Ph 529 LECTURE 10

CHARGE CONJUGATION ≡ C

SUPPOSE ALL PARTICLES IN THE UNIVERSE WERE REPLACED BY THEIR ANTI-PARTICLES, LEAVING POSITIONS, VELOCITIES AND ANGULAR MOMENTA UNCHANGED. WOULD THE SUBSEQUENT BEHAVIOR OF THE UNIVERSE BE ANY DIFFERENT?

ANTIPARTICLES WERE INTRODUCED AS A CONSEQUENCE OF THE DIRAC EQUATION. THIS EQUATION WOULD INDICATE THAT THE ANTI-UNIVERSE BEHAVES EXACTLY AS THE ORDINARY UNIVERSE. THIS IS THE IDEA OF CHARGE CONJUGATION INVARIANCE. THE NAME "CHARGE CONJUGATION" RATHER THAN 'PARTICLE-ANTIPARTICLE' CONJUGATION COMES FROM THE DAYS WHEN e^+-e^- WAS THE ONLY EXAMPLE OF A PARTICLE-ANTIPARTICLE PAIR.

COMMON SENSE DOES NOT YIELD A DEFINITIVE OPINION ABOUT CHARGE CONJUGATION INVARIANCE - AS THERE SEEM TO BE VERY FEW ANTIPARTICLES AROUND. BUT THIS LAST OBSERVATION INDICATES THAT THE KNOWN UNIVERSE IS NOT CHARGE CONJUGATION SYMMETRIC, WHICH SUGGESTS A VIOLATION SOMEWHERE. NONETHELESS, THE STRONG AND ELECTROMAGNETIC INTERACTIONS APPEAR TO BE COMPLETELY INVARIANT UNDER CHARGE CONJUGATION.

1. TESTS OF CHARGE CONJUGATION INVARIANCE.

THE EXPERIMENTAL EVIDENCE FOR CHARGE CONJUGATION INVARIANCE IN THE STRONG AND ELECTROMAGNETIC INTERACTIONS IS NOT REMARKABLY PRECISE. FOR THE STRONG INTERACTION THE BEST LIMITS COME FROM A COMPARISON OF THE ENERGY SPECTRA OF MULTIPARTICLE FINAL STATES IN $\bar{p}p$ ANNIHILATION,

$$\bar{p}p \to K^0 K^- \pi^+ \quad \text{OR} \quad \bar{K}^0 K^+ \pi^-$$

$$\bar{p}p \to K^0 K^- \pi^0 \pi^+ \quad \text{OR} \quad \bar{K}^0 K^+ \pi^0 \pi^-$$

LIMITS OF ABOUT 1% ACCURACY AGAINST ANY VIOLATION HAVE BEEN SET BY PAIS, P.R.L. 3, 242 (1959) AND DOBRZYNSKI ET AL, PHYS. LETT. 22, 105 (1966).

FOR THE ELECTROMAGNETIC INTERACTION THERE ARE 2 SORTS OF TESTS. CERTAIN PARTICLE DECAYS ARE FORBIDDEN IF C INVARIANCE HOLDS, AS DISCUSSED SHORTLY. THESE INCLUDE

$$\pi^0 \to 3\gamma, \quad \text{EXPT. LIMIT: } < 4\times10^{-7} \text{ OF ALL } \pi^0 \text{ DECAYS}$$

$$e^+e^- \to {}^1S_0 \to 3\gamma, \quad \text{LIMIT: } < 5\times10^{-4}$$

$$\eta \to \begin{cases}\pi^0\gamma \\ \pi^0 e^+e^-\end{cases}, \quad \text{LIMIT: } < 5\times10^{-5}$$

ANOTHER TEST CONCERNS POSSIBLE ASYMMETRIES IN THE FINAL STATE OF THE DECAY $\eta \to \pi^+ \pi^- \pi^0$ OR $\eta \to \pi^+ \pi^- \gamma$

AFTER SOME INITIAL EXCITEMENT IN THE MID 1960'S THESE DECAYS WERE SHOWN TO BE SYMMETRIC TO ABOUT 0.2% ACCURACY.

THE WEAK INTERACTION SHOWS MARKED C VIOLATION AS WELL AS P VIOLATION. THIS IS EVIDENT FROM OUR ARGUMENT ABOUT THE DECAY OF POLARIZED MUONS: $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ DISCUSSED IN LECTURE 3.

THE DECAY CONFIGURATION WHICH GIVES AN ELECTRON OF MAXIMUM POSSIBLE ENERGY IS

CASE I YES

NO CASE II

CASE II IS STRONGLY SUPPRESSED BY THE LEFT-HANDED (PARITY VIOLATING) NATURE OF THE WEAK INTERACTION.

IF C INVARIANCE HELD, WE COULD SIMPLY CHANGE ALL PARTICLES TO ANTI PARTICLES, WHILE KEEPING MOMENTA AND <u>SPINS</u> FIXED.

CASE III = C [CASE I]

THIS CASE, HOWEVER, IS SUPPRESSED, AS THE WEAK INTERACTION OF ANTI PARTICLES IS RIGHT HANDED. THE FAVORED CONFIGURATION FOR μ^+ DECAY TO THE MAXIMUM ENERGY e^+ IS

CASE IV

IT IS INTERESTING TO NOTE THAT CASE IV CAN BE OBTAINED FROM CASE I BY THE COMBINED TRANSFORMATION CP, WHICH REVERSES CHARGES AND MOMENTA, BUT LEAVES SPIN DIRECTIONS UNCHANGED.

$$\text{IV} = P(\text{III}) = PC(\text{I})$$

BETWEEN 1956 AND 1964 IT WAS THOUGHT THAT PERHAPS WHILE THE WEAK INTERACTION VIOLATES P AND C INVARIANCE IT MIGHT OBEY CP INVARIANCE. HOWEVER IN 1964 CRONIN & FITCH, P.R.L. <u>13</u>, 380 (1964), SHOWED THAT CERTAIN K^0 DECAYS VIOLATE CP INVARIANCE ALSO. SUBSEQUENTLY T VIOLATION IN THESE DECAYS WAS DEMONSTRATED INDIRECTLY, SO THAT THE COMBINED SYMMETRY CPT REMAINS THE ONLY COMPLETELY VALID DISCRETE SYMMETRY KNOWN TODAY.

HOWEVER, THE CP VIOLATION IS VERY SLIGHT, AND THUS FAR IS OBSERVED ONLY IN K DECAYS. SO ON THE WHOLE CP REMAINS A USEFUL SYMMETRY WHEN THINKING ABOUT THE WEAK INTERACTION.

2. THE CHARGE CONJUGATION QUANTUM NUMBER.

WE CAN SAY THAT THE CHARGE CONJUGATION OPERATION APPLIED TO A PROTON YIELDS AN ANTI PROTON:

$$C|p\rangle = |\bar{p}\rangle$$

LIKEWISE

$$C|n\rangle = |\bar{n}\rangle$$

$$C|\pi^+\rangle = |\pi^-\rangle$$

NONE OF THESE EXAMPLES ARE EIGENSTATES OF C, AND SO CANNOT BE SAID TO HAVE AN ASSOCIATED QUANTUM NUMBER.

HOWEVER, IF A PARTICLE IS ITS OWN ANTIPARTICLE, WE CAN ASSIGN A QUANTUM NUMBER (EIGENVALUE), USUALLY CALLED C.

i.e. $\underset{\text{OPERATOR}}{C}|\pi^0\rangle = \underset{\text{EIGENVALUE}}{C_{\pi^0}}|\pi^0\rangle \qquad C|\gamma\rangle = C_\gamma|\gamma\rangle \qquad C|\eta\rangle = C_\eta|\eta\rangle$ ETC

SINCE $C[C|\pi^0\rangle] = |\pi^0\rangle$ WE EXPECT $C_{\pi^0}^2 = 1 \Rightarrow C_{\pi^0} = \pm 1$.

THE SIGNS OF THE C QUANTUM NUMBERS ARE ESTABLISHED STARTING WITH THE PHOTON. IF THE OPERATOR C IS APPLIED TO ALL ELECTRIC CHARGES (AND MAGNETIC MOMENTS) THE ELECTROMAGNETIC FIELDS AND POTENTIALS ALL CHANGE SIGN. AS THE PHOTON WAVE FUNCTION IS THE QUANTISATION OF THE 4-POTENTIAL A_μ WE THEN EXPECT

$$C|\gamma\rangle = -|\gamma\rangle \qquad \text{OR} \qquad C_\gamma = -1$$

ANOTHER WAY TO SEE THIS IS TO NOTE THAT THE INTERACTION ENERGY $\gamma_\mu A_\mu$ BETWEEN A CURRENT AND A PHOTON SHOULD BE C INVARIANT.

THEN AS $C\gamma_\mu = -\gamma_\mu$ WE EXPECT $CA_\mu = -A_\mu$ ALSO.

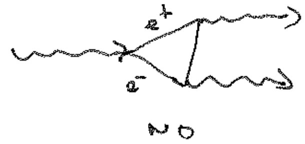
NOTE THAT CLASSICAL CHARGE CONJUGATION WOULD NOT CHANGE THE POLARIZATION OF AN ELECTROMAGNETIC WAVE.

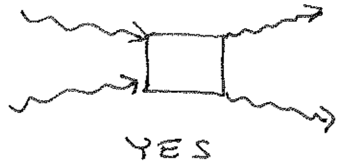
A STRICTLY QUANTUM MECHANICAL RESULT CONCERNS A STATE OF n PHOTONS:

$$C|n\gamma\rangle = (-1)^n |n\gamma\rangle$$

THAT IS, THE C QUANTUM NUMBER IS MULTIPLICATIVE.

AN IMMEDIATE CONSEQUENCE IS THAT NO INTERACTION CAN CONVERT AN EVEN NUMBER OF PHOTONS INTO AN ODD NUMBER OF PHOTONS (UNLESS EXTERNAL PARTICLES ARE PRESENT) (FURRY, 1937)





DELBRUCK SCATTERING OR 'LIGHT-BY-LIGHT' SCATTERING

NOW THE C QUANTUM NUMBER MAY BE ESTABLISHED FOR PARTICLES WHICH DECAY TO PHOTONS. THE DECAYS

$\pi^0 \to \gamma\gamma \qquad \eta \to \gamma\gamma \qquad \eta'(958) \to \gamma\gamma$ ARE ALL OBSERVED

HENCE $\quad C_{\pi^0} = C_\eta = C_{\eta'} = +1$

THEN, AS MENTIONED ON P. 169, $\pi^0 \not\to 3\gamma$ & $\eta \not\to \pi^0\gamma$ ACCORDING TO C INVARIANCE. (NOTE THAT $\gamma \to 3\gamma$ IS CONSISTENT WITH C INVARIANCE!)

3. CHARGE CONJUGATION OF PARTICLE - ANTIPARTICLE PAIRS.

A STATE CONSISTING OF A PARTICLE AND ITS ANTIPARTICLE CAN HAVE A DEFINITE C QUANTUM NUMBER EVEN THO NEITHER PARTICLE NOR ANTIPARTICLE IS ITSELF A C EIGENSTATE.

IT IS NOT IMMEDIATELY OBVIOUS THAT APPLYING C TO A STATE $|a\bar{a}\rangle$ BRINGS US TO $|a\bar{a}\rangle$ AGAIN IF THAT STATE HAS ORBITAL ANGULAR MOMENTUM AND SPIN. RATHER $C|a\bar{a}\rangle = |\bar{a}a\rangle$ WHICH SWAPS THE SPINS, AND THE POSITIONS IN SPACE OF PARTICLE AND ANTIPARTICLE. HOWEVER THE NEW STATE DIFFERS FROM THE ORIGINAL STATE ONLY BY VARIOUS PHASE FACTORS OF ± 1. IN CALCULATING THE C QUANTUM NUMBER WE SIMPLY KEEP TRACK OF ALL THE PHASE CHANGES NEEDED TO RESTORE THE STATE $|\bar{a}a\rangle$ BACK TO $|a\bar{a}\rangle$.

FOR EXAMPLE $\qquad \uparrow a \quad \bar{a}\downarrow \xrightarrow{C} \uparrow\bar{a} \quad a\downarrow$

TO RESTORE THE FINAL STATE BACK TO THE INITIAL WE PROCEED IN 2 STEPS: FIRST INTERCHANGE THE PARTICLE AND ANTIPARTICLE IN SPACE, CARRYING THE SPINS ALONG AS IS

$\uparrow\bar{a} \quad a\downarrow \longrightarrow \downarrow a \quad \bar{a}\uparrow$ $\qquad$ SPATIAL INTERCHANGE

THEN SWAP THE SPINS

$\downarrow a \quad \bar{a}\uparrow \longrightarrow \uparrow a \quad \bar{a}\uparrow$ $\qquad$ SPIN INTERCHANGE

NOTE THAT THE SPATIAL INTERCHANGE IS EXACTLY EQUIVALENT TO A PARITY OPERATION, AS PARTICLE AND ANTIPARTICLE HAVE EQUAL MASSES. HENCE THE PHASE FACTOR ASSOCIATED WITH THE SPATIAL INTERCHANGE IS

$$(-1)^{\ell} \qquad \text{BOSON-ANTIBOSON PAIR}$$

$$(-1)^{\ell+1} \qquad \text{FERMION-ANTIFERMION PAIRS}$$

WHERE ℓ = ORBITAL ANGULAR MOMENTUM (p163)

REGARDING THE PHASE FACTOR ASSOCIATED WITH SPIN INTERCHANGE WE MUST RECALL SOME FACTS ABOUT COMPOSITION OF SPIN STATES.

SUPPOSE THE PARTICLE IS A BOSON $\Rightarrow$ INTEGRAL SPIN. COMBINATION OF INTEGER SPIN STATES YIELDS WAVE FUNCTIONS WHICH ARE SYMMETRIC WITH RESPECT TO SPIN INTERCHANGE FOR EVEN TOTAL S; AND ANTISYMMETRIC FOR ODD S. HENCE THE SPIN INTERCHANGE PHASE FACTOR IS $(-1)^S$ AND SO

$$C = (-1)^{\ell+s} \qquad \text{BOSON-ANTIBOSON.}$$

FOR FERMIONS THE SPIN IS HALF INTEGER. HENCE THE SYMMETRIC SPIN COMBINATIONS OF TWO PARTICLES HAVE ODD INTEGERS, WHILE THE ANTISYMMETRIC COMBINATIONS HAVE EVEN S. HENCE THE PHASE FACTOR IS $(-1)^{S+1}$, BUT

$$C = (-1)^{\ell+S} \qquad \text{FERMION-ANTIFERMION ALSO.}$$

4. POSITRONIUM DECAY.

AN INTERESTING EXAMPLE OF A BOUND PARTICLE-ANTIPARTICLE STATE IS e^+e^- – POSITRONIUM. IN THE SEQUEL WE WILL CONSIDER NEUTRAL MESONS AS QUARK-ANTIQUARK BOUND STATES WITH MANY FEATURES OF THEIR DECAYS SIMILAR TO THAT OF POSITRONIUM.

THERE ARE 2 SIMPLE STATES OF POSITRONIUM, BOTH OF ORBITAL ANGULAR MOMENTUM 0, NAMELY THE SINGLET SPIN STATE $^1S_0$ AND THE TRIPLET SPIN STATE, $^3S_1$.

FROM SEC 3
$$C(^1S_0) = (-1)^{0+0} = 1$$
$$C(^3S_1) = (-1)^{0+1} = -1$$

HENCE $^1S_0 \to 2\gamma$ WHILE $^3S_1 \to 3\gamma$ ARE THE C ALLOWED 'DECAYS' UPON ANNIHILATION OF THE POSITRON AND THE ELECTRON. (WHY CAN'T WE HAVE $^3S_1 \to 1\gamma$?).

IT IS WORTH DIGRESSING TO MAKE A SIMPLE ESTIMATE OF THE LIFETIME OF THESE POSITRONIUM STATES. CALCULATIONS OF A BOUND STATE DECAY USE A SOMEWHAT DIFFERENT LINE OF ARGUMENT FROM THAT PRESENTED IN LECTURE 3. HERE WE HAVE A WAVE FUNCTION FOR THE STATE: $\psi(\bar{r}_1 - \bar{r}_2)$. FOR THE DECAY TO OCCUR, THE POSITRON MUST ANNIHILATE WITH THE ELECTRON, WHICH REQUIRES THAT THEY MEET AT THE ORIGIN $\bar{r} = 0$. FOR THE 2γ DECAY WE ALSO EXPECT A FACTOR OF e FOR EACH PHOTON IN THE MATRIX ELEMENT

HENCE $\quad \text{RATE} = \Gamma \sim e^4 |\psi(0)|^2 = \alpha^2 |\psi(0)|^2$

NOW ψ^2 IS A PROBABILITY PER UNIT VOLUME, AND SO HAS DIMENSIONS $1/(\text{LENGTH})^3$. FOR ELECTRONS AND POSITRONS ANY CHARACTERISTIC LENGTH IS SURELY OF THE FORM $L \sim 1/M_e$ ACCORDING TO DIMENSIONAL ARGUMENTS WHEN $\hbar = c = 1$. BUT POSITRONIUM IS A KIND OF ATOM, SO THE CHARACTERISTIC LENGTH SHOULD ALSO BE THE BOHR RADIUS. IT IS MEMORABLE THAT

$$a_0 = \frac{1}{\alpha M_e} \qquad \left(= \frac{\hbar^2}{M_e e^2}\right)$$

[COMPARE THIS WITH THE OTHER 2 CHARACTERISTIC LENGTHS ASSOCIATED WITH AN ELECTRON:

$$\text{COMPTON WAVELENGTH} = \frac{1}{M_e} \qquad \left(= \frac{\hbar}{M_e c}\right)$$

$$\text{CLASSICAL ELECTRON RADIUS} = \frac{\alpha}{M_e} \qquad \left(= \frac{e^2}{M_e c^2}\right)$$

]

HENCE $\quad \text{RATE} \sim \alpha^2 |\psi(0)|^2 \sim \alpha^5 M_e^3$

BUT RATE HAS DIMENSIONS ENERGY, OR MASS. TO FIX UP THE DIMENSIONS THE ONLY PLAUSIBLE MANEUVER IS TO DIVIDE BY M_e^2.

FINALLY, $\quad \text{RATE}\,({}^1S_0 \to 2\gamma) \sim \alpha^5 M_e$

SIMILARLY $\quad \text{RATE}\,({}^3S_1 \to 3\gamma) \sim \alpha^6 M_e$

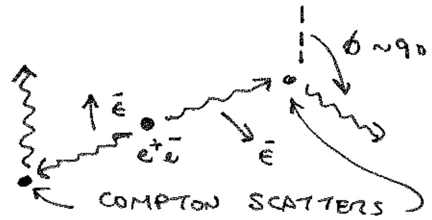
THEN $\quad \tau_{2\gamma} = \dfrac{1}{\text{RATE}} \sim \dfrac{1}{\alpha^5 M_e} \sim \dfrac{10^{10}}{1\,\text{MeV}} \left(10^{-21}\,\text{MeV sec}\right) \sim 10^{-11}\,\text{SEC}$

EXPERIMENTALLY, $\tau_{2\gamma} \sim 1.25 \times 10^{-10}$ SEC, SO WE'RE OFF BY 2π OR SO.

WE ESTIME $\tau_{3\gamma} \sim \dfrac{\tau_{2\gamma}}{\alpha} \sim 10^{-9}$ SEC, COMPARED TO 1.5×10^{-7} SEC. IN FACT.

Both the $^1S_0$ and $^3S_1$ states have negative parity, according to our rules about fermion-antifermion pairs (p163). Hence the 2γ decay of the $^1S_0$ must have the photons in a state of crossed polarization, as for $\pi^0 \to 2\gamma$ (p. 166).

In e^+e^- annihilation the γ's have energy 511 keV $= m_e c^2$. At this energy the Compton scattering of photons off atomic electrons shows strong correlation with the direction of the electric polarization vector $\vec{\varepsilon}$ of the photon.

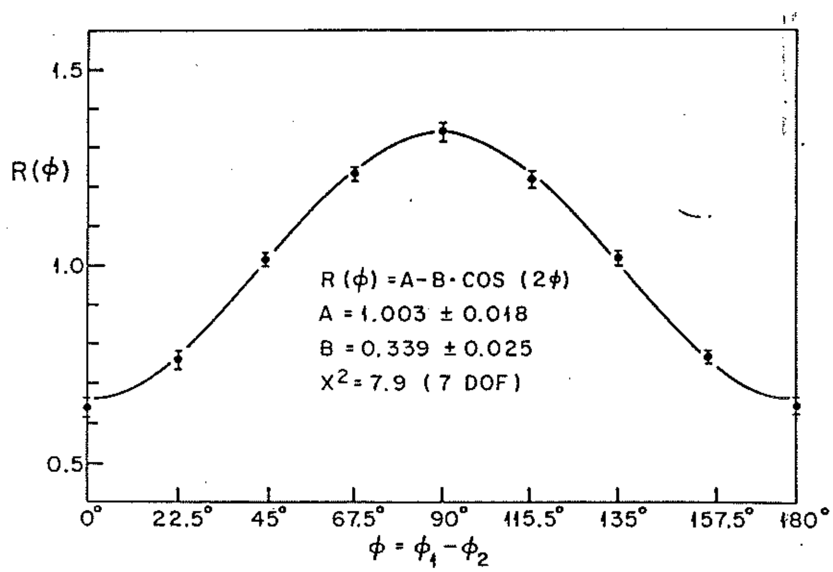


Favored configuration in $e^+e^- \to 2\gamma$ followed by double Compton scattering

If ϕ = azimuthal angle between the photons after each has Compton scattered, we expect a distribution like

$$N(\phi) = A - B \cos(2\phi)$$

This was first well verified by Wu & Shaknov, P.R. <u>77</u>, 136 (1950). The data below are from a generals experiment done at Princeton a few years ago.



While this experiment tests P and C conservation in the electromagnetic interaction, it really tests a much deeper aspect of quantum mechanics: the correlation of a 2 particle wave function over large distances in space and time. In particular, the 2 photon polarizations must be orthogonal no matter what azimuthal direction one chooses to look along. If you observe one polarization at angle ϕ, the other must be at $\phi + 90°$. You can even choose ϕ after the decay has occurred! This experiment has played an important role in the Einstein-Podolsky-Rosen controversy, giving evidence in favor of the 'standard' interpretation of quantum mechanics.

CPT INVARIANCE

WHILE ALL THREE DISCRETE SYMMETRIES C, P AND T ARE VIOLATED BY THE WEAK INTERACTION TO SOME EXTENT, THE COMBINED SYMMETRY CPT APPEARS TO BE WELL OBEYED BY NATURE.

THIS IS COMFORTING AS THERE IS A MUCH STRONGER PREJUDICE IN FAVOR OF CPT INVARIANCE THAN IN ANY OF THE INDIVIDUAL CASES – THE SO-CALLED CPT THEOREM (LÜDERS, 1954). CRUDELY, THE COMBINED TRANSFORM PT TAKES $x_\mu = (t, \vec{r})$ INTO $-x_\mu = (-t, -\vec{r})$, VIA A 'PROPER' LORENTZ TRANSFORMATION OF UNIT DETERMINANT. BECAUSE OF THIS ONE WOULD EXPECT PT TO BE A GOOD TRANSFORMATION, IF IT DID NOT TAKE US INTO AN EXPERIMENTALLY UNREALIZABLE SITUATION. IF WE ADD THE C TRANSFORMATION TO THIS, WE REACH A NEGATIVE ENERGY STATE MOVING BACKWARDS IN TIME (USING THE MOST PRIMITIVE DEFINITION OF C). BUT THIS LAST SITUATION IS CLOSELY IDENTIFIED WITH OUR CONCEPTION OF ANTI PARTICLES, FOLLOWING FEYNMAN. HENCE WE INFER THAT THE INTRINSIC PROPERTIES OF PARTICLE AND ANTI PARTICLE SHOULD BE IDENTICAL (OTHER THAN SIGN OF CHARGE) IF CPT INVARIANCE HOLDS, INDEPENDENT OF HOW PARTICULAR INTERACTIONS TRANSFORM UNDER C, P AND T.

THE LIFETIMES OF π^+ & π^-, μ^+ AND μ^-, K^+ & K^- ARE IDENTICAL TO ABOUT 10^{-3} (NOT A CRITICAL TEST SINCE CP VIOLATION OCCURS AT ABOUT THE 10^{-3} LEVEL.)

THE MAGNETIC MOMENTS OF THE μ^+ & μ^- ARE KNOWN TO BE IDENTICAL TO ABOUT 1 IN 10^9. THIS IS GOOD EVIDENCE THAT ELECTROMAGNETIC EFFECTS ARE CPT INVARIANT.

THE MOST SENSITIVE TEST COMES FROM K^0 DECAYS, WHERE $M_{K_L^0} - M_{K_S^0} \sim 10^{-14} M_{K^0}$. THE LONG AND SHORT-LIVED K^0 MESONS ARE MIXTURES OF THE K^0 AND $\bar{K}^0$, SO WE INFER $M_{K^0} - M_{\bar{K}^0} \lesssim 10^{-14} M_{K^0}$. AS STATED EARLIER, A DETAILED ANALYSIS OF THE CP VIOLATION IN K^0 DECAYS SHOWS NO EVIDENCE FOR CPT VIOLATION, TO ACCURACY $\sim 10^{-3}$

A FOOTNOTE: ALTHOUGH THE LIFETIMES OF A PARTICLE & ITS ANTIPARTICLE MUST BE EQUAL IF CPT HOLDS, THIS NEED NOT BE TRUE FOR PARTICULAR DECAY MODES (S. OKUBO, P.R. 109, 984 (1958)). HE GAVE THE EXAMPLE THAT $\Gamma_{\Sigma^+ \to p \pi^0}$ IS NOT NECESSARILY EQUAL TO $\Gamma_{\bar{\Sigma}^- \to \bar{p} \pi^0}$.

IN PRACTICE, THE ONLY KNOWN EXAMPLE IS $\Gamma_{K^0 \to \pi^+ e^- \bar{\nu}} = \Gamma_{\bar{K}^0 \to \pi^+ e^- \bar{\nu}} \neq \Gamma_{K^0 \to \pi^- e^+ \nu} = \Gamma_{\bar{K}^0 \to \pi^- e^+ \nu}$. THIS IS EVIDENCE FOR CP VIOLATION, BUT NOT CPT VIOLATION (p 336, LECTURE 18)

BARYON CONSERVATION

WE NOW TURN OUR ATTENTION TO THE PHENOMENOLOGY OF THE HADRONS, THOSE PARTICLES WHICH CAN INTERACT VIA THE STRONG INTERACTION.

IN THE 1950'S AS MORE AND MORE HADRONS WERE DISCOVERED, THEY WERE OBSERVED TO FALL INTO 2 CATEGORIES:

- BARYONS, SUCH AS $p, n, \Lambda, \Sigma, \Delta, \ldots \Omega^-$
- MESONS, SUCH AS $\pi, \eta, K, \rho, \omega \ldots \; \psi, \Upsilon \ldots$

A DISTINCTION IS THAT THE NUMBER OF BARYONS IN A REACTION SEEMS TO BE CONSERVED, WHILE THE NUMBER OF MESONS IS NOT. IN THIS ACCOUNTING WE ASSIGN BARYON NUMBER +1 TO EACH BARYON, 0 TO EACH MESON, AND −1 TO EACH ANTI BARYON. THEN WE SAY THAT BARYON NUMBER IS AN ADDITIVE QUANTUM NUMBER.

ALTHOUGH BARYON NUMBER CONSERVATION IS NOT KNOWN TO BE VIOLATED, PEOPLE ARE MADLY SEARCHING FOR PROTON DECAY, OR $n \to \bar{n}$ TRANSITIONS. ALSO, AS WHEELER IS FOND OF REMARKING, BLACK HOLES HAVE VERY LITTLE RESPECT FOR BARYON NUMBER.

SIMPLE EXAMPLES OF BARYON NUMBER CONSERVATION INCLUDE

$p + p \to d + \pi^+$ — THE DEUTERON = $^3S_1$ STATE OF $n\,p$, HAS $N_B = 2$.

$p + p \to p + p + p + \bar{p}$ — THIS IS THE SIMPLEST PP REACTION TO PRODUCE AN ANTIPROTON

$\pi^- + p \to K + \Lambda$ — THIS INDICATES THAT THE Λ HAS $N_B = +1$

NOWADAYS WE SAY THAT QUARKS HAVE BARYON NUMBER $N_B = 1/3$, SO ALL qqq STATES (THE BARYONS) HAVE $N_B = 1$. MESONS ARE $q\bar{q}$ STATES, SO HAVE $N_B = -1$.

IF YOU WISH, WE CAN MAKE A SYMMETRY TRANSFORMATION ASSOCIATED WITH BARYON NUMBER N. JUST DEFINE A (UNITARY) OPERATOR

$$U_N = e^{iN\theta}$$

THEN $U_N|\text{BARYON}\rangle = e^{i\theta}|\text{BARYON}\rangle \qquad U_N|n \text{ BARYONS}\rangle = e^{in\theta}|n \text{ BARYONS}\rangle$

$$U_N|\text{MESON, OR LEPTON}\rangle = |\text{MESON, OR LEPTON}\rangle$$

$$U_N|\text{ANTIBARYON}\rangle = e^{-i\theta}|\text{ANTI BARYON}\rangle$$

IF BARYON CONSERVATION HOLDS, PHYSICS IS INVARIANT UNDER THE PHASE CHANGE IN THE WAVE FUNCTIONS INDUCED BY THE OPERATOR U_N.

THUS WE HAVE COOKED UP A NICE NEW SYMMETRY, SO-CALLED $U(1)_N$. WHAT GOOD IS IT?

ISOSPIN

A VERY USEFUL SYMMETRY, ISOSPIN, WAS INTRODUCED BY HEISENBERG IN 1932, SHORTLY AFTER CHADWICK'S DISCOVERY OF THE NEUTRON. IT IS IMAGINED THAT THE PROTON AND NEUTRON ARE REALLY 2 STATES OF A MORE FUNDAMENTAL 2 COMPONENT ENTITY – THE NUCLEON

$$N = \begin{pmatrix} p \\ n \end{pmatrix}$$

THE INTERESTING SUPPOSITION IS THAT THE STRENGTH OF THE STRONG INTERACTION IS ACTUALLY INDEPENDENT OF THE COMPONENT OF THE NUCLEON (I.E. PROTON OR NEUTRON). HISTORICALLY THIS IDEA IS ALSO CALLED "CHARGE INDEPENDENCE OF THE NUCLEAR FORCE". IT IS CLEAR FROM THE OUTSET THAT THE ELECTROMAGNETIC INTERACTION WILL VIOLATE THE ISOSPIN SYMMETRY (CONTRARY TO PERKINS P31); IT TURNS OUT THAT THE WEAK INTERACTION ALSO VIOLATES ISOSPIN.

THE CLAIM IS THAT AT LEAST THE STRONG INTERACTION IS INVARIANT UNDER A NEW CLASS OF TRANSFORMATIONS: ALL POSSIBLE 2x2 MATRIX OPERATORS ON THE 2 COMPONENT STATE $\begin{pmatrix} p \\ n \end{pmatrix}$ WHICH CONSERVE PROBABILITY.

THAT IS, IF WE START FROM SOME NORMALISED STATE $a|p\rangle + b|n\rangle$, AND TRANSFORM TO SOME OTHER NORMALISED STATE $a'|p\rangle + b'|n\rangle$, THE FORM OF THE STRONG INTERACTIONS OF EITHER STATE WOULD BE THE SAME. (THE OSTENSIBLE BEHAVIOR WOULD DIFFER ACCORDING TO A CHARGE-SENSITIVE OBSERVER – ONE TIME WE HAVE PROTONS, ANOTHER TIME NEUTRONS, BUT REACTION RATES, ANGULAR DISTRIBUTIONS ETC, WOULD BE IDENTICAL.)

THE SET OF ALL 2x2 MATRICES WHICH PRESERVE NORMALISATION IS CALLED THE $U(2)$ GROUP. IT IS POSSIBLE TO PULL AN OVERALL PHASE FACTOR OUT OF EACH 2x2 MATRIX UNTIL IT HAS THE FORM

$U = e^{i\theta} S$, WHERE $\det(S) = +1$. WE RECOGNIZE THE PHASE FACTOR AS THAT ASSOCIATED WITH BARYON CONSERVATION! THE SET OF THE MATRICES S FORM THE SO-CALLED $SU(2)$ SUBGROUP, AND WE SAY THAT ISOSPIN IS AN $SU(2)$ SYMMETRY.

OPPORTUNITIES FOR JARGON LOVERS ABOUND: THE COMBINED SYMMETRIES OF BARYON CONSERVATION AND ISOSPIN COULD BE CALLED

$$SU(2)_I \times U(1)_N.$$

THINKING BACK TO THE CASE OF SPIN, CAN WE IDENTIFY A SYMMETRY ASSOCIATED WITH OVERALL PHASE INVARIANCE OF BOTH MEMBERS OF A SPIN 1/2 DOUBLET, SUCH AS $\begin{pmatrix} e\uparrow \\ e\downarrow \end{pmatrix}$? THIS IS CHARGE CONSERVATION, ANOTHER $U(1)$ SYMMETRY. NOW WE HAVE TOTAL SYMMETRY

$$SU(2)_S \times SU(2)_I \times U(1)_Q \times U(1)_N \times \ldots\ldots$$

Even without the benefit of all this jargon Heisenberg recognized that $\binom{p}{n}$ is just like the 2 spin components of a spin 1/2 object. Hence 'isospin', concocted out of isotope + spin.

Picturesquely, each nucleon has a kind of internal coordinate space attached to it, and a pointer indicating up or down in this space: up → proton, down → neutron. When 2 nucleons interact, their pointers can be rearranged according to rules exactly the same as for composition of spin angular momentum.

Rules of special importance are the isospin conservation laws: both the total I, and I_z (or I_3) component of the isospin of a system of particles are conserved in a strong interaction.

I = label of the multiplet of the combined isospin states

I_3 = total isospin component along the 3rd axis in isospin space (which of course has no relation to the z axis of 'real' space)

The strength of the strong interaction may depend upon I, but not upon I_3 (charge independence).

To take advantage of these rules in practice, one must use the proper decomposition of the given multi-nucleon state into eigenstates of I and I_3. This involves use of tables of Clebsch-Gordon coefficients, such as that appended to this lecture.

Example: Compare $pp \to pp$, $pn \to pn$ and $nn \to nn$ scattering

Clearly $|pp\rangle = |\underset{I}{1}, \underset{I_3}{1}\rangle$ while $|nn\rangle = |1,-1\rangle$

But $|pn\rangle = \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{\sqrt{2}}|0,0\rangle$

The hypothesis of charge independence is that the strong interaction matrix element does not depend on I_3. So we may write $M_1 = \langle 1, I_3 | M | 1, I_3\rangle$ and $M_0 = \langle 0,0 | M | 0,0\rangle$

Then $\sigma_{pp\to pp} = \sigma_{nn\to nn} \sim M_1^2$

But $\sigma_{pn\to pn} \sim \frac{1}{4}|M_1 + M_0|^2$

Isospin invariance alone does <u>not</u> require that $M_1 = M_0$

INTERESTINGLY, AT ENERGIES ABOVE A FEW HUNDRED MEV, $\sigma_{pp} \sim \sigma_{pn}$

AN EXPLANATION OF THIS FACT (SEE NEXT SECTION) MUST INVOKE SOME FEATURE BEYOND ISOSPIN CONSERVATION!

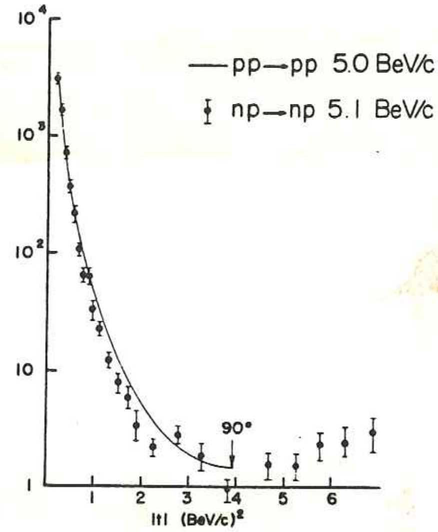


Figure 2.18. Comparison of *pn* and *pp* elastic scattering differential cross sections: from M. Perl, CERN Report 68-7.

ANOTHER ASPECT OF THE ISOSPIN IDEA IS THAT IT ALLOWS AN EXTENSION OF FERMI STATISTICS ARGUMENTS TO COLLECTIONS OF NUCLEONS. NORMALLY THESE ARGUMENTS APPLY ONLY TO IDENTICAL PARTICLES. NOW WE CONSIDER THE PROTON AND NEUTRON AS ASPECTS OF A SINGLE PARTICLE, IN THIS SENSE ANY MULTI-NUCLEON SYSTEM IS A SYSTEM OF IDENTICAL PARTICLES. THE OVERALL WAVE FUNCTION MUST THEN BE ANTI SYMMETRIC, WHERE NOW THE WAVE FUNCTION INCLUDES A FACTOR REFERRING TO THE CONFIGURATION IN ISOSPIN SPACE:

$$\psi_{TOTAL} = \psi_{SPACE} \cdot \psi_{SPIN} \cdot \psi_{ISOSPIN}$$

FOR 2 NUCLEONS, THE $I=1$ COMBINATION IS SYMMETRIC IN ISOSPIN SPACE, THE $I=0$ COMBINATION ANTI SYMMETRIC.
SO THE POSSIBLE SPACE + SPIN WAVE FUNCTIONS MAY BE CATEGORIZED:

$$I=1 \quad {}^1S_0 \quad {}^3P_{0,1,2} \quad {}^1D_2 \quad \ldots$$

$$I=0 \quad {}^3S_1 \quad {}^1P_1 \quad {}^3D_{1,2,3} \quad \ldots$$

THE ${}^3S_1$ DEUTERON HAS NO PARTNERS IN PP OR nn STATES, CONSISTENT WITH ISOSPIN CONSERVATION. NOTE HOWEVER THAT A TOO NIAVE INTERPRETATION OF 'CHARGE INDEPENDENCE' WOULD SUGGEST THE EXISTENCE OF SUCH PARTNERS.

PION ISOSPIN

THE ISOSPIN CONCEPT TAKES ON ADDITIONAL INTEREST IN HIGH ENERGY PHYSICS IN THAT IT IS READILY APPLIED TO THE PI MESON FAMILY. THE TRIPLET $\begin{pmatrix}\pi^+\\ \pi^0 \\ \pi^-\end{pmatrix}$ SEEMS A NATURAL CANDIDATE FOR AN $I=1$ STATE. THE η MESON OF MASS 548 MeV/c² IS A CANDIDATE FOR A RELATED $I=0$ STATE.

THE QUESTION ARISES AS TO WHETHER THE π^+ OR THE π^- SHOULD BE THE $I_3=+1$ COMPONENT. A CONVENTION IS ESTABLISHED BY RELATING ELECTRIC CHARGE TO I_3:

$$Q = I_3 + B/2$$

WHERE B = BARYON NUMBER (CALLED N_B ON p 177)
NOTE THAT ACCORDING TO THIS CONVENTION THE ANTI NUCLEON ISO-DOUBLET WILL BE OF THE FORM $\begin{pmatrix}\bar{n}\\ \bar{p}\end{pmatrix}$.

EVIDENCE THAT THE PION HAS ISOSPIN 1 COMES FROM THE OBSERVATION

$$\frac{\sigma_{p+n\to d+\pi^0}}{\sigma_{p+p\to d+\pi^+}} \sim \frac{1}{2}$$ [HILDEBRAND, P.R. 89, 1090 (1953)]

ASSUMING $I_\pi = 1$, BOTH $|d\pi^0\rangle$ AND $|d\pi^+\rangle$ HAVE $I=1$, AS $I_d=0$

AGAIN $|pp\rangle = |1,1\rangle$ WHILE $|pn\rangle = \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{\sqrt{2}}|0,0\rangle$ (IN ISOSPIN SPACE)

HENCE $\sigma_{pn\to d\pi^0} \sim \frac{1}{2}\,|\langle 1,0|M_1|1,0\rangle|^2 = \frac{1}{2}M_1^2$

WHILE $\sigma_{pp\to d\pi^+} \sim |\langle 1,1|M_1|1,1\rangle|^2 = M_1^2$

EXERCISE: SHOW $\dfrac{\sigma_{p+d\to\pi^+ + t}}{\sigma_{p+d\to\pi^0+He^3}} = 2$ NOTING $\begin{pmatrix}He^3\\ t\end{pmatrix}$ FORM AN ISO-DOUBLET

ANOTHER CLASSIC EXAMPLE CONCERNS $\pi N \to \pi N$ SCATTERING. THERE ARE 6 POSSIBLE INITIAL STATES, WHICH BELONG TO AN $I=3/2$ QUARTET AND AN $I=1/2$ DOUBLET. BY ISOSPIN CONSERVATION THE $I=3/2$ STATES CAN ONLY SCATTERING INTO OTHER $I=3/2$ STATES, AND THE CROSS SECTION WILL BE INDEPENDENT OF I_3. SIMILARLY FOR THE $I=1/2$ STATES. THE ONLY PROBLEM IS THAT A LABORATORY STATE SUCH AS $\pi^- p$ IS NOT A PURE ISOSPIN STATE.

A FAMOUS COMPARISON INVOLVES THE 3 REACTIONS

$\pi^+ p \to \pi^+ p$ (a)

$\pi^- p \to \pi^- p$ (b)

$\pi^- p \to \pi^0 n$ (c)

DATA ON THESE REACTIONS COLLECTED IN THE 1950'S SHOWED EVIDENCE FOR RESONANCE-LIKE BEHAVIOR OF THE CROSS SECTION AS A FUNCTION OF CM ENERGY. THIS WAS INTERPRETED AS EVIDENCE FOR PRODUCTION OF 2 FAMILIES OF EXCITED STATES OF THE NUCLEON. THESE RESONANCES ARE SHORT-LIVED 'PARTICLES' WHICH CAN HAVE QUANTUM NUMBERS OF THEIR OWN.

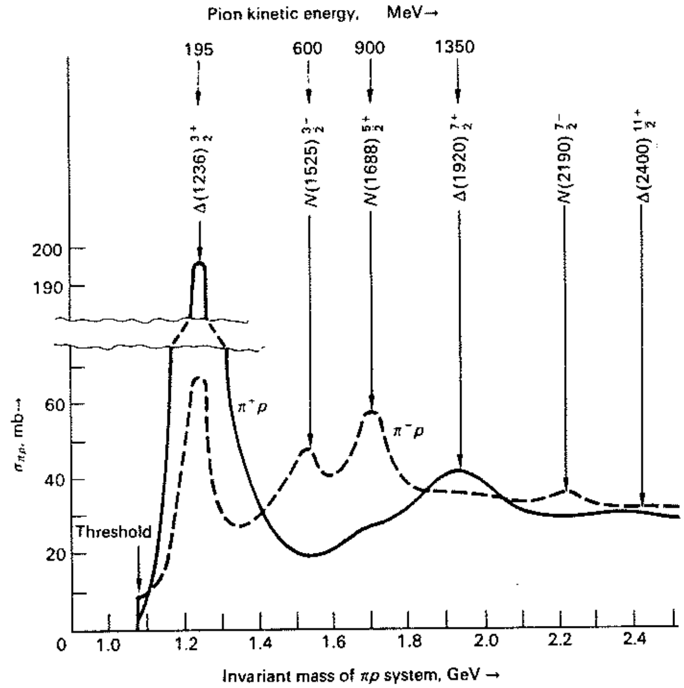


Fig. 4.6 Variation of total cross section for π^+ and π^- mesons on protons, with incident pion energy. The symbol Δ refers to resonances of $I = \frac{3}{2}$; N refers to $I = \frac{1}{2}$. The positions of only a few of the known states, together with their spin-parity assignments, are given.

WE INDICATE HOW THE ISOSPIN OF THE VARIOUS RESONANCES WAS DETERMINED.

WE RELATE THE INITIAL STATES OF THE SCATTERING TO ISOSPIN EIGENSTATES

$$\pi^+ p = |3/2, 3/2\rangle$$

$$\pi^- p = \sqrt{1/3}\,|3/2, -1/2\rangle - \sqrt{2/3}\,|1/2, -1/2\rangle \quad \text{USING THE C-G TABLES}$$

$$\pi^0 n = \sqrt{2/3}\,|3/2, -1/2\rangle + \sqrt{1/3}\,|1/2, -1/2\rangle$$

SO $\sigma_{\pi^+ p \to \pi^+ p} \sim M_{3/2}^2$ WHERE $M_{3/2} \equiv \langle 3/2, I_3 | M_{STRONG} | 3/2, I_3\rangle$ ETC

$$\sigma_{\pi^- p \to \pi^- p} \sim \left| \tfrac{1}{3} M_{3/2} - \tfrac{2}{3} M_{1/2} \right|^2$$

$$\sigma_{\pi^- p \to \pi^0 n} \sim \left| \tfrac{2}{3} M_{3/2} + \tfrac{1}{3} M_{1/2} \right|^2$$

THUS IF $M_{3/2} \gg M_{1/2}$ THEN $\sigma_a : \sigma_b : \sigma_c = 9:1:2$

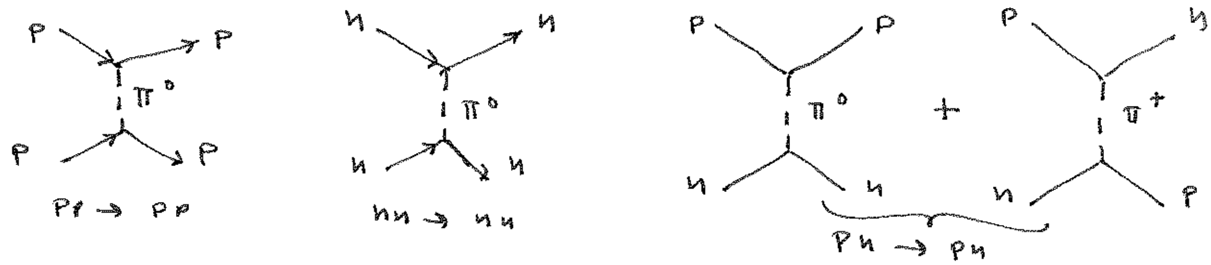
WHILE IF $M_{3/2} \ll M_{1/2}$ THEN $\sigma_a : \sigma_b : \sigma_c = 0:2:1$

COMPARISON WITH THE DATA ALLOWS US TO CATEGORIZE THE RESONANCES $\Delta(1236)$, $\Delta(1920)$ AND $\Delta(2400)$ AS $I = 3/2$ STATES, WHILE $N^*(1525)$, $N^*(1688)$ AND $N^*(2190)$ ARE $I = 1/2$. THESE RESONANCES HAVE BARYON NUMBER 1, AND SO WE EXPECT THE 4 Δ COMPONENTS TO BE Δ^{++}, Δ^+, Δ^0 AND Δ^- ACCORDING TO $Q = I_3 + B/2$.

HOW COULD WE IDENTIFY THE SPINS OF THESE RESONANCES? BASICALLY ONE OBSERVES THE ANGULAR DISTRIBUTIONS OF THE FINAL STATE PARTICLES AND COMPARES WITH THE GENERAL FORMS ALLOWED BY ANGULAR MOMENTUM CONSERVATION. THIS INTERESTING BUT DETAILED PROCESS IS CALLED A PARTIAL WAVE ANALYSIS. WE WILL TRY TO SKETCH THE BASIC FEATURES OF THIS IN LECTURE 11.

AS A 3RD EXAMPLE OF PION ISOSPIN WE RETURN TO THE QUESTION OF N-N SCATTERING (pp 179-80). WE ARE SEEKING AN EXPLANATION OF THE OBSERVED CHARGE INDEPENDENCE: $\sigma_{pp} \sim \sigma_{pn}$.

A SIMPLE MESON THEORY MODEL IS THAT ELASTIC N-N SCATTERING TAKES PLACE VIA SINGLE PION EXCHANGE, IN ANALOGY TO e-P ELASTIC SCATTERING.



AT EACH πNN VERTEX THERE IS A COUPLING CONSTANT $C_{\pi NN}$ — THE ANALOGY OF CHARGE. OF COURSE $C_{\pi NN}$ DOES NOT DEPEND DIRECTLY ON ELECTRIC CHARGE ACCORDING TO THE CHARGE INDEPENDECE OF THE NUCLEAR INTERACTION, BUT IT DOES DEPEND ON THE PARTICULAR πNN COMBINATION ACCORDING TO THE RULES OF ISOSPIN.

$C_{\pi^0 pp}$ INVOLVES A TRANSITION $p \leftrightarrow p\pi^0$ OR $|\tfrac12\ \tfrac12\rangle \leftrightarrow \sqrt{\tfrac23}\,|\tfrac32\ \tfrac12\rangle - \sqrt{\tfrac13}\,|\tfrac12\ \tfrac12\rangle$

$C_{\pi^0 nn}$ $\quad n \leftrightarrow n\pi^0$ OR $|\tfrac12\ {-\tfrac12}\rangle \leftrightarrow \sqrt{\tfrac23}\,|\tfrac32\ {-\tfrac12}\rangle + \sqrt{\tfrac13}\,|\tfrac12\ {-\tfrac12}\rangle$

$C_{\pi^+ pn}$ $\quad p \leftrightarrow n\pi^+$ $\quad |\tfrac12\ \tfrac12\rangle \leftrightarrow \sqrt{\tfrac13}\,|\tfrac32\ \tfrac12\rangle + \sqrt{\tfrac23}\,|\tfrac12\ \tfrac12\rangle$

ISOSPIN IS CONSERVED, SO ALL COUPLINGS ARE ONLY TO THE 2ND TERM OF THE WAVE FUNCTIONS ON THE RHS ABOVE

HENCE $\quad C_{\pi^0 pp} \sim -C \qquad C \equiv \sqrt{\tfrac13}\,\langle \tfrac12\ I_3 | M_{1/2} | \tfrac12\ I_3\rangle$

$C_{\pi^0 nn} \sim +C$

$C_{\pi^+ pn} \sim \sqrt{2}\,C$

AND $\quad \sigma_{pp\to pp} \sim |C_{\pi^0 pp}\cdot C_{\pi^0 pp}|^2 \sim (-c)^4 = c^4$

$\sigma_{nn\to nn} \sim |C_{\pi^0 nn}\,C_{\pi^0 nn}|^2 \sim c^4$

$\sigma_{pn\to pn} \sim |C_{\pi^0 pp}\,C_{\pi^0 nn} + C_{\pi^+ pn}\,C_{\pi^+ pn}|^2 = |-c^2 + (\sqrt{2}c)^2|^2 = c^4$

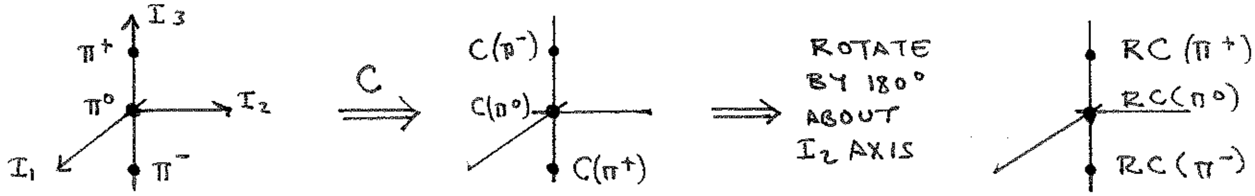
THIS PREDICTION IS AN INTERESTING SUCCESS FOR THE MESON THEORY.

G-PARITY

IN OUR DISCUSSION OF CHARGE CONJUGATION WE FOUND A QUANTUM NUMBER C WHICH COULD BE ASSIGNED TO NEUTRAL STATES WHICH ARE THEIR OWN ANTI PARTICLE. THIS IN TURN GAVE US SELECTION RULES, SUCH AS FOR THE POSSIBLE DECAYS OF $e^+e^- \to n\gamma$.

ISOSPIN ESTABLISHES A RELATION AMONG THE CHARGED AND NEUTRAL PIONS. CAN WE TAKE ADVANTAGE OF THIS TO ESTABLISH RULES FOR SUCH DECAYS AS $N\bar{N} \to n\pi$? TO THIS END LEE & YANG [NUOVO CIMENTO $\underline{3}$, 749 (1956)] INTRODUCED THE CONCEPT OF G-PARITY. THIS IS A WELL-DEFINED TRANSFORMATION IN THE INTERNAL ISOSPIN SPACE - BUT IT IS NOT VERY INTUITIVE.

WE DESIRE AN EXTENSION OF THE CHARGE CONJUGATION OPERATION SUCH THAT π^+ AND π^- CAN BE EIGENSTATES, AS WELL AS π^0. WE PICTURE THIS IN THE ISOSPIN SPACE OF THE PION:



THE G-PARITY TRANSFORMATION $\equiv$ C · ROTATION BY 180° ABOUT I_2

WE SEE THAT $G|\pi^\pm\rangle \sim |\pi^\pm\rangle$, SO THAT $|\pi^+\rangle$ AND $|\pi^-\rangle$ ARE EIGENSTATES. WHAT IS THE CORRESPONDING G QUANTUM NUMBER?

THIS IS BEST ESTABLISHED STARTING WITH THE π^0. RECALL THAT $C|\pi^0\rangle = +|\pi^0\rangle$ AS DEDUCED FROM THE DECAY $\pi^0 \to \gamma\gamma$. REGARDING THE π^0 AS THE STATE $|I=1, I_3=0\rangle$, ITS PROPERTIES UNDER A ROTATION IN ISOSPIN SPACE ARE LIKE THOSE OF $Y_1^0 \sim \cos\theta$ (OR, IF YOU PREFER, RECALL THE SPIN 1 ROTATION MATRIX, P 118)

THUS $R_y(180°)|\pi^0\rangle = -|\pi^0\rangle$

AND $G|\pi^0\rangle = -|\pi^0\rangle$

WE THEN EXPECT $G|\pi^\pm\rangle = -|\pi^\pm\rangle$, AS G SHOULD, BY DEFINITION, TREAT ALL MEMBERS OF AN ISOSPIN MULTIPLET EQUIVALENTLY.

TECHNICALLY WE ARRANGE THIS BY DEFINING $C|\pi^\pm\rangle = -|\pi^\mp\rangle$ RATHER THAN $+|\pi^\mp\rangle$ AS MIGHT BE FIRST EXPECTED.
THIS DEFINITION IS NEEDED BECAUSE $R_y(180°)\left[Y_1^{\pm 1}\right] = +Y_1^{\mp 1}$ AS YOU MIGHT VERIFY.

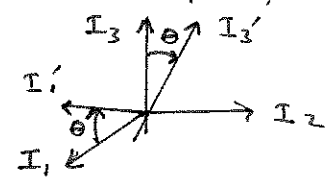
THE G QUANTUM NUMBER IS MULTIPLICATIVE: $G|n\pi\rangle = (-1)^n|n\pi\rangle$
HENCE THERE CAN BE NO TRANSITION FROM A STATE OF AN ODD NUMBER OF PIONS TO A STATE OF AN EVEN NUMBER OF PIONS (ANALAGOUS TO FURRY'S THEOREM FOR γ'S, p. 172).

WE CONTINUE THE LINE OF THOUGHT THAT THE G-PARITY OPERATION GENERATES RULES FOR PION TRANSITIONS SIMILAR TO THE RULES FOR PHOTON TRANSITIONS DUE TO C INVARIANCE. THE CASE OF IMMEDIATE INTEREST CONCERNS TRANSITIONS $N\bar{N} \to n\pi$. TOWARDS THIS END WE NEED TO UNDERSTAND HOW THE NUCLEON AND ANTINUCLEON TRANSFORM UNDER G-PARITY.

ISOSPIN WAS INTRODUCED BY ASSIGNING THE NUCLEON AS A 2 COMPONENT STATE OF $I = 1/2$: $\begin{pmatrix} p \\ n \end{pmatrix}$. WE WISH TO SHOW THAT THE ANTINUCLEON ISOSPINOR IS ACTUALLY $\begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}$ RATHER THAN OTHER IMAGINABLE POSSIBILITIES. THE MOTIVATION FOR THIS ASSIGNMENT IS THAT THE ANTINUCLEON ISOSPINOR SHOULD TRANSFORM UNDER ROTATIONS IN ISOSPIN SPACE JUST LIKE THE NUCLEON ISOSPINOR.

IT IS SUFFICIENT TO CONSIDER A ROTATION ABOUT THE I_2 AXIS BY ANGLE θ. WE REFER TO THE SPIN 1/2 ROTATION MATRIX (p 119)

THEN
$$|p'\rangle = \cos\theta/2\,|p\rangle + \sin\theta/2\,|n\rangle$$
$$|n'\rangle = -\sin\theta/2\,|p\rangle + \cos\theta/2\,|n\rangle$$



WHERE $|p'\rangle$ AND $|n'\rangle$ ARE THE ISOSPIN COMPONENTS ALONG THE I_3' AXIS AT ANGLE θ TO THE I_3 AXIS.

IF WE APPLY CHARGE CONJUGATION TO THIS TRANSFORMATION, THEN
$$|\bar{p}'\rangle = \cos\theta/2\,|\bar{p}\rangle + \sin\theta/2\,|\bar{n}\rangle$$
$$|\bar{n}'\rangle = -\sin\theta/2\,|\bar{p}\rangle + \cos\theta/2\,|\bar{n}\rangle$$

ACCORDING TO THE CONVENTION $Q = I_3 + B/2$ (p 181) WE WANT THE $\bar{n}$ TO BE THE $I_3 = +1/2$ COMPONENT. THIS SUGGESTS REWRITING THE ANTINUCLEON TRANSFORMATION:
$$|\bar{n}'\rangle = \cos\theta/2\,|\bar{n}\rangle + \sin\theta/2\,(-|\bar{p}\rangle)$$
$$-|\bar{p}'\rangle = -\sin\theta/2\,|\bar{n}\rangle + \cos\theta/2\,(-|\bar{p}\rangle)$$

THUS ROTATIONS OF THE SPINOR $\begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}$ HAVE THE SAME TRANSFORMATION LAWS AS DO THOSE OF $\begin{pmatrix} p \\ n \end{pmatrix}$, JUSTIFYING THE MINUS SIGN FOR THE $\bar{p}$ COMPONENT. IN PARTICULAR: $R_y(180^\circ)\begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} n \\ -p \end{pmatrix}$; $R_y(180^\circ)\begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix} = \begin{pmatrix} -\bar{p} \\ -\bar{n} \end{pmatrix}$

SO $G\begin{pmatrix} p \\ n \end{pmatrix} = \begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}$ WHILE $G\begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix} = -\begin{pmatrix} p \\ n \end{pmatrix}$ AND $G^2\begin{pmatrix} p \\ n \end{pmatrix} = -\begin{pmatrix} p \\ n \end{pmatrix}$

THE LAST RESULT IS NOT SURPRISING IF WE RECALL THAT A SPIN 1/2 OBJECT MUST BE ROTATED BY 720° TO RESTORE THE ORIGINAL PHASE.

NOTE THAT G AND NOT C TAKES US FROM THE NUCLEON DOUBLET $\begin{pmatrix} p \\ n \end{pmatrix}$ TO THE 'CORRECT' ANTINUCLEON DOUBLET $\begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}$, WHERE 'CORRECT' REFERS TO BEHAVIOR UNDER ISOSPIN TRANSFORMATIONS.

WE CAN NOW WORK OUT THE EFFECT OF THE G OPERATION ON $N\bar{N}$ COMBINATIONS. FIRST WE NOTE THAT WE HAVE AN ISO TRIPLET AND AN ISO SINGLET

	$I = 1$	$I = 0$
$I_3 = 1$	$\lvert p\bar{n}\rangle$	
$I_3 = 0$	$\frac{1}{\sqrt{2}}\left(-\lvert p\bar{p}\rangle + \lvert n\bar{n}\rangle\right)$	$-\frac{1}{\sqrt{2}}\left(\lvert p\bar{p}\rangle + \lvert n\bar{n}\rangle\right)$
$I_3 = -1$	$-\lvert n\bar{p}\rangle$	

THE SIGN CHANGES DUE TO THE $I_3 = -1/2$ ANTINUCLEON STATE BEING $-\lvert\bar{p}\rangle$ ARE OF SOME IMPORTANCE LATER.

BY DIRECT APPLICATION WE FIND THAT ALL THE ABOVE STATES ARE G-PARITY EIGENSTATES, WITH EIGENVALUES $(-1)^I$

OF COURSE WE MUST ALSO CONSIDER THE SPACE AND SPIN PARTS OF THE $N\bar{N}$ WAVE FUNCTION. THESE GIVE EIGEN FACTORS $(-1)^{\ell+S}$ UNDER G, DUE TO THE EFFECT OF CHARGE CONJUGATION (p 173).

ALTOGETHER $\quad G(N\bar{N}) = (-1)^{\ell+S+I}$

WE NOW SURVEY SEVERAL APPLICATIONS OF G-PARITY

1. MESON DECAYS $\qquad$ [SEE PERKINS p 402 ff FOR EXPERIMENTAL DATA]

a. PSEUDOSCALAR MESONS.

BESIDES THE PION THERE ARE 2 MORE PSEUDOSCALAR ($P=-1$) MESONS OF NOTE, THE $\eta(548)$ AND THE $\eta'(958)$.

BOTH DECAY TO $\gamma\gamma$ WHICH SUGGESTS THEY HAVE SPIN ZERO, AND $C=+1$. THE DECAY $\eta \to \pi^+\pi^-\pi^0$ SUGGESTS THAT THE η HAS NEGATIVE PARITY, AND THEN $\eta' \to \eta\pi^+\pi^-$ INDICATES THAT THE η' ALSO HAS NEGATIVE PARITY.

THEY HAVE NO OBSERVED CHARGED PARTNERS, SO WE INFER THAT THEY HAVE $I = 0$.

THEN $G = C R_y(180) = +1$ FOR BOTH η AND η'

HENCE THE η AND η' CAN DECAY STRONGLY ONLY TO AN EVEN NUMBER OF PIONS. HOWEVER ANGULAR MOMENTUM AND PARITY CONSERVATION FORBID $\eta \to \pi\pi$ (IS $\eta \to 4\pi$ FORBIDDEN?) HOWEVER, $\eta \to 3\pi$ IS OBSERVED. THIS IS A G VIOLATION, AND SO CANNOT BE A STRONG INTERACTION DECAY. AS IT OCCURS WITH ROUGHLY EQUAL RATE AS $\eta \to \gamma\gamma$ WE CONCLUDE THAT $\eta \to \pi^+\pi^-\pi^0$ MUST BE DUE TO AN ELECTROMAGNETIC INTERACTION.

THE DOMINANT DECAY OF THE $\eta' \to \eta\pi\pi$ DOES CONSERVE G, AND SO IS A STRONG INTERACTION PROCESS. NOTE THAT $\eta' \to \gamma\gamma$ HAPPENS ONLY 2 % OF THE TIME.

b. VECTOR MESONS

AN EXTENSIVE FAMILY OF MESONS CAN BE PRODUCED IN THE REACTION

$e^+e^- \to \gamma \to$ MESON : $\rho(760)$, $\omega(780)$, $\phi(1020)$, $J/\psi(3100)$, $\psi(3700)$ $\Upsilon(9400)$...

FROM ANALYSIS OF THE PRODUCTION CROSS SECTIONS (LECT 7 & 11) WE INFER THAT A SINGLE PHOTON EXISTS IN THE INTERMEDIATE STATE $(J = 1)$, SO IN TURN THE MESONS HAVE SPIN 1; THEIR PARITY IS -1, AS FOR THE PHOTON; LIKEWISE $C = -1$.

ONLY THE ρ MESON IS OBSERVED TO HAVE CHARGED PARTNERS $\rho^\pm$ (IN OTHER REACTIONS). SO $I_\rho = 1$, WHILE $I = 0$ FOR ALL OTHERS.

IT FOLLOWS THAT THE G PARITY OF THE ρ IS $+1$, BUT $G = -1$ FOR THE OTHER VECTOR MESONS. THUS ρ CAN DECAY STRONGLY TO EVEN NUMBERS OF PIONS. HOWEVER $\rho \to \pi^0\pi^0$ IS FORBIDDEN. WHY?

THE OTHER VECTOR MESONS CAN DECAY ONLY TO ODD NUMBERS OF PIONS. FOR EXAMPLE, THE DOMINANT STRONG DECAYS OF THE J/ψ PARTICLE ARE TO 5π AND 7π!

2. $N\bar{N} \to n\pi$

YOU MAY WISH TO PRACTICE YOU SKILLS WITH SELECTION RULES BY VERIFYING THE TABLE ON THE NEXT PAGE.

Selection Rules for $\bar{p} + p \rightarrow N\pi$

State	Spin Parity	C	I	G	$2\pi^0$	$\pi^+ - \pi^-$	$3\pi^0$	$\pi^+ + \pi^- + \pi^0$	$4\pi^0$	$\pi^+ + \pi^- + 2\pi^0$	$2\pi^+ + 2\pi^-$	$5\pi^0$	$\pi^+ + \pi^- + 3\pi^0$	$2\pi^+ + 2\pi^- + \pi^0$
$^1S_0$	0−	+	0	+	×	×	−	−				−	−	−
			1	−	×	×				−	−			
$^3S_1$	1−	−	0	−	×	−	×		×	−	−	×		
			−	+	×		×	−	×			×		
$^1P_1$	1+	−	0	−	×	×	×		×	−	−	×		
			1	+	×	×	×	−	×			×	−	−
$^3P_0$	0+	+	0	+			×	×				−	−	−
			1	−	−	−	×	×	−	−	−			
$^3P_1$	1+	+	0	+	×	×	−	−				−	−	−
			1	−	×	×			−	−	−			
$^3P_3$	2+	+	0	+			−	−				−	−	−
			1	−	−	−			−	−	−			

Selection Rules for $\bar{p} + n \rightarrow N\pi$

State	Spin Parity	I	G	$\pi^- + \pi^0$	$2\pi^- + \pi^+$	$\pi^- + 2\pi^0$	$2\pi^- + \pi^+ + \pi^0$	$\pi^- + 3\pi^0$	$3\pi^- + 2\pi^+$	$2\pi^- + \pi^+ + 2\pi^0$	$\pi^- + 4\pi^0$
$^1S_0$	0−	1	−	×			−	−			
$^3S_1$	1−	1	+		−	−			−	−	−
$^1P_2$	1+	1	+	×	−	−			−	−	−
$^3P_0$	0+	1	−	−	×	×	−	−			
$^3P_1$	1+	1	−	×			−	−			
$^3P_2$	2+	1	−	−			−	−			

The × means strictly forbidden, and the − means forbidden so far as the isotopic spin is a good quantum number.

3. $NN \rightarrow NN$ SCATTERING REVISITED. (p183)

THE CLAIM IS THAT $\sigma_{pp} = \sigma_{nn} = \sigma_{pn}$ IN THE ONE PION EXCHANGE APPROXIMATION. WE CAN VERIFY THIS BY LOOKING AT THESE REACTIONS FROM THE "PION'S POINT OF VIEW." THIS IS ALSO CALLED THE t-CHANNEL OR 'CROSSED CHANNEL' VIEW (p32). HERE WE THINK OF EACH VERTEX IN THE DIAGRAMS ON p183 AS A TRANSITION $\pi \leftrightarrow N\bar{N}$. THE RELATIVE STRENGTHS OF THE VERTICES ARE AGAIN GOVERNED BY ISOSPIN SYMMETRY. REFER TO OUR LIST OF THE $I=1$ AND $I=0$ $N\bar{N}$ WAVE FUNCTIONS ON p186. THEN IF WE WRITE $C_{\pi^+ p\bar{n}} \equiv C$ WE INFER $C_{\pi^0 p\bar{p}} = \frac{-c}{\sqrt{2}}$

AND $C_{\pi^0 nn} = +c/\sqrt{2}$

THEN $\sigma_{pp} \sim (C_{\pi^0 p\bar{p}})^4 = c^4/4$; $\sigma_{nn} \sim (C_{\pi^0 nn})^4 = c^4/4$

AND $\sigma_{pn} \sim \left(C^2_{\pi^+ p\bar{n}} + C_{\pi^0 p\bar{p}} C_{\pi^0 n\bar{n}} \right)^2 \sim \left(c^2 - c^2/2 \right)^2 = c^4/4$

AGAIN ALL CROSS SECTIONS ARE EQUAL.

NOTE THE NECESSITY OF THE SIGN CHANGE OF THE $p\bar{p}$ TERM IN THE $I=1$, $I_3=0$ COUPLING.

STRANGENESS AND ISOSPIN

A FAMILY OF MESONS AND BARYONS WAS DISCOVERED IN THE 1950'S WHICH OBEYED RESTRICTIONS CONSISTENT WITH THE EXISTENCE OF A NEW QUANTUM NUMBER S = STRANGENESS. THIS QUANTUM NUMBER IS CONSERVED IN THE STRONG AND ELECTROMAGNETIC INTERACTIONS. PERKINS SEC 4.4 IS A GOOD INTRODUCTION TO THIS FAMILY. THE STRANGE PARTICLES CAN BE GROUPED INTO VARIOUS ISOSPIN MULTIPLETS, EACH MULTIPLET HAVING A FIXED VALUE OF S. SIMPLE EXAMPLES ARE

MESONS: $S = 1 \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$; $S = -1 : \begin{pmatrix} \bar{K}^0 \\ -K^- \end{pmatrix}$

HYPERONS: $S = -1 : (\Lambda^0)$ & $\begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}$; $S = -2 : \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}$ (THESE ARE BARYONS)

S = 0 FOR THE OTHER MESONS & BARYONS WE HAVE DISCUSSED.

THE RULE RELATING THE I_3 COMPONENT TO ELECTRIC CHARGE WAS EXTENDED TO INCLUDE STRANGENESS BY GELL-MANN & NISHIJIMA P.R. 93, 933 (1953)

$$Q = I_3 + \frac{B+S}{2} \qquad \left[B + S \equiv Y = \text{HYPERCHARGE}\right]$$

S IS AN ADDITIVE QUANTUM NUMBER, LIKE Q AND B.

WE NOTE THAT IN RETROSPECT THE QUANTUM NUMBER S WOULD HAVE BEEN BETTER DEFINED AS THE NEGATIVE OF THE ABOVE ASSIGNMENTS. THEN $S' = -S$ WOULD SIMPLY COUNT THE NUMBER OF STRANGE QUARKS CONTAINED IN THE PARTICLE. HOWEVER, THE QUARK MODEL DID EMERGE OUT OF THE STRUGGLE TO CATEGORIZE STATES OF ISOSPIN AND STRANGENESS IN TERMS OF A MORE COMPREHENSIVE SYMMETRY.

ISOSPIN VIOLATION

a. ELECTROMAGNETISM.

THE MASSES OF PARTICLES WITHIN AN ISOSPIN MULTIPLET TYPICALLY DEPEND ON THE ELECTRIC CHARGE (UNLESS LIKE π^+ & π^- THEY ARE ANTI PARTICLES OF EACH OTHER). THIS IS CERTAINLY AN ISOSPIN VIOLATION.

PARTICLE DECAYS INVOLVING PHOTONS ALSO SHOW PATTERNS WHICH CAN BE CALLED ISOSPIN VIOLATION. FOR EXAMPLE $\eta' \to \rho^0 \gamma$ & $\omega \gamma$ ALSO. RECALL THAT $I_{\eta'} = I_\omega = 0$ WHILE $I_\rho = 1$. SO WE CANNOT ASSIGN THE PHOTON A SINGLE ISOSPIN I_γ WHICH WOULD MAKE BOTH OF THESE DECAYS ISOSPIN CONSERVING. HOWEVER, A SURVEY OF SUCH DECAYS INDICATES THAT THE PHOTON TRANSITIONS ARE ALWAYS OF THE TYPE $\Delta I = 0$ or 1

$\Delta I = 1$: $\rho \to \eta\gamma$, $\omega \to \pi\gamma$, $\eta' \to \rho\gamma$ .... $\Sigma^0 \to \Lambda\gamma$ ...

$\Delta I = 0$: $\rho \to \pi\gamma$, $\omega \to \eta\gamma$, $\eta' \to \omega\gamma$

[NOTE THAT THERE ARE SOME APPARENT EXCEPTIONS IN HYPERON DECAY WHICH SEEM TO BE $\Delta I = 1/2$: $\Sigma^+ \to p\gamma$, $\Xi^0 \to \Lambda^0\gamma$ ... HOWEVER THESE DECAYS OCCUR LESS OFTEN THAN PARITY VIOLATING DECAYS OF THE SAME HYPERONS, AND SO ARE INTERPRETED AS WEAK DECAYS WITH PHOTON EMISSION AS A CORRECTION. THE $\Delta I = 1/2$ TRANSITION IS CHARACTERISTIC OF THE WEAK INTERACTION, AS DISCUSSED BELOW.]

THE OBSERVED PATTERN HAS LED PEOPLE TO SAY THAT THE PHOTON HAS AN ISOSCALAR PART AND AN ISOVECTOR PART. AS LOOSE SUPPORT FOR THIS INTERPRETATION, WE NOTE THE GELL-MANN-NISHIJIMA RELATION

$$Q = \underbrace{\frac{B+S}{2}}_{\text{"ISOSCALAR PART"}} + \underbrace{I_3}_{\text{"ISOVECTOR PART"}}$$

WE HAVE ALSO ENCOUNTERED THIS IDEA IN OUR STUDY OF THE NUCLEON FORM FACTORS (P 102) WHERE WE WROTE

$$G^p = G^S + G^V$$
$$G^n = G^S - G^V$$

ALSO, WE FOUND $e^+e^- \to \gamma \to$ VECTOR MESONS WITH BOTH $I = 0$ AND $I = 1$ MESON FINAL STATES.

b. WEAK INTERACTIONS.

A SURVEY OF THE DATA REVEALS VARIOUS PATTERNS OF ISOSPIN VIOLATION

- "NON-LEPTONIC" DECAYS: $\Lambda \to p\pi^-$, $K^0 \to \pi^+\pi^-$ ETC

  WHERE $\Delta I = 1/2$, $\Delta I_3 = \pm 1/2$

- "SEMI-LEPTONIC" DECAYS: $n \to p e \bar{\nu}_e$, $K^\pm \to \pi^0 e^\pm \nu_e (\bar{\nu}_e)$

  THE FIRST DECAY HAS $\Delta I = 0$ BUT $\Delta I_3 = 1$

  THE SECOND HAS $\Delta I = 1/2$, $\Delta I_3 = \pm 1/2$

  [NOTE THAT I IS DEFINED ONLY FOR HADRONS]

- <u>RARE</u> DECAYS INCLUDE $\pi^\pm \to \pi^0 e \nu$ $\quad \Delta I = 0 \quad \Delta I_3 = \pm 1$

  $\Sigma^\pm \to \Lambda e \nu$ $\quad \Delta I = 1 \quad \Delta I_3 = \pm 1$

A GOOD FIRST APPROXIMATION IS THE $\underline{\Delta I = 1/2 \text{ RULE}}$

THIS IS SHORTHAND FOR $\Delta I = 1/2$, $\Delta I_3 = \pm 1/2$ IN CASES WHERE $\Delta I \neq 0$.

THIS ALLOWS MANY SIMPLE RATE ESTIMATIONS.

EXAMPLE: $\Lambda \to \begin{cases} p\pi^- \\ n\pi^0 \end{cases}$ THE 2 FINAL STATES MIGHT BE $I = 1/2$ OR $3/2$. THE $\Delta I = 1/2$ RULE SUGGESTS ONLY THE $I = 1/2$ FINAL STATES ARE IMPORTANT. FROM THE C-G TABLES,

$$|1/2, -1/2\rangle = \sqrt{1/3}\,|n\pi^0\rangle - \sqrt{2/3}\,|p\pi^-\rangle$$

SO WE INFER $\frac{\text{RATE } \Lambda \to p\pi^-}{\text{RATE } \Lambda \to n\pi^0} = 2$ DATA: $\frac{64.2\%}{35.8\%}$

EXERCISES:

COMPARE $\frac{\text{RATE } K^0 \to \pi^+\pi^-}{\text{RATE } K^0 \to \pi^0\pi^0}$

SHOW $\frac{\text{RATE } K^0 \to \pi^- e^+ \nu_e}{\text{RATE } K^+ \to \pi^0 e^+ \nu_e} \sim 2$ $\frac{\text{RATE } \bar{K}^0 \to \pi^+ e^- \bar{\nu}_e}{\text{RATE } K^- \to \pi^0 e^- \bar{\nu}_e} \sim 2$

NOTE THAT $K^0 \not\to \pi^+ e^- \bar{\nu}_e$ ACCORDING TO THE RULE $\Delta I_3 = \pm 1/2$

$\bar{K}^0 \not\to \pi^- e^+ \nu_e$

THEN IF $K_L^0 \equiv \frac{K^0 - \bar{K}^0}{\sqrt{2}}$ $\frac{\text{RATE } K_L^0 \to \pi^\pm e^\mp \bar{\nu}_e\,(\nu_e)}{\text{RATE } K^\pm \to \pi^0 e^\pm \nu_e\,(\bar{\nu}_e)} \sim 2$ ALSO

LOOK UP THE EXPERIMENTAL FACTS IN THE APPENDIX TO PERKINS....

A FAMOUS EXCEPTION TO THE $\Delta I = 1/2$ RULE IS WORTH NOTING.

CONSIDER $K^+ \to \pi^+\pi^0$. WHILE THE $\pi^+\pi^0$ STATE HAS $I_3 = 1$ IT CANNOT HAVE $I = 1$. THIS FOLLOWS FROM BOSON STATISTICS AND ANGULAR MOMENTUM CONSERVATION — THE $\pi^+\pi^0$ HAS $l = 0$ SINCE THE KAON IS SPINLESS. HENCE THE ISOSPIN PART OF THE 2π WAVE FUNCTION MUST BE SYMMETRIC $\Rightarrow I = 0$ OR 2 ONLY. THUS $K^+ \to \pi^+\pi^0$ IS A $\Delta I = 3/2$ TRANSITION. EXPERIMENTALLY

$$\frac{\text{RATE } K^+ \to \pi^+\pi^0}{\text{RATE } K^0 \to \pi\pi} \sim 1.5 \times 10^{-3}$$

THIS GIVES EVIDENCE THAT WHILE THE $\Delta I = 1/2$ RULE IS NOT EXACT, VIOLATIONS ARE INDEED SUPPRESSED.

WE WILL CONSIDER ISOSPIN VIOLATION OF THE WEAK INTERACTION FURTHER IN LECTURE 17.

43. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2×1/2

		5/2
		+5/2
+2	+1/2	1

		5/2	3/2
		+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2×1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1

2×1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

3/2×1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

1×1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

2×3/2

		7/2
		+7/2
+2	+3/2	1

		7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

		7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

		7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

		7/2	5/2	3/2	1/2
		−1/2	−1/2	−1/2	−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

		7/2	5/2	3/2
		−3/2	−3/2	−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

		7/2	5/2
		−5/2	−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

		7/2
		−7/2
−2	−3/2	1

3/2×3/2

		3
		+3
+3/2	+3/2	1

		3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

		3	2	1
		+1	+1	+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

		3	2	1	0
		0	0	0	0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

		3	2	1
		−1	−1	−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

		3	2
		−2	−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

		3
		−3
−3/2	−3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×2

		4
		+4
+2	+2	1

		4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

		4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

		4	3	2	1
		+1	+1	+1	+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

		4	3	2	1	0
		0	0	0	0	0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

		4	3	2	1
		−1	−1	−1	−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

		4	3	2
		−2	−2	−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

		4	3
		−3	−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

		4
		−4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2} \qquad d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2 \qquad d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 43.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).