})]^2. \end{aligned} \tag{55}$$

Using eqs. (53) and (55) in eqs. (50) and (52), we have,

$$\mathbf{E} = \frac{e(\mathbf{r} - r\boldsymbol{\beta})}{\gamma^2(r - \mathbf{r}\cdot\boldsymbol{\beta})^3}, \qquad \mathbf{B} = \boldsymbol{\beta}\times\mathbf{E} = \frac{e\boldsymbol{\beta}\times\mathbf{r}}{\gamma^2(r - \mathbf{r}\cdot\boldsymbol{\beta})^3} = \frac{\hat{\mathbf{r}}\times e(\mathbf{r} - r\boldsymbol{\beta})}{\gamma^2(r - \mathbf{r}\cdot\boldsymbol{\beta})^3} = \hat{\mathbf{r}}\times\mathbf{E}, \tag{56}$$

which agree with the Liénard-Weichert fields for a uniformly moving charge, p. 234, Lecture 19 of the Notes.[3]

[2] The result $r' = \gamma(r - \mathbf{r}\cdot\boldsymbol{\beta})$ looks like a Lorentz transformation, but actually is not, as r' is the present distance in the $'$ frame, while r is the retarded distance in the lab frame.

[3] The Lieńard-Wiechert fields can be deduced via a Lorentz transformation, as in kirkmcd.princeton.edu/examples/lw_potentials.pdf.

11. Two electrons, of charge $-e$, with equal velocity $\mathbf{v}$ move side by side, separated by distance a. Midway between them is an infinite sheet of surface charge density σ.

In the rest frame of the sheet, it supports a static electric field $\mathbf{E}_\sigma = \text{sign}(z)2\pi\sigma\,\hat{\mathbf{z}}$, taking the sheet to lie in the plane $z = 0$.

Assuming the electrons move with constant velocity $v\,\hat{\mathbf{x}}$, the fields at the electron with positive z due to the other electron are,

$$\mathbf{E}_e = -\frac{\gamma e\,\hat{\mathbf{z}}}{a^2}, \qquad \mathbf{B}_e = \frac{\gamma e\; v\,\hat{\mathbf{y}}}{a^2 c}, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{57}$$

recalling prob. 10 above. The force on that electron is,

$$\mathbf{F} = -e\left(\mathbf{E}_e + \mathbf{E}_\sigma + \frac{\mathbf{v}}{c} \times \mathbf{B}_e\right) = \left[\frac{\gamma e^2}{a^2}\left(1 - \frac{v^2}{c^2}\right) - 2\pi e\sigma\right]\hat{\mathbf{z}}\,. \tag{58}$$

This force is zero for surface charge density,

$$\sigma = \frac{e}{2\pi\gamma a^2}\,. \tag{59}$$

In the rest frame of the electrons, the fields at the electron with positive z due to the other electron are,

$$\mathbf{E}'_e = -\frac{e\,\hat{\mathbf{z}}}{a^2}, \qquad \mathbf{B}'_e = 0, \tag{60}$$

while the electric fields due to the surface charge density σ, which has velocity $-v\,\hat{\mathbf{x}}$, is, according to the Lorentz transformation of the fields, p. 221, Lecture 18 of the Notes,

$$\mathbf{E}'_\sigma = \gamma\mathbf{E}_\sigma = 2\pi\gamma\sigma\,\hat{\mathbf{z}}. \tag{61}$$

The force on this charge (in its rest frame) is,

$$\mathbf{F}' = -e\,(\mathbf{E}'_e + \mathbf{E}'_\sigma) = \left(\frac{e^2}{a^2} - 2\pi e\gamma\sigma\right)\hat{\mathbf{z}}\,, \tag{62}$$

which is zero for $\sigma = e/2\pi\gamma a^2$, as found in eq. (47) via the analysis in the rest frame of the sheet.