Analysis of 3 Body Decays

In this lecture we present 2 additional methods for extracting information about particle quantum numbers by analysis of particle production and decay. These techniques go somewhat beyond the simplest application of symmetry principles, but they are based on these principles rather than any detailed understanding of the interactions involved. The first technique concerns the decay of a particle into 3 other particles, as interpreted by the so-called Dalitz-plot analysis.

While application of our rules about spin, parity, etc., are straightforward for a 2 body decay such as $K^+ \to \pi^+ \pi^0$ (so long as we remember all relevant rules!), there is more difficulty in dealing with 3 body decays such as $K^+ \to \pi^+ \pi^+ \pi^-$ or $\omega, \eta \to \pi^+ \pi^- \pi^0$, the classic examples. This is because the orbital angular momentum state can be compounded out of, say, the relative l between the π^+ and π^-, and the l' between the π^0 and the $\pi^+\pi^-$ system. As a guide, it is suggested that we first examine the general features of the 3 body decay distribution subject to the restrictions of any symmetry principles which may apply.

We begin with the relativistic form of the calculation of the decay rate Γ for a particle of mass M_i to n particles

$$d\Gamma = \frac{1}{2M_i} \frac{1}{2S_i+1} \sum_{\text{spins}} |\mathcal{M}_{fi}|^2 \underbrace{\frac{d^3\bar{p}_1}{(2\pi)^3 2E_1} \cdots \cdots \frac{d^3\bar{p}_n}{(2\pi)^3 2E_n} (2\pi)^4 \delta^4\Big(P_i - \sum_{j=1}^{n} P_j\Big)}_{\text{phase space factor}}$$

[Compare with the expression for the scattering cross section, p.79]

Note the relativistic normalisation factors $\frac{1}{2M_i} \cdot \frac{1}{2E_1} \cdots \frac{1}{2E_n}$

S_i = spin of initial state, which state is assumed to be unpolarized

P_i, P_j are the energy-momentum 4 vectors in the rest frame of the initial particle, $P_i = (M_i, 0, 0, 0)$.

To obtain the total decay rate Γ, integrate $d\Gamma$ over all final state momenta. In general, the matrix element $\mathcal{M}_{fi}$ will depend on these momenta. We wish to explore the approximation that $\mathcal{M}_{fi}$ is independent of the particle momenta,

so that all kinematic dependence of Γ is in the phase-space factor. We will also consider cases where the dependence of $\mathcal{M}_{fi}$ on the P_j can be 'guessed' from symmetry arguments.

1. 2 BODY PHASE SPACE.

FOR A 2 BODY DECAY WE HAVE ALREADY EVALUATED THE PHASE SPACE FACTOR ON p. 80. IT IS WORTH NOTING THE NUMBER OF DEGREES OF FREEDOM OF THE FINAL STATE. THE 2 PARTICLES ARE DESCRIBED BY 2 ENERGY-MOMENTUM 4 VECTORS, WITH 8 COMPONENTS IN TOTAL. BUT WE KNOW THE 2 PARTICLE MASSES, AND ALSO THE ENERGY-MOMENTUM 4-VECTOR OF THE INITIAL STATE. HENCE THERE ARE ONLY 2 DEGREES OF FREEDOM, WHICH ARE SUITABLY CHOOSEN TO BE THE ANGLES θ, φ OF THE BACK-TO-BACK 3-MOMENTUM VECTORS OF THE FINAL-STATE PARTICLES. WE ARE MEASURING θ & φ IN THE REST FRAME OF THE INITIAL STATE PARTICLE. CORRESPONDINGLY, IF WE INTEGRATE OVER THE $\delta^4(P_i - \sum P_j)$ IN THE PHASE SPACE FACTOR, WE OBTAIN (p. 80)

$$\text{PHASE SPACE FACTOR} = \frac{1}{(2\pi)^2}\frac{P_f\, d\Omega_f}{4 M_i^2} \qquad (P_f \text{ FIXED})$$

$$\text{AND} \quad \frac{d\Gamma}{d\Omega} = \frac{P_f}{32\pi^2 M_i^2}\frac{\sum_{\text{SPINS}} |\mathcal{M}_{fi}|^2}{2S_i+1}$$

IF $\mathcal{M}_{fi}$ HAS NO ANGULAR DEPENDENCE, THEN $\Gamma = \frac{P_f}{8\pi M_i^2}\frac{\sum_{\text{SPINS}} |\mathcal{M}_{fi}|^2}{2S_i+1}$

2. 3 BODY PHASE SPACE & THE DALITZ PLOT

THE 3 BODY PHASE SPACE FACTOR CAN ALSO BE SIMPLIFIED BY ANALYTIC PROCEDURES. WE FIRST NOTE THAT THIS TIME WE HAVE $3\times 4 - 3 - 4 = 5$ DEGREES OF FREEDOM. AN INTERESTING CHOICE OF VARIBLES MIGHT BE (ENERGY OF PARTICLE 1, $\theta_2, \varphi_2, \theta_3, \varphi_3$ OF PARTICLES 2 & 3).

HOWEVER IF WE THINK A BIT ANOTHER SET OF VARIABLES MAY BE MORE 'NATURAL'. IN THE INITIAL STATE REST FRAME, THE FINAL STATE 3-MOMENTA OBEY

$$\bar{P}_1 + \bar{P}_2 + \bar{P}_3 = 0$$

THAT IS, ALL 3 PARTICLES LIE IN A PLANE IN SPACE, THE 'DECAY PLANE'. THE DIRECTION OF THE NORMAL, $\hat{n}$, TO THE DECAY PLANE IS 2 OF THE NATURAL VARIABLES. FURTHER, THE AZIMUTHAL ORIENTATION OF THE TRIAD $(\bar{P}_1, \bar{P}_2, \bar{P}_3)$ ABOUT $\hat{n}$ IS A 3RD NATURAL VARIABLE. IT IS EASY TO IMAGINE THAT $\mathcal{M}_{fi}$ WILL NOT DEPEND ON THESE VARIABLES. THEN WE ARE LEFT WITH 2 NON-TRIVIAL VARIABLES, WHICH TURN OUT TO BE USEFULLY TAKEN TO BE THE ENERGIES OF 2 OF THE PARTICLES (THE INSIGHT OF DALITZ, PHIL. MAG. 44 1068 (1953)).

WE PURSUE THIS INSIGHT TO EVALUATE THE PHASE SPACE FACTOR. SEE PERKINS SEC 4.6 FOR AN ALTERNATE DERIVATION, WHICH IS PERHAPS QUICKER.

FIRST NOTE THAT $\frac{d^3\vec{p}}{E} = p^2\frac{dp\,d\Omega}{E} = p\,dE\,d\Omega$ USING $E^2 = p^2 + m^2$

THEN THE 3 BODY PHASE SPACE FACTOR IS

$$\frac{1}{2^8\pi^5}\, p_1\, dE_1\, d\Omega_1 \underbrace{\int \frac{d^3\vec{p}_2}{E_2}\,\frac{d^3\vec{p}_3}{E_3}\,\delta^4(P_i - \textstyle\sum P_j)}$$

JUST LIKE THE PHASE SPACE FACTOR FOR THE 2 BODY PROCESS $P_i + (-P_1) \to P_2 + P_3$

IN THE CENTER OF MASS FRAME OF PARTICLES 2 AND 3 WE CAN IMMEDIATELY EVALUATE THE INTEGRAL AS $\frac{p_2^* d\Omega_2^*}{M^*}$ USING p 192

THE $*$ $\Rightarrow$ QUANTITY MEASURED IN THE 2+3 C.M. FRAME

$M^{*2} = (p_2+p_3)^2 \equiv M_{23}^2$ WHERE M_{23} = INVARIANT MASS OF PARTICLES 2 & 3

TO BE OF USE, WE MUST TRANSFORM OUR RESULT TO THE INITIAL-STATE REST FRAME. THIS TRANSFORMATION TAKES THE 4-VECTOR P_2+P_3 FROM $(M_{23}, \vec{0})$ TO $(E_2+E_3, \underbrace{\vec{P}_2+\vec{P}_3}_{-\vec{P}_1})$.

THUS THE 3-MOMENTUM TRANSFORMATION IS $\vec{0} = \gamma\left(-\vec{P}_1 + \vec{\beta}(E_2+E_3)\right) \Rightarrow \vec{\beta} = \frac{\vec{P}_1}{E_2+E_3}$

A USEFUL ENERGY TRANSFORMATION IS $E_2+E_3 = \gamma(M_{23} - \vec{\beta}\cdot\vec{0}) \Rightarrow \gamma = \frac{E_2+E_3}{M_{23}}$

A CLEVER TRICK IS THAT $E_2 = \gamma(E_2^* - \vec{\beta}\cdot\vec{p}_2^*) \equiv \gamma(E_2^* - \beta p_2^* \cos\theta^*)$

SO $dE_2 = -\gamma\beta p_2^*\, d\cos\theta^*$

AND $\frac{p_2^* d\Omega^*}{M^*} = \frac{dE_2\, d\phi_2^*}{\gamma\beta}\,\frac{\gamma}{E_2+E_3} = \frac{dE_2\, d\phi_2^*}{p_1}$ (IGNORING THE - SIGN)

RATHER MAGICALLY WE HAVE ARRIVED AT

3 BODY PHASE SPACE FACTOR $= \frac{1}{2^8\pi^5}\, dE_1\, dE_2\, d\Omega_1\, d\phi_2^*$

ASSUMING $\mathcal{M}_{fi}$ IS INDEPENDENT OF Ω_1 AND ϕ_2^*, WE INTEGRATE OVER THESE TO YIELD FACTOR $= \frac{1}{32\pi^3}\, dE_1\, dE_2$

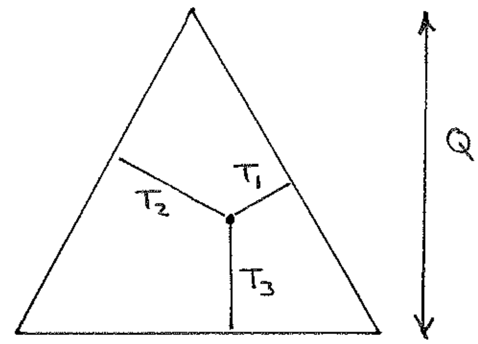
THE STRIKING RESULT IS THAT 3 BODY PHASE VOLUME IS UNIFORMLY DISTRIBUTED IN AN IMAGINARY PLANE WITH ANY 2 PARTICLE ENERGIES AS AXES.

Dalitz gave a nice geometrical interpretation of this relation. The kinetic energies of the final state particles

$$T_j = E_j - m_j \qquad j = 1,2,3$$

obey $\quad T_1 + T_2 + T_3 = M_i - m_1 - m_2 - m_3 \equiv \underline{Q}$

This constraint implies that any allowed set of energies T_j can be represented as a point inside an equilateral triangle of altitude Q. T_j = distance from that point to side j.



Note that $\quad d\,\text{Area} = dT_1\,dT_2\,\dfrac{\sqrt{3}}{2}$

$$= dE_1\,dE_2\,\frac{\sqrt{3}}{2}$$

so phase volume is uniformly distributed over the triangular Dalitz plot.

At last, we have a useful technique for analysis of 3 body decays. Plot the points corresponding to the T_j for a series of observed decays using a Dalitz plot. Any departure from a uniform distribution is an indication of significant structure in the matrix element $\mathcal{M}_{fi}$.

Query: How can a relativistic analysis be based on kinetic energies? We will answer this in section 5 below.

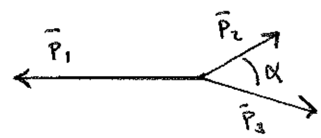
Exercise: show that phase space vanishes for the decay $\gamma \to 3\gamma$ of a photon. This is why the photon is stable.

3. Dalitz-plot boundaries and projections.

Because $\bar{P}_1 + \bar{P}_2 + \bar{P}_3 = 0$, not every point on the Dalitz triangle is physically accessible. In particular, regions near the corners are excluded, because in the limit $T_1, T_2 \to 0$, $\bar{P}_1 + \bar{P}_2 + \bar{P}_3 \to \bar{P}_3 \neq 0$

The physically allowed region lies within some boundary curve inscribed within the Dalitz triangle. It is easy to see that the boundary touches the triangle at the midpoint of each side, corresponding to $\begin{cases} T_1 = 0, & T_2 = T_3 \\ \bar{P}_1 = 0 & \bar{P}_2 = -P_3 \end{cases}$ etc.

The general shape of the boundary is determined by momentum conservation:

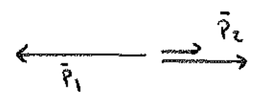


$\sum \bar{P}_j = 0$

From the sketch, $\quad P_1^2 = P_2^2 + P_3^2 + 2P_2P_3\cos\alpha$

Limiting cases are clearly associated with $\cos\alpha = \pm 1$

i.e. when all 3 momenta are colinear



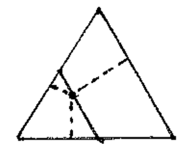
etc.

$$|\bar{P}_1| = |\bar{P}_2| + |\bar{P}_3|$$

a. RELATIVISTIC LIMIT : $M_i \gg m_j \Rightarrow T_j \rightarrow P_j$

THEN THE COLINEAR RELATION $P_1 = P_2 + P_3$ BECOMES $T_1 = T_2 + T_3$

THIS IS JUST THE STRAIGHT LINE JOINING THE MIDPOINTS OF SIDES 2 AND 3. HENCE THE ALLOWED REGION IS THE SMALL EQUILATERAL TRIANGLE OF HEIGHT $Q/2$ INSIDE THE ORIGINAL TRIANGLE.

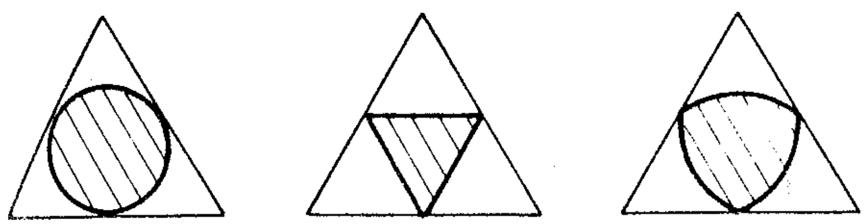


b) NON-RELATIVISTIC LIMIT: $T_j \rightarrow P_j^2/2M_j$

IF $M_1 = M_2 = M_3$ THEN THE BOUNDARY EQUATION $P_1^2 = P_2^2 + P_3^2 \pm 2P_2P_3$

BECOMES $T_1 = T_2 + T_3 \pm 2\sqrt{T_1 T_2}$ OR $4T_2T_3 = (T_1 - T_2 - T_3)^2$

CLEARLY THE T_j ARE LINEAR FUNCTIONS OF THE X-Y AXES OF THE DALITZ PLOT, SO WE HAVE A CLOSED BOUNDARY CURVE WHICH IS A QUADRATIC FUNCTION OF x & y $\Rightarrow$ ELLIPSE. THEN SINCE $M_1 = M_2 = M_3$ THIS ELLIPSE MUST BE SYMMETRIC UNDER ROTATIONS BY $120°$ $\Rightarrow$ CIRCLE!



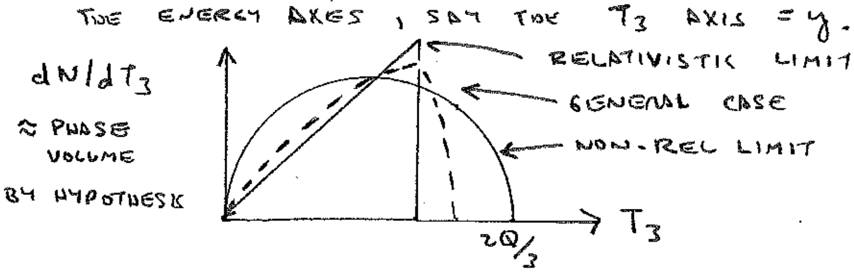
(a) NON-RELATIVISTIC (ALL MASSES EQUAL)

(b) RELATIVISTIC LIMIT

(c) GENERAL CASE

ALLOWED REGIONS INSIDE THE DALITZ PLOT

WE CAN EXTRACT SOME RESULTS FROM THE SHAPES OF THE ALLOWED REGIONS AT ONCE. SUPPOSE WE ARE INTERESTED ONLY IN THE ENERGY DISTRIBUTION OF A SINGLE PARTICLE. THEN IF $\mathcal{M}_{fi}$ = CONSTANT, WE SIMPLY PROJECT THE ALLOWED REGION OF THE DALITZ PLOT ONTO ONE OF THE ENERGY AXES, SAY THE T_3 AXIS $= y$.



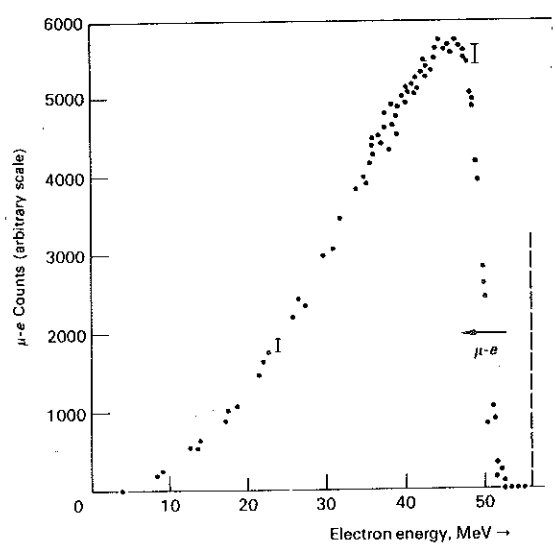
EXAMPLE: IN THE WEAK DECAY

$\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu$

WE CAN ONLY PRACTICALLY MEASURE E_e. HERE $Q = M_\mu - M_e \gg M_e$

SO WE ARE IN THE RELATIVISTIC LIMIT.

TO THE EXTENT THAT $\mathcal{M}_{fi}$ = CONSTANT, $dN/dE_e \rightarrow$ TRIANGULAR DISTRIBUTION.



4. 2 BODY VS. 3 BODY DECAY RATES.

IF $\mathcal{M}_{fi} \sim$ CONSTANT, THE TOTAL DECAY RATE IS PROPORTIONAL TO THE TOTAL PHASE VOLUME. THE TOTAL ALLOWED AREA ON THE DALITZ PLOT VARIES BETWEEN

$$\frac{\pi Q^2}{9} \text{ (NON-REL)} \qquad \text{AND} \qquad \frac{\pi\sqrt{3}}{12} Q^2 \text{ (EXTREME REL)}$$

FROM P195. $\int dE_1 dE_2 = \frac{2}{\sqrt{3}} \cdot$ AREA OF DALITZ PLOT $\sim \frac{Q^2}{3}$ ROUGHLY

SO THAT TOTAL 3 BODY PHASE VOLUME IS $\sim \frac{Q^2}{96\pi^3} \sim \frac{Q^2}{3000}$

IT IS INTERESTING TO COMPARE THIS TO TOTAL 2-BODY PHASE VOLUME WHICH IS $\frac{1}{4\pi}\frac{P_f}{M_i} \sim \frac{P_f}{10 M_i}$ (P 193)

WE CAN NOW ESTIMATE THE RELATIVE DECAY RATES FOR THE 2 OR 3 BODY DECAYS OF A PARTICLE, ASSUMING $\mathcal{M}_{fi}$ IS BASICALLY THE SAME IN BOTH CASES (A POSSIBLY DOUBTFUL ASSUMPTION).

NOTE THAT THE 2 & 3 BODY PHASE VOLUMES DO NOT HAVE THE SAME DIMENSIONS, SO WE CANNOT COMPARE THEM AT ONCE. BY DIMENSIONAL ARGUMENTS WE INFER THAT $|\mathcal{M}_{fi}|^2$ FOR THE 3 BODY DECAY HAS AN EXTRA FACTOR OF DIMENSIONS $1/E^2$ COMPARED TO THE 2 BODY CASE. WE PROCEED BY SUPPOSING THE 'NATURAL' ENERGY SCALE OF A PARTICLE DECAY IS THE MASS OF THE PARTICLE ITSELF, M_i.

THEN $$\frac{\Gamma_{3\,BODY}}{\Gamma_{2\,BODY}} \sim \frac{Q^2}{3000\, M_i^2} \cdot \frac{10 M_i}{P_f} \approx \frac{Q^2}{300\, M_i P_f}$$

$\left[\text{ASSUMING } \mathcal{M}_{i\to 3} \approx \frac{1}{M_i}\mathcal{M}_{i\to 2}\right]$

EXAMPLE: COMPARE $K^+ \to 2\pi$ OR 3π, AND $K^0 \to 2\pi$ OR 3π

FOR THE 2 BODY DECAYS $E_\pi = M_K/2 \Rightarrow P_f \approx M_K/2$

FOR THE 3 BODY DECAYS $Q = M_K - 3M_\pi \sim 85$ MeV $\sim M_K/6$

THE 'PHASE SPACE' ESTIMATE IS $\frac{\Gamma_{K\to 3\pi}}{\Gamma_{K\to 2\pi}} \sim \frac{1}{300}\left(\frac{1}{6}\right)^2 \cdot 2 \sim 2\times 10^{-4}$

THIS IS QUITE SIGNIFICANT REDUCTION IN PHASE SPACE FOR THE 3 BODY DECAY!

FROM THE DATA (APPENDIX TO PERKINS) $\frac{\Gamma_{K^+\to 3\pi}}{\Gamma_{K^+\to 2\pi}} = \frac{5\times 10^6 \sec^{-1}}{1.5\times 10^7 \sec^{-1}} \sim \frac{1}{3}$ WHILE $\frac{\Gamma_{K^0\to 3\pi}}{\Gamma_{K^0\to 2\pi}} = \frac{6\times 10^6 s^{-1}}{10^{10} s^{-1}} \sim 6\times 10^{-4}$

(CAN YOU ACTUALLY EXTRACT THESE NUMBERS FROM THE DATA TABLES?)

WE CONCLUDE THAT THE LOW RATE OF $K^0 \to 3\pi$ IS ESSENTIALLY DUE TO PHASE SPACE SUPPRESSION, AND NOT DUE TO ANY EFFECT IN THE MATRIX ELEMENT. HOWEVER, IT WOULD APPEAR THAT $K^+ \to 2\pi$ IS ACTUALLY SUPPRESSED IN THE MATRIX ELEMENT. As we noted on p.191, $K^+ \to \pi^+\pi^0$ VIOLATES THE $\Delta I = 1/2$ RULE FOR WEAK DECAYS, AND SO IS EXPECTED TO BE SUPPRESSED (EMPIRICALLY).

EXAMPLE $\phi \to 2K$ or 3π $[\phi \not\to 2\pi$ BY G PARITY!$]$

IN $\phi \to 2K$, $E_K = M_\phi/2 = 510$ MeV $\Rightarrow T_K = 15$ MeV $\Rightarrow P_K \sim \sqrt{2M_K T_K} \sim 120$ MeV

IN $\phi \to 3\pi$, $Q = M_\phi - 3M_\pi = 1020 - 410 = 610$ MeV

THE PHASE SPACE ESTIMATE IS $\dfrac{\Gamma\, \phi\to 3\pi}{\Gamma\, \phi \to 2K} \sim \dfrac{1}{300}\dfrac{610^2}{(1020)(120)} \sim .01$

EXPERIMENTALLY THE RATIO IS $\sim 1/5$

OUR ARGUMENT ABOUT Γ_3/Γ_2 MUST BE VERY CRUDE!

5. DALITZ PLOTS AND PARTICLE PRODUCTION.

WE DIGRESS TO NOTE A POWERFUL APPLICATION OF THE DALITZ PLOT TECHNIQUE, USEFUL IN DEMONSTRATING THE EXISTENCE OF NEW PARTICLES. CONSIDER REACTIONS WITH 3 BODY FINAL STATES SUCH AS $\pi N \to N + 2\pi$, OR $\bar{p}p \to 3\pi$. (THE FORMER DOES NOT VIOLATE FURRY'S THEOREM FOR π'S IF WE THINK OF IT AS DUE TO ONE PION EXCHANGE). IT IS POSSIBLE THAT A SHORT-LIVED PARTICLE WAS PRODUCED DURING THE REACTION, WHICH THEN DECAYED TO $N\pi$ OR 2π. THIS BEHAVIOR WOULD BE ASSOCIATED WITH A MATRIX ELEMENT QUITE DIFFERENT FROM 'CONSTANT'. HENCE A DALITZ-PLOT ANALYSIS OF THE 3 PARTICLE FINAL STATE MAY HELP US REMOVE THE LESS INTERESTING PHASE-SPACE 'BACKGROUND', REVEALING STRUCTURE POSSIBLY DUE TO NEW PARTICLE PRODUCTION.

SUPPOSE FINAL-STATE PARTICLES 1 AND 2 ARE DUE TO THE DECAY OF THE NEW PARTICLE. THE INVARIANT MASS OF THE NEW PARTICLE IS

$$M_{new}^2 = M_{12}^2 = (P_1 + P_2)^2 = (P_i - P_3)^2 \qquad i = \text{INITIAL STATE}$$

$$= E_{CM}^2 + M_3^2 - 2E_{CM}E_3 \qquad \text{IN CM FRAME}$$

HENCE A PLOT OF NUMBER OF EVENTS VS. E_3 OR T_3 SHOULD SHOW AN ACCUMULATION AT A VALUE CORRESPONDING TO M_{12} OF THE NEW PARTICLE.

(THIS RELATION SHOWS HOW KINETIC ENERGY IS SIMPLY RELATED TO RELATIVISTIC INVARIANTS OF THE REACTION, SO T IS AN O.K. VARIABLE AFTER ALL.)

AN AMUSING EXAMPLE IS THE REACTION

$$\bar{p}p \to \Lambda \pi \Sigma \quad \text{or} \quad \bar{\Lambda} \pi \Sigma$$

SHOWN IS A DALITZ PLOT WITH AXES $M^2_{\Sigma\pi} \sim E_\Lambda$

AND $M^2_{\Lambda\pi} \sim E_\Sigma$

THEREFORE PHASE VOLUME IS UNIFORM OVER THE ALLOWED KINEMATIC REGION. THE PROJECTIONS OF THIS OVULAR REGION ONTO THE AXES ARE THEN THE 'PHASE-SPACE' CURVES AS SHOWN. DEPARTURES OF THE DATA FROM THESE CURVES ARE INTERPRETED AS NEW PARTICLES.

HERE 1 PLOT YIELDS 3 NEW PARTICLES!

RABI: ONCE THE DISCOVERY OF A NEW PARTICLE WAS AWARDED WITH A NOBEL PRIZE; NOW PERHAPS THE AWARD SHOULD BE A $10,000 FINE.

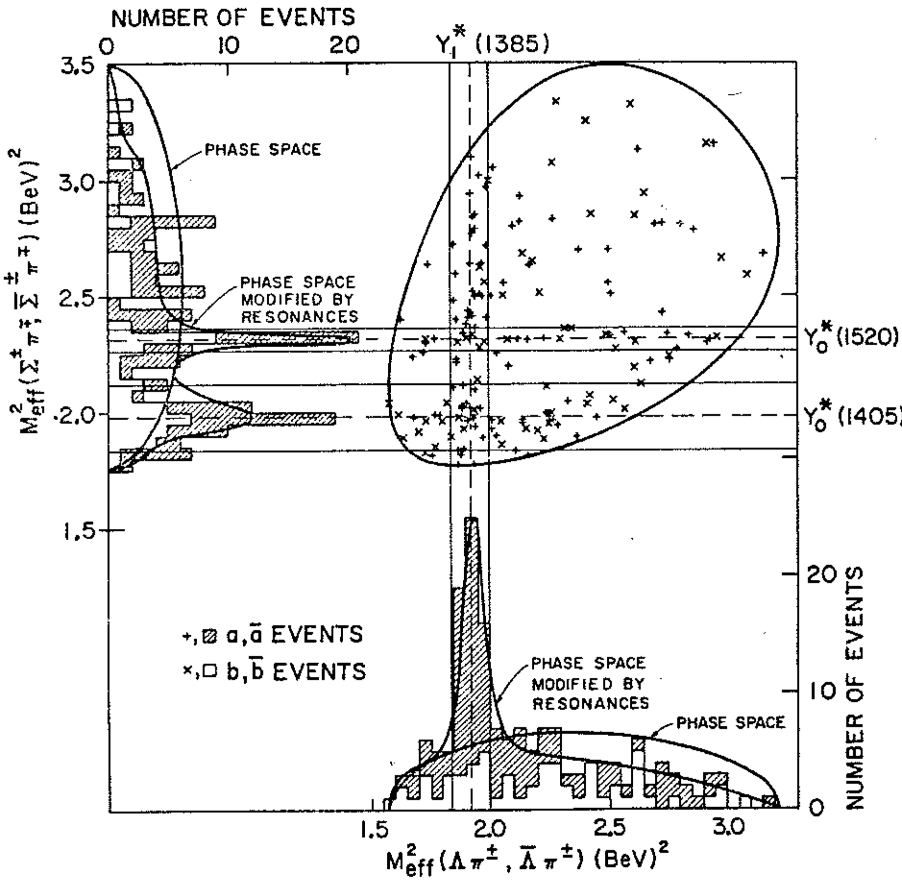


Fig. 24.2. Dalitz plot for squares of effective masses of $\Sigma^\pm\pi$ and $\bar{\Sigma}^\pm\pi$ vs $\Lambda\pi$ and $\bar{\Lambda}\pi$. Bands 2Γ in width have been drawn through the plot, from 1355 to 1415 MeV for the Σ (1385) or Y*, 1355 to 1455 MeV for the Λ (1405) or Y* (1405) and 1505 to 1535 MeV for the Λ (1520) or Y* (1520). Best-fitting phase-space distributions modified by Y* resonances are drawn in addition to simple phase-space distributions. The modified phase-space curves are computed with the usually accepted positions and widths of the Y* resonances, but with a mass of 1389 MeV and a Γ of 26 MeV for the Σ^* (BALTAY 1963)

THE TECHNIQUE OF 'BUMP HUNTING' IS READILY EXTENDED TO REACTIONS WITH MORE THAN 3 FINAL STATE PARTICLES. SIMPLY PLOT THE NUMBER OF EVENTS AS A FUNCTION OF THE INVARIANT MASS OF ANY DESIRED SUB GROUP OF FINAL STATE PARTICLES. IN GENERAL THERE IS NO ANALYTIC OR GRAPHICAL TECHNIQUE TO CALCULATE THE PHASE VOLUME, BUT COMPUTERS CAN ALWAYS DO THE INTEGRALS NUMERICALLY. ANOTHER TECHNIQUE IS TO COMPARE TWO SIMILAR PARTICLE COMBINATIONS. IF ONE GROUPING SHOWS NO 'BUMP', IT MAY BE USED AS THE ESTIMATE FOR THE PHASE VOLUME OF ANOTHER GROUPING.

A CLASSIC EXAMPLE CONCERNS 3π GROUPS IN THE REACTION $\bar{p}p \to 5\pi$. THE INVARIANT MASS DISTRIBUTIONS FOR $\pi^+\pi^+\pi^-$, $\pi^+\pi^+\pi^0$, $\pi^-\pi^-\pi^+$ AND $\pi^-\pi^-\pi^0$ SHOWED NO BUMP, WHILE $\pi^+\pi^-\pi^0$ SHOWED EVIDENCE FOR THE $\omega(780)$ MESON AS A BUMP ABOVE A 'PHASE SPACE' BACKGROUND SHAPE DERIVED FROM THE OTHER 4 COMBOS. WE INFER AT ONCE THAT THE ω HAS ISOSPIN ZERO.

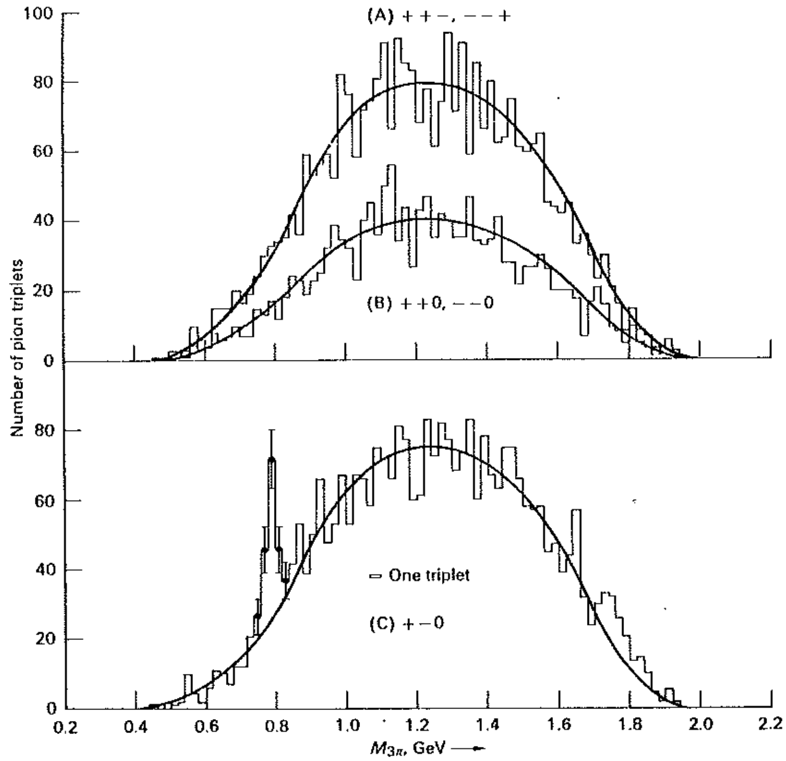


Fig. 4.25 Mass spectra of three-pion combinations selected from events of the type $p + \bar{p} \rightarrow \pi^+ + \pi^+ + \pi^- + \pi^- + \pi^0$. (A) Combinations of charge ± 1; (B) combinations of charge ± 2; (C) neutral combinations. The curves refer to the phase-space distribution, i.e. the spectrum expected if there is no strong pion-pion interaction in the final state. (After Maglic *et al.* 1961.)

6. SPIN-PARITY ANALYSIS OF 3 BODY DECAYS.

AT LAST WE CONSIDER THE ORIGINAL MOTIVATION OF THE DALITZ-PLOT ANALYSIS, WHICH IS TO PROVIDE SPIN AND PARITY DETERMINATION OF A PARTICLE BY OBSERVING ITS 3-BODY DECAY. WE DO THIS BY NOTING THAT THE MATRIX ELEMENT MAY FORBID POPULATION OF CERTAIN REGIONS OF THE DALITZ PLOT, DEPENDING ON THE SPIN AND PARITY OF THE INITIAL PARTICLE. WE CONSIDER THE 3 CLASSIC EXAMPLES

a. $K^+ \rightarrow \pi^+\pi^+\pi^-$

IT IS CONVENIENT TO GROUP THE 3 π's INTO A 2+1 CONFIGURATION

LET $\bar{q} = P_1^* - P_2^*$ = RELATIVE MOMENTUM OF THE $2\pi^+$, IN THEIR C.M. FRAME

l_q = ORBITAL ANGULAR MOMENTUM OF THE $2\pi^+$ SYSTEM
BECAUSE WE HAVE 2 IDENTICAL BOSONS, l_q MUST BE EVEN

$\bar{P}_3$ = MOMENTUM OF THE π^- IN THE K^+ REST FRAME

l_3 = ORBITAL ANGULAR MOMENTUM OF THE π^- ABOUT THE $2\pi^+$ SYSTEM

J = SPIN OF K^+ = $|l_q - l_3| \cdots l_q + l_3$

The parity of the 3π system is then $P_{3\pi} = (-1)^{l_q + l_3 + 1}$ ← intrinsic

We now consider the form of the matrix element for various spin and parity combinations, labelled $J^P = 0^+, 0^-, 1^+, 1^- \ldots$

$J^P = 0^+$ $\quad J = 0 \Rightarrow l_q = l_3 = 0 \Rightarrow P_{3\pi} = -1 \Rightarrow 0^+$ is impossible

$J^P = 0^-$ $\quad$ Again $l_q = l_3 = 0$. In turn this implies that the matrix element cannot depend directly on $\bar{q}$ or $\bar{P}_3$ (i.e. $\bar{q} \cdot \bar{P}_3$ not allowed). $\therefore \mathcal{M}_{fi}$ will be essentially uniform over the Dalitz plot. [If $l_q = n$, then $\mathcal{M}_{fi}$ depends on $P_n(\cos\theta_q) \sim (\cos\theta_q)^n \sim (\hat{z}\cdot\bar{P}_q)^n$ etc.]

$J^P = 1^+$ $\quad$ The matrix element must transform like a vector, since $J = 1$. By this we mean the initial state has vector polarization $\bar{\epsilon}$, so the matrix element must be a vector $\bar{m}$ leading to an overall scalar interaction $\bar{\epsilon} \cdot \bar{m}$. We also note that since the 3π state has negative intrinic parity, the vector $\bar{m}$ must also have negative parity, leading to an overall 1^+ state. Recall that an ordinary vector (polar vector) has negative parity.

With l_q even, we might have $l_3 = 1, l_q = 0$ or $l_3 = 1 \ldots l_q = 2 \ldots$ to have overall positive parity

The simplest case is clearly $l_3 = 1$, $l_q = 0$. This indicates the matrix element might depend on $\bar{P}_3$ but not $\bar{q}$

$$\bar{m} \sim \bar{P}_3$$

This vanishes when $T_3 \to 0$, i.e. at the bottom of the Dalitz plot

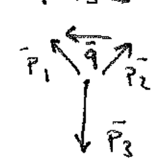
 ← forbidden if $J^P = 1^+$ (for simplest l_3, l_q)

$J^P = 1^-$ $\quad$ This time the matrix element must transform like a positive-parity axial vector. The overall parity is negative so the simplest orbital angular momentum configuration is $l_3 = l_q = 2$, since l_q must be even.

Hence $\bar{m}$ depends on $\bar{P}_3$ and $\bar{q}$ quadratically, but so as to form an axial vector

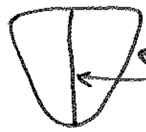
i.e. $\bar{m} \sim (\bar{P}_3 \cdot \bar{q})(\bar{P}_3 \times \bar{q})$

Now $(\bar{P}_3 \cdot \bar{q})$ vanishes when $\bar{q}$ is $\perp$ to $\bar{P}_3 \Rightarrow T_1 = T_2$ according to the picture



THIS OCCURS ALL ALONG THE VERTICAL MIDLINE OF THE DALITZ PLOT.

$\vec{q} \times \vec{p}_3$ VANISHES WHEN $\vec{q}$ & $\vec{p}_3$ LINE UP, WHICH HAPPENS WHEN ALL 3 MOMENTA ARE COLINEAR. ON p195 WE NOTED THAT THIS HAPPENS EVERYWHERE ON THE BOUNDARY OF THE DALITZ PLOT



FORBIDDEN IF $J^P = 1^-$

THE DATA SHOW AN ESSENTIALLY UNIFORM DALITZ PLOT, WHICH INDICATES THAT THE 3π'S ARE PRODUCED IN A 0^- STATE.

SO CERTAINLY THE K^+ HAS SPIN 0. BUT WE CAN SAY THAT IT HAS NEGATIVE PARITY ONLY IF THE DECAY PROCESS IS PARITY CONSERVING, WHICH IT ISN'T.

RECALL THE NEGATIVE-PARITY ASSIGNMENT FOR THE K IS MADE VIA THE STRONG INTERACTION.

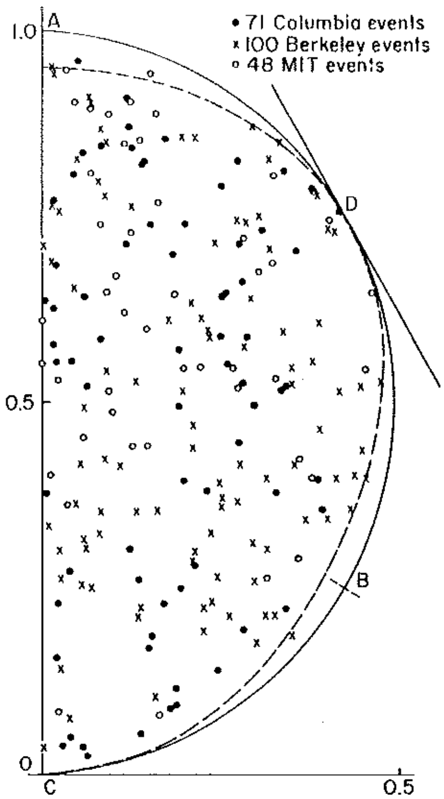


FIG. 7.13. The Dalitz plot for 219 decays $K^+ \to \pi^+\pi^+\pi^-$. The boundary of the plot is the broken line. The solid line is the inscribed circle to the triangle and is the boundary for a nonrelativistic treatment of the kinematics. (Orear *et al.*, 1956.)

6. $\omega(780) \to \pi^+\pi^-\pi^0$

THIS PARTICLE WAS FOUND IN THE REACTION $\bar{p}p \to 5\pi$ AND IS MEASURED TO HAVE A BROAD WIDTH $\Gamma \sim 10\ \text{MEV} \Rightarrow \tau = 1/\Gamma \sim 10^{-22}$ SEC.

THIS RAPID DECAY RATE INDICATES THAT THE STRONG INTERACTION IS INVOLVED, SO THAT PARITY AND ISOSPIN WILL BE CONSERVED. WE SAW THAT THE ω HAS NO CHARGED PARTNERS AND SO HAS $I = 0$.

COMPARED TO THE K^+ CASE WE NO LONGER HAVE THE SPECIAL RESTRICTION OF 2 IDENTICAL PARTICLES IN THE FINAL STATE, AND SO NO PARTICULAR 2+1 GROUPING IS FAVORED.

ON THE OTHER HAND AN $I=0$ 3π STATE CAN ONLY BE MADE OUT OF A 2+1 GROUPING WITH $I=1$ FOR THE PAIR, ACCORDING TO THE RULES OF SPIN ADDITION. HENCE THE MATRIX ELEMENT MUST BE ANTI-SYMMETRIC WITH RESPECT TO THE INTERCHANGE OF ANY 2 PIONS (AS THE $I=1$ 2π STATE IS ANTISYMMETRIC). WE CAN NOW CATALOG THE POSSIBLE MATRIX ELEMENTS FOR SPIN 0 OR 1 (GELL-MANN)

$J^P = 0^+$ EXCLUDED AS BEFORE

$J^P = 0^-$ THE MATRIX ELEMENT IS A SCALAR, BUT ANTI SYMMETRIC UNDER PARTICLE INTERCHANGE. A PLAUSIBLE FORM DEPENDS ON THE PION ENERGIES

$$\mathcal{M} \sim (E_1 - E_2)(E_2 - E_3)(E_3 - E_1)$$

THIS VANISHES ON THE MIDLINES OF THE DALITZ PLOT

$J^P = 1^+$ THE MATRIX ELEMENT IS AN ORDINARY VECTOR, AND ANTISYMMETRIC UNDER PARTICLE INTERCHANGE

$$\mathcal{M} \sim \bar{P}_1 (E_2 - E_3) + \bar{P}_2 (E_3 - E_1) + \bar{P}_3 (E_1 - E_2)$$

THIS VANISHES WHEN ALL 3 ENERGIES ARE EQUAL; I.E. AT THE CENTER OF THE DALITZ PLOT. ALSO IT VANISHES IF $\bar{P}_j = 0$ AND THE OTHER 2 ENERGIES ARE EQUAL.

$J^P = 1^-$ THE MATRIX ELEMENT IS AN AXIAL VECTOR

$$\mathcal{M} \sim \bar{P}_1 \times \bar{P}_2 + \bar{P}_2 \times \bar{P}_3 + \bar{P}_3 \times \bar{P}_1 = 3\, \bar{P}_j \times \bar{P}_k \quad \text{USING} \quad \bar{P}_1 + \bar{P}_2 + \bar{P}_3 = 0$$

THIS VANISHES WHENEVER $\bar{P}_k \parallel \bar{P}_j$, WHICH IS EXACTLY THE CONDITION FOR THE BOUNDARY OF THE DALITZ PLOT.

THE DATA SHOW DEPLETION OF THE DALITZ PLOT POPULATION NEAR THE BOUNDARY.

THE EVENTS ARE FROM THE REACTION $\pi^+ p \to \omega \Delta$ AND SO INCLUDE SOME CONTAMINATION OF π'S FROM THE Δ DECAY. SO THE POPULATION AT THE BOUNDARY DOESN'T VANISH.

BUT $J^P = 1^-$ IS STRONGLY INDICATED.

$e^+e^- \to \omega$ CONFIRMS THIS.

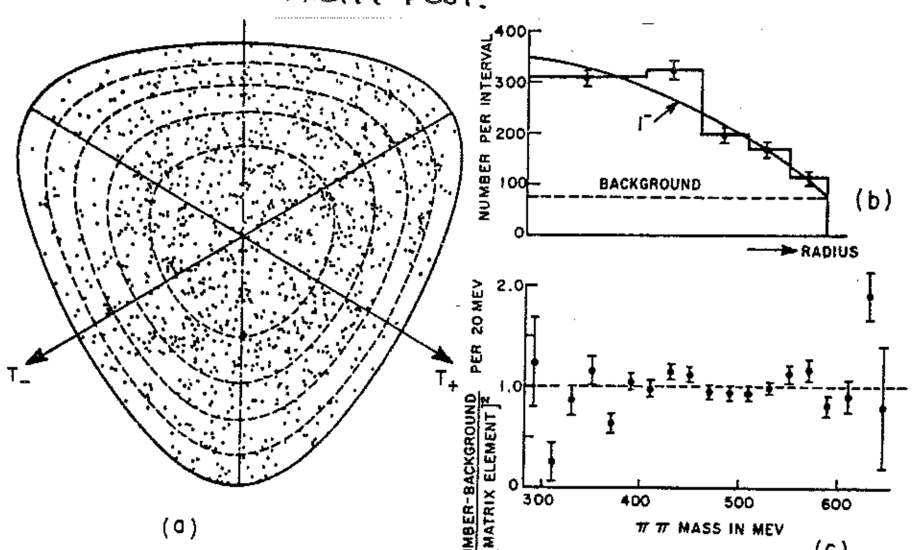


Fig. 15.2. (a) The Dalitz plot for 1100 omegas (including a background of 375 nonresonant triplets). (b) The density of points on the Dalitz plot compared to the expected density for a 1^- ω plus a uniformly distributed background. (c) The dependence of the $\pi\pi$ interaction in the $I = 1$ $J = 1$ state as a function of energy. This was obtained by summing the $\pi^+\pi^-$, $\pi^+\pi^0$, and $\pi^-\pi^0$ mass spectra for pion pairs from the ω decays, subtracting a background, and dividing by the distribution expected for 1^- decay into $\pi^+\pi^-\pi^0$. Since two of the three mass combinations are independent, an error corresponding to $(2/3N)^{1/2}$, where N is the number of pairs per interval before background subtraction, was assigned to each point (ALFF 1962)

c. $\eta(548) \to \pi^+\pi^-\pi^0$

The fact that $\eta \to \gamma\gamma$ as well as 3π tells us quickly that $J^P = 0^-$ is the probable assignment for the η. (ie $\eta \to \gamma\gamma \Rightarrow J \neq 1$, $\eta \to 3\pi \Rightarrow$ not 0^+). A Dalitz-plot analysis of the 3π decay confirms this.

We have already noted that $\eta \to 3\pi$ violates G parity and cannot be a strong decay. It is consistent that the decay is due to an electromagnetic interaction – which at least conserves charge conjugation. This tells us that the 3π final state cannot have $I = 0$. That is, $I = 0,\ G = -1 \Rightarrow C = -1$, but $\eta \to \gamma\gamma \Rightarrow C_\eta = +1$.

So we need only consider the $I = 1$ 3π states. Again C helps us, if we note $C(\pi^+\pi^-\pi^0) = C(\pi^+\pi^-)\,C(\pi^0) = (-1)^{l_q}$ where l_q = orbital angular momentum of the $\pi^+\pi^-$ (boson-antiboson) system. Then $C_\eta = +1 \Rightarrow l_q$ must be even. Hence the form of the $\pi^+\pi^-\pi^0$ matrix elements will be exactly like those we found for $K^+ \to \pi^+\pi^+\pi^-$!

The population of the η Dalitz plot appears to be rather uniform, consistent with $J^P = 0^-$.

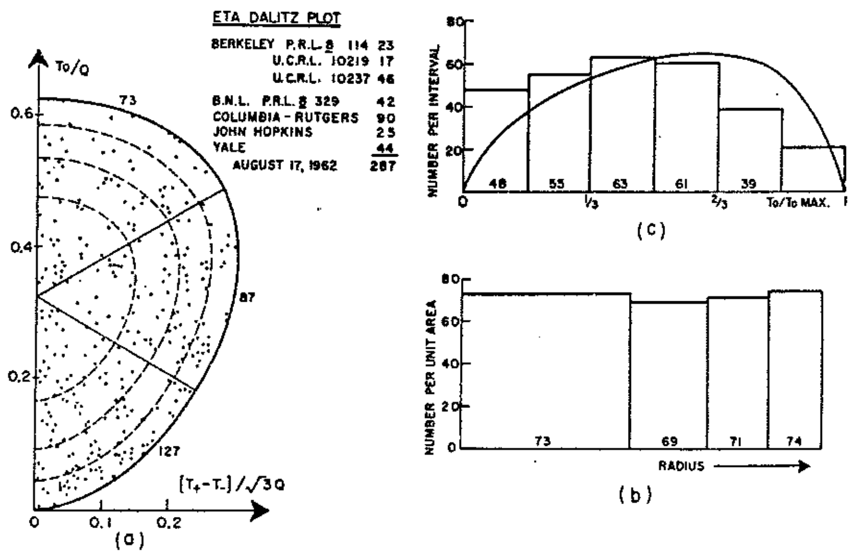
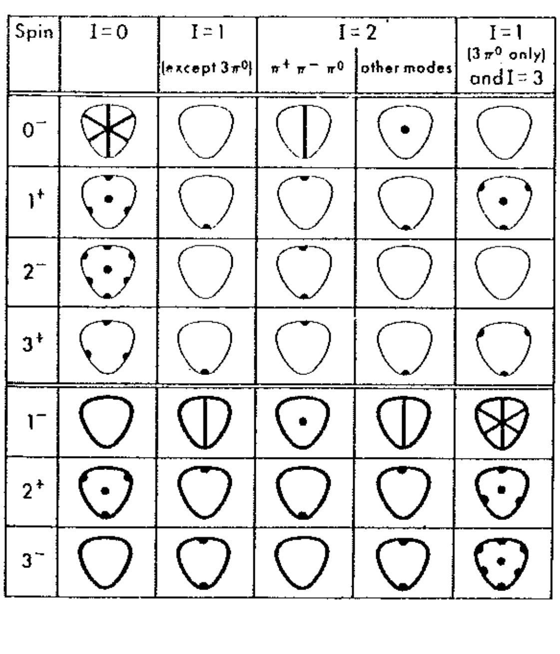


Fig. 16.2. The Dalitz plot and projections for published η decays into $\pi^+\pi^-\pi^0$. (a) shows the distribution of points, (b) the radial density, and (c) the projection of the points on the π^0 axis. The solid line in (c) corresponds to uniform population (Alff 1962)



← A catalog of forbidden regions in 3π Dalitz plots with I and J up to 3 has been given by Zemach, P.R. B133, 1201 (1964).

PARTIAL-WAVE ANALYSIS

[WE FOLLOW PERKINS SECS 4.7-4.9 FAIRLY CLOSELY.]

WE LEAVE PARTICLE DECAYS FOR A WHILE, AND TAKE UP AN ANALYSIS OF PARTICLE SCATTERING WHICH IS INDEPENDENT OF THE DETAILED FORM OF THE INTERACTION. CONSIDER ELASTIC SCATTERING OF 2 SPINLESS PARTICLES, $a+b \to a+b$. IN THIS SCATTERING, ORBITAL ANGULAR MOMENTUM, l, IS A CONSERVED QUANTUM NUMBER, ALTHOUGH IN GENERAL THE INITIAL STATE DOES NOT HAVE A DEFINITE l. HOWEVER, IF WE INTEGRATE THE SCATTERING CROSS SECTION, $d\sigma/d\Omega$, OVER ANGLES TO GET σ, THE STATES OF DIFFERING l CANNOT MIX, AND WE WRITE

$$\sigma = \sum_l \sigma_l$$

THIS IS THE BASIC IDEA OF THE <u>PARTIAL WAVE ANALYSIS</u>. SUCH AN ANALYSIS WILL ONLY BE RELEVANT WHEN A FEW VALUES OF l DOMINATE THE SERIES. THIS WILL BE THE CASE IF $a+b$ COMBINE TO FORM A SHORT-LIVED 'RESONANCE' OF A PARTICULAR ANGULAR MOMENTUM. THEN THE REACTION $a+b \to a+b$ CAN BE THOUGHT OF AS $a+b \to c$ FOLLOWED BY THE 'DECAY' $c \to a+b$.

THE BASIC PROCEDURES OF THE PARTIAL WAVE ANALYSIS CAN BE UNDERSTOOD FROM A NON-RELATIVISTIC VIEW OF THE SCATTERING, AS INTRODUCED IN LECTURE 5. WE WORK IN THE C.M. FRAME, WHERE EACH PARTICLE HAS MOMENTUM k. THE INITIAL STATE IS A PLANE WAVE $e^{ikz}(e^{-i\omega t})$ MOVING ALONG THE z-AXIS. (STRICTLY SPEAKING, THIS DESCRIBES ONLY PARTICLE a. PARTICLE b IS DESCRIBED BY A WAVE e^{-ikz}, ETC. BUT IT IS SUFFICIENT TO FOLLOW PARTICLE a)

THE FINAL STATE INCLUDES A SMALL CORRECTION TO THE INCIDENT PLANE WAVE, NAMELY THE SCATTERED SPHERICAL WAVE.

$$\psi_f \approx e^{ikz} + \frac{e^{ikr}}{r} f(\theta)$$

WE SAW IN LECTURE 5, THAT $d\sigma/d\Omega = |f(\theta)|^2$

A GENERAL FORM FOR $f(\theta)$ IS SUGGESTED BY EXPANDING THE PLANE WAVE IN TERMS OF SPHERICAL WAVES:

$$e^{ikz} = \frac{1}{2ikr} \sum_l (2l+1) \left[e^{ikr} - (-1)^l e^{-ikr} \right] P_l(\cos\theta) \qquad (kr \gg 1)$$

OUTGOING ↗ ↖ INCOMING WAVES

PLAUSIBLY, $\psi_{f,\,\mathrm{OUTGOING}} = \frac{e^{ikr}}{2ikr} \sum_l (2l+1)\, a_l\, P_l(\cos\theta)$ $\qquad a_l$ COMPLEX

AND $f(\theta) = \frac{1}{k} \sum_l (2l+1) \frac{a_l - 1}{2i} P_l(\cos\theta)$

A KEY RESTRICTION IS THAT $|a_\ell| \le 1$ IN ORDER TO CONSERVE PROBABILITY. THE OUTGOING PROBABILITY FLUX CANNOT BE GREATER THAN THAT IF NO SCATTERING OCCURRED AT ALL. PROBABILITY FLUX DENSITY IS GIVEN BY $\vec{J} = \hat{r}\, v\, |\psi_{f,\text{outgoing}}|^2 = \hat{r}\, \frac{k}{m} |\psi_{f,\text{out}}|^2$

AND TOTAL FLUX $= \int r^2 J_r \, d\Omega = \frac{\pi}{mk} \sum_\ell (2\ell+1) |a_\ell|^2$

BUT TOTAL FLUX IF NO SCATTER IS $\frac{\pi}{mk} \sum_\ell (2\ell+1)$

$\Rightarrow |a_\ell|^2 \le 1$

OFTEN PEOPLE WRITE $a_\ell = \eta_\ell e^{2i\delta_\ell}$ $\quad 0 \le \eta_\ell \le 1$ & REAL, $\delta_\ell \equiv$ PHASE SHIFT

THEN $\sigma_{\text{elastic}} = \int \frac{d\sigma}{d\Omega} d\Omega = \int |f(\theta)|^2 d\Omega = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \left| \frac{\eta_\ell e^{2i\delta_\ell} - 1}{2i} \right|^2$

IF $\eta_\ell < 1$ THE OUTGOING FLUX IS LESS THAN THE INCOMING FLUX, SO SOMETHING ELSE MUST BE HAPPENING BESIDES ELASTIC SCATTERING. WE LUMP ALL OTHER POSSIBILITIES TOGETHER UNDER THE TITLE 'ABSORPTION', OR 'INELASTIC SCATTERING.'

FOR THE SIMPLE CASE WHEN NO ABSORPTION OCCURS, $\eta_\ell = 1$, AND

$$\sigma_{\text{elastic}} = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2 \delta_\ell$$

WE CAN DEFINE AN 'ABSORPTION CROSS SECTION' BY MEANS OF PROBABILITY CONSERVATION.

$v\, \sigma_{\text{abs}}$ = RATE OF ABSORPTION = FLUX IF NO SCATTER − OUTGOING FLUX OF ELASTIC SCATTERERS

$$= \frac{\pi}{mk} \sum_\ell (2\ell+1) \left(1 - |a_\ell|^2\right)$$

SO $\sigma_{\text{abs}} = \frac{\pi}{k^2} \sum_\ell (2\ell+1)(1 - \eta_\ell^2)$

SOME LIMITING CASES ARE WORTH NOTING

$\sigma_{\text{elastic},\ell} \le \frac{4\pi}{k^2}(2\ell+1)$ LIMIT ACHIEVED IF $\eta_\ell = 1$, $\delta_\ell = 90°$

$\sigma_{\text{abs},\ell} \le \frac{\pi}{k^2}(2\ell+1)$ LIMIT ACHIEVED IF $\eta_\ell = 0 \Leftrightarrow$ 'TOTAL ABSORPTION'

BUT EVEN WHEN $\eta_\ell = 0$, $\sigma_{\text{elastic},\ell} = \frac{\pi}{k^2}(2\ell+1) = \sigma_{\text{abs},\ell}$

THIS PARADOXICAL RESULT MAY BE FAMILIAR FROM CLASSICAL OPTICS: IF AN OBJECT ABSORBS ALL THE LIGHT THAT HITS IT, THERE IS STILL A DIFFRACTION SCATTERING AROUND THE OBJECT, WITH SCATTERING CROSS SECTION EXACTLY THAT OF THE GEOMETRICAL CROSS SECTION OF THE OBJECT.

ANOTHER RESULT WITH A CLASSICAL-OPTICS ANALOGY IS THE <u>OPTICAL THEOREM</u> (FIRST PROVED IN HIGH-ENERGY PHYSICS, HOWEVER)

WE DEFINE $\sigma_{TOTAL} \equiv \sigma_{ELASTIC} + \sigma_{ABS} = \frac{2\pi}{k^2} \sum_l (2l+1)\left(1 - \eta_l \cos 2\delta_l\right)$

NOTE THAT $\mathrm{Im}\, f(\theta = 0°) = \frac{1}{2k} \sum_l (2l+1)(1 - \eta_l \cos 2\delta_l)$

SO $\sigma_{TOT} = \frac{4\pi}{k} \mathrm{Im}\left[f(0)\right]$ FOR WHAT IT'S WORTH.

2. <u>BREIT-WIGNER RESONANCE</u>

WE NOTED THAT THE MAXIMUM CROSS SECTION IN A PARTICULAR PARTIAL WAVE l IS ACHIEVED WHEN $\delta_l \to 90°$. BUT PHYSICALLY, WE MIGHT EXPECT A BIG CROSS SECTION IF PARTICLES a AND b COMBINE TO FORM A KIND OF RESONANT INTERMEDIATE STATE, WHICH THEN DECAYS BACK TO $a+b$. A CONNECTION BETWEEN THESE 2 POINTS OF VIEW WAS PROVIDED BY BREIT AND WIGNER.

DEFINE $f_l = \frac{\eta e^{2i\delta} - 1}{2i}$ $\equiv$ SCATTERING AMPLITUDE FOR THE lTH PARTIAL WAVE

WE RESTRICT OURSELVES TO THE CASE $\eta = 1$ $\to$ NO ABSORPTION.

THEN $f_l = e^{i\delta} \sin\delta = \frac{1}{\cot\delta - i}$

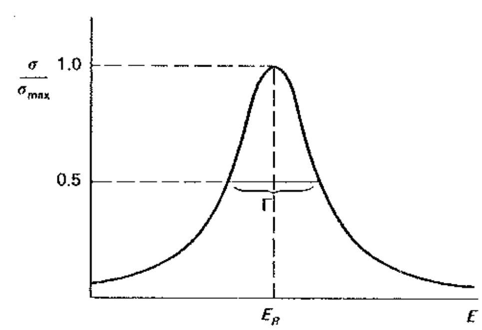
THE PHASE SHIFT δ IS A FUNCTION OF THE C.M. ENERGY, AND WHEN $\delta(E) = 90°$ WE SAY $E = E_R$ = ENERGY (OR BETTER, MASS) OF THE RESONANT STATE. THE INSIGHT OF BREIT AND WIGNER IS THAT GOOD THINGS HAPPEN IF WE EXPAND $\cot\delta(E)$ ABOUT E_R.

$$\cot\delta(E) = \underbrace{\cot\delta(E_R)}_{=0} + (E - E_R)\frac{d}{dE}\cot\delta(E_R)$$

$$= -(E - E_R)\frac{2}{\Gamma} \quad \text{WHEN WE DEFINE} \quad \frac{2}{\Gamma} = -\frac{d}{dE}\cot\delta(E_R)$$

THEN $f_l = \frac{\Gamma/2}{(E_R - E) - i\Gamma/2}$

AND $\sigma_{ELASTIC,\,l}(E) = \frac{\pi}{k^2}(2l+1)\frac{\Gamma^2}{(E - E_R)^2 + \Gamma^2/4}$

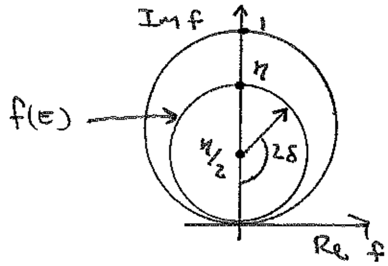


WE SEE THAT Γ IS THE FULL WIDTH AT HALF MAX OF THE RESONANCE CURVE $\sigma(E)$. THEN THE MEAN LIFETIME OF THE RESONANT STATE IS GIVEN BY $\tau = 1/\Gamma$, AS DISCUSSED IN LECTURE 1.

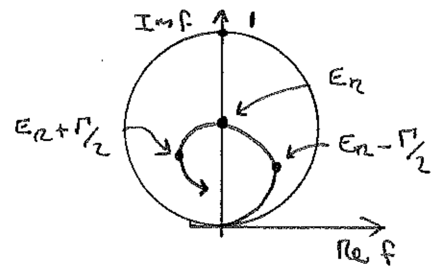
WE DID NOT REALLY NEED THE PARTIAL-WAVE ANALYSIS TO INTERPRET THE GIANT PEAKS SEEN IN $\sigma_{\pi N \to \pi N}$ AS RESONANCES (p. 182)

BUT MANY OTHER RESONANCES HAVE BEEN FOUND BY DETAILED FITS OF $\sigma(E)$ FOR $\eta_l(E)$ AND $\delta_l(E)$. THE RESULTS OF THESE INTRICATE ANALYSES CAN BE PRESENTED IN A NICE GEOMETRICAL WAY, ON THE ARGAND DIAGRAM

$$f = \frac{\eta e^{2i\delta} - 1}{2i}$$



$\eta(E)$ = CONSTANT



$\eta(E)$ VARYING

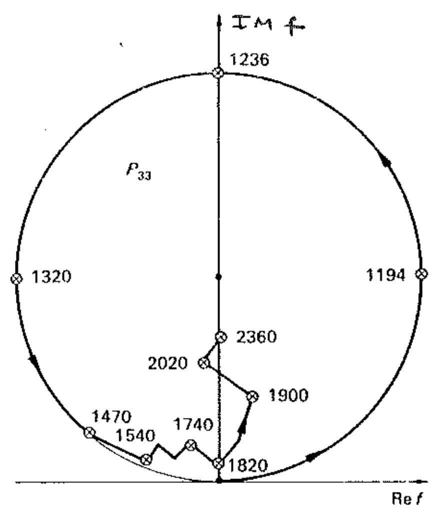
IF $\eta(E)$ = CONSTANT, $f(E)$ TRACES OUT A CIRCLE OF RADIUS $\eta/2$ AS SHOWN. THE RESONANT ENERGY E_R IS THAT FOR WHICH THE CURVE $f(E)$ CROSSES THE IMAGINARY AXIS. THE WIDTH Γ CAN ALWAYS BE CALCULATED AS

$$\Gamma = \frac{-2}{\frac{d}{dE} \cot \delta(E_R)}$$

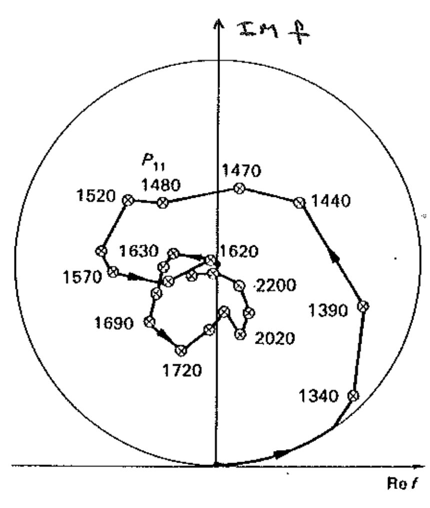
BUT TO A FIRST APPROXIMATION, $E_R \pm \Gamma/2$ ARE THE ENERGIES AT WHICH $\delta = 45^\circ$ OR 135°, CORRESPONDING TO THE EXTREMES OF THE CURVE $f(E)$ WITH RESPECT TO THE REAL AXIS.

THE CLASSIC EXAMPLE IS THE $\Delta(1236)$ πN RESONANCE, FOR WHICH $\eta \sim 1$.

NOTE THAT THE SIMPLE PRESCRIPTION FOR READING Γ OFF THE ARGAND DIAGRAM WOULD YIELD $\Gamma \approx 84$ OR 168 MeV. THE CONSENSUS IS THAT $\Gamma \sim 115$ MeV, AS READ DIRECTLY OFF A PLOT OF $\sigma(E)$, p. 209.



A CASE OF 'RESONANCE' NOT DIRECTLY VISIBLE IN $\sigma_{\pi N}$ (p. 182) IS THE $N^*(1470)$. THIS IS DETERMINED TO HAVE $I = 1/2$, $J^P = 1/2^+$ FROM THE PARTIAL-WAVE ANALYSIS. THIS STATE, THE SO-CALLED 'ROPER RESONANCE' HAS AN INTERESTING INTERPRETATION IN THE QUARK MODEL.



3. Resonance and Spin

In case the initial state particles a and b have spin, the resonance cross section is given by

$$\sigma_{EL}(E) = \frac{\pi}{k^2} \frac{(2S_R+1)}{(2S_a+1)(2S_b+1)} \frac{\Gamma^2}{(E-E_R)^2 + \Gamma^2/4}$$

As a 'mnemonic derivation', we note that the spin factors in the denominator are consistent with our prescription to average over initial state spins for unpolarized beam and target particles. Likewise the factor $2S_R+1$, which counts the number of spin states of the resonance, can be thought of as resulting from the sum over final state spins. (This argument won't bear too close scrutiny)

The peak cross section for $\pi + N \to \Delta(1236)$ is $8\pi/k^2$. According to the spin factors listed above,

$$\sigma_{peak} = \frac{4\pi}{k^2} \frac{(2S_\Delta+1)}{2}$$

So $S_\Delta = 3/2$

This argument assumes that $\eta = 1$, which is not self-evident, but can be inferred from the Argand diagram on p 208.

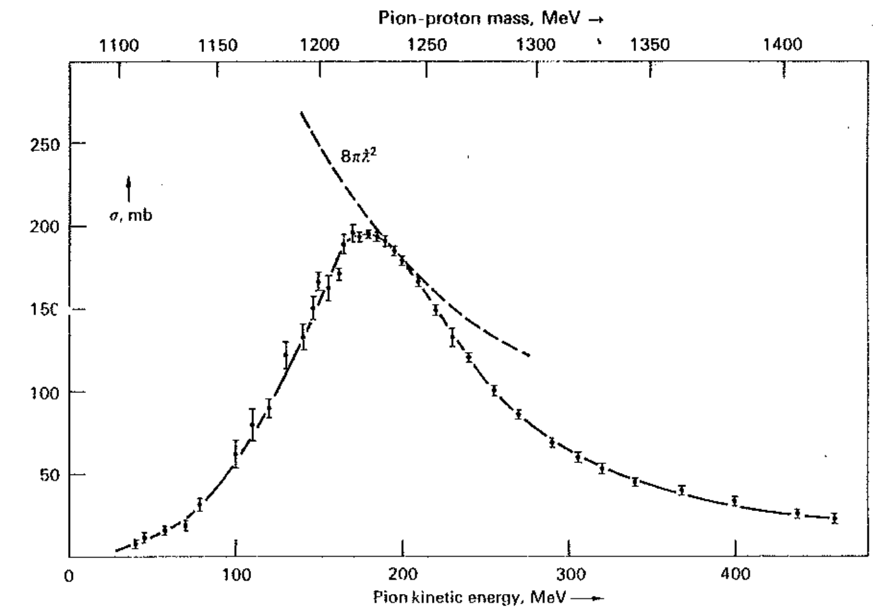
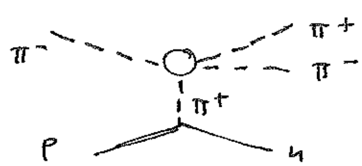


Fig. 4.17 The π^+p total cross section as a function of kinetic energy of the incident pion, or the π^+p mass, in the region of the 1236 MeV, $I = \frac{3}{2}, J^P = \frac{3}{2}^+$ resonance. Not all experimental points have been included. The maximum cross section, $8\pi\lambda\!\!\!^{-2}$, allowed by conservation of probability is shown dashed.

Another example concerns the reaction $\pi^- p \to \pi^+ \pi^- n$. This is interpreted as due to one pion exchange:



If we believe the meson theory, the lower vertex factor can be calculated, leaving us with a 'measurement' of the reaction $\pi^-\pi^+ \to \pi^-\pi^+$

The cross section so extracted has a big bump at ~ 760 MeV, with $\sigma_{peak} \sim 8\pi/k^2$. The peak is identified as the ρ meson, which is also observed in $\pi^+ p \to \pi^+ \pi^0 p$, so has isospin 1. Now $I = 1$ 2π states can only have odd spin according to Bose statistics.
If $S_\rho = 1$, we expect $\sigma_{peak} \sim 12\pi/k^2$, while if $S_\rho = 3$, $\sigma_{peak} \sim 28\pi/k^2$.
So $S_\rho = 1$, as verified by other techniques.

4. PARTIAL WIDTHS

SUPPOSE A RESONANCE C CAN DECAY IN MANY WAYS. THEN IT HAS A TRANSITION RATE, OR PARTIAL WIDTH Γ_f TO EACH FINAL STATE f. THE TOTAL DECAY RATE IS

$$\Gamma_t = \sum_f \Gamma_f$$

IF A POSSIBLE FINAL STATE IS THE 2 PARTICLES a & b, WE COULD PRODUCE RESONANCE C IN THE REACTION

$$a + b \to c \to a + b$$

or $$a + b \to c \to \text{ANY ALLOWED FINAL STATE}$$

WHAT ARE THE BREIT-WIGNER CROSS SECTION FORMULAE FOR THESE REACTIONS?

THE ENERGY BEHAVIOR OF THE CROSS SECTION IS GOVERNED BY THE TOTAL WIDTH, ACCORDING TO THE SIGNIFICANCE OF Γ_t AS DISCUSSED IN LECTURE 1.

$$\sigma(E) \sim \frac{1}{(E-E_R)^2 + \Gamma_t^2/4}$$

THE FACTORS OF Γ IN THE NUMERATOR OF THE EXPRESSION FOR σ REPRESENT THE COUPLING OF THE RESONANCE TO THE INITIAL AND FINAL STATES. CLEARLY THE CROSS SECTION FOR $a+b \to c \to$ ANYTHING IS BIGGER THAN THAT FOR $a+b \to c \to a+b$. WE CONCLUDE

$$\sigma_{a+b \to c \to a+b} = \frac{\pi}{k^2} \cdot \text{SPIN FACTOR} \cdot \frac{\Gamma_{ab}^2}{(E-E_R)^2 + \Gamma_t^2/4}$$

$$\sigma_{a+b \to c \to \text{ANYTHING}} = \frac{\pi}{k^2} \cdot \text{SPIN FACTOR} \cdot \frac{\Gamma_{ab}\Gamma_t}{(E-E_R)^2 + \Gamma_t^2/4}$$

$$\sigma_{a+b \to c \to d+e} = \pi/k^2 \cdot \text{SPIN FACTOR} \cdot \frac{\Gamma_{ab}\Gamma_{de}}{(E-E_R)^2 + \Gamma_t^2/4}$$

THESE RELATIONS FIND APPLICATION IN THE REACTIONS $e^+e^- \to$ VECTOR MESONS, AS OBSERVED IN e^+e^- COLLISIONS AT STORAGE RINGS. ON pp 106-107 WE PRESENTED AN ANALYSIS FOR $e^+e^- \to \rho^0 \to \pi^+\pi^-$

SINCE $\rho^0 \to \pi^+\pi^-$ ~100% OF THE TIME, $\Gamma_{\pi^+\pi^-} \sim \Gamma_t$

THEN $$\sigma = \frac{\pi}{k^2} \cdot \frac{3}{4} \frac{\Gamma_{ee}\Gamma_t}{(E-M_\rho)^2 + \Gamma_t^2/4} \qquad (k = E_e)$$

SO $$\sigma_{PEAK} = \frac{3\pi}{E_e^2} \frac{\Gamma_{ee}}{\Gamma_t} = \frac{12\pi}{M_\rho^2} \frac{\Gamma_{ee}}{\Gamma_t}$$

COMPARISON WITH EXPERIMENT YIELDS $\Gamma_{ee}/\Gamma_t \sim 6 \times 10^{-5}$

Γ_t IS READ DIRECTLY FROM THE SHAPE OF $\sigma(E)$: $\Gamma_\rho \sim 150$ MeV, $\Gamma_\omega \sim 10$ MeV, $\Gamma_\phi \sim 4$ MeV

5. RESONANCE-DECAY ANGULAR DISTRIBUTIONS

THE PARTIAL-WAVE ANALYSIS GIVES A SYSTEMATIC EXPANSION OF $d\sigma/d\Omega$ IN TERMS OF SQUARES OF LEGENDRE POLYNOMIALS.

IF THE CROSS SECTION IS DOMINATED BY A RESONANCE IN A PARTICULAR PARTIAL WAVE, THE ANGULAR DISTRIBUTION CAN BE CALCULATED MORE QUICKLY BY A 'STRAIGHTFORWARD' APPROACH.

FOR EXAMPLE, THE REACTION $\pi N \to \Delta(1236) \to \pi N$. WE HAVE ARGUED THAT THE SPIN OF THE Δ IS $3/2$, BUT WE WOULD LIKE TO CONFIRM THIS ASSIGNMENT BY OBSERVING THE ANGULAR DISTRIBUTION OF THE FINAL STATE. THE πN ORBITAL ANGULAR MOMENTUM MUST BE $l = 1$ OR 2, CORRESPONDING TO POSITIVE OR NEGATIVE PARITY FOR THE Δ. WE HAVE A PREJUDICE IN FAVOR OF $l = 1$ BASED ON THE LOW ENERGY IDEA OF THE 'ANGULAR MOMENTUM BARRIER'. I MAKE NO ATTEMPT TO JUSTIFY THIS, BUT NOTE THAT THE CLAIM IS THAT REACTIONS WITH ENERGIES BARELY ABOVE THRESHOLD PREFER THE LOWEST POSSIBLE ORBITAL ANGULAR MOMENTUM l.

SO SUPPOSE WE PROCEED UNDER THE ASSUMPTION THAT THE πN ORBITAL ANGULAR MOMENTUM IS $l = 1$ IN $\Delta(1236)$ PRODUCTION AND DECAY. THE INITIAL πN STATE CAN HAVE $J_z = \pm 1/2$ ONLY, DUE TO THE NUCLEON SPIN, AS $l_z = 0$ FOR A PLANE WAVE ALONG THE Z AXIS. HENCE WE CAN ACTUALLY PRODUCE ONLY 2 OF THE 4 SPIN STATES OF THE Δ

NAMELY: $|3/2, \pm 1/2\rangle$

THIS STATE THEN COUPLES TO THE FINAL πN STATE, WHOSE WAVE FUNCTION IS $$|3/2 \ 1/2\rangle = \sqrt{2/3}\, Y_1^0(\theta,\varphi)\,|1/2 \ 1/2\rangle + \sqrt{1/3}\, Y_1^1(\theta,\varphi)\,|1/2 \ {-1/2}\rangle$$

USING THE C.G TABLES. (THERE IS NO NEED TO CALCULATE THE $|3/2 \ {-1/2}\rangle$, WHICH IS RELATED TO THE ABOVE BY A PARITY TRANSFORMATION)

$$\text{SO} \quad d\sigma/d\Omega \sim \tfrac{2}{3}|Y_1^0|^2 + \tfrac{1}{3}|Y_1^1|^2 \sim \frac{1 + 3\cos^2\theta}{8\pi}$$

[WE SQUARE BEFORE ADDING AS THE FINAL STATE SPINS ARE DISTINGUISHABLE]

EXERCISE: VERIFY THAT IF THE πN STATE WERE PURE d WAVE, THE SAME ANGULAR DISTRIBUTION WOULD HOLD.

THE DATA (p 212) AGREE WELL WITH $1 + 3\cos^2\theta$ AT $E = E_R$, BUT DIFFER SIGNIFICANTLY OFF RESONANCE. THIS IS GOOD CONFIRMATION THAT $J = 3/2$ FOR THE Δ, BUT DOES NOT STRICTLY PROVE $l_{\pi N} = 1$. TO DEMONSTRATE THIS, WE WOULD HAVE TO CONSIDER THE INTERFERENCE BETWEEN PARTIAL WAVES, A MORE LENGTHY PROCEDURE...

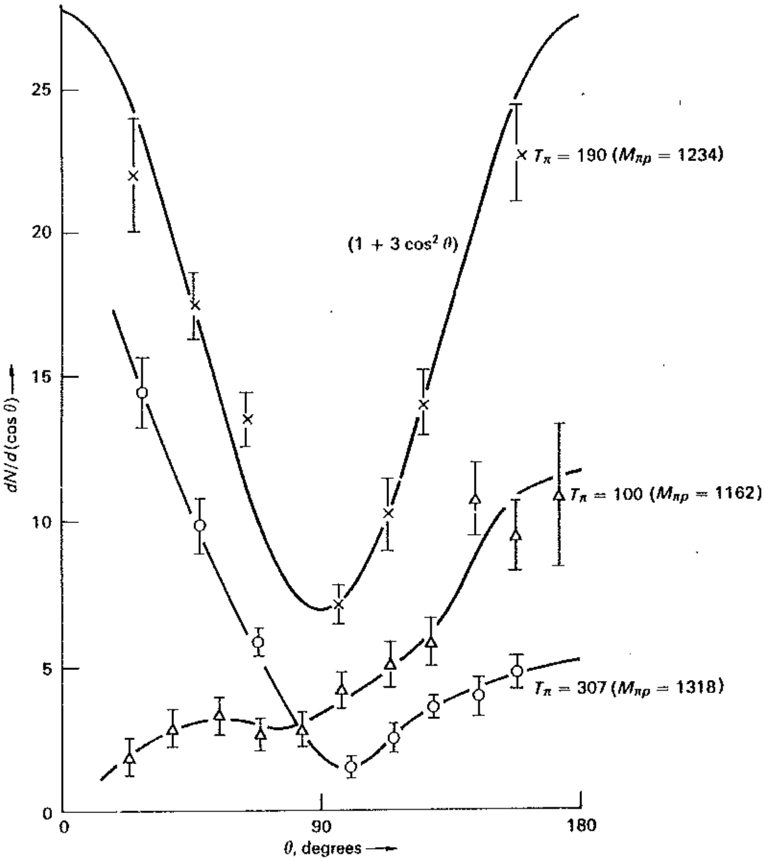


Fig. 4.20 The angular distribution of the scattered pion, relative to the incident pion, in $\pi^+ p$ elastic scattering, as measured in the center-of-mass frame. In the region of the Δ-resonance of mass 1236 MeV ($T_\pi = 190$ MeV), the distribution has the form $1 + 3\cos^2\theta$, as in (4.52).

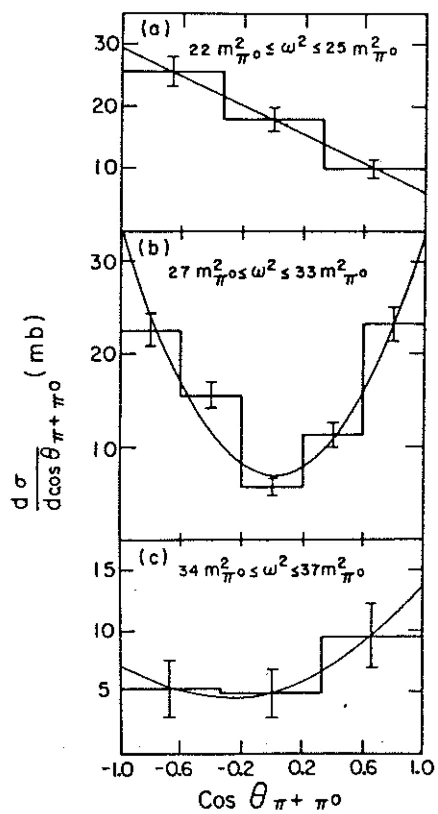
A simple example of a resonance analysis taking interference into account is $\pi\pi$ scattering, as extracted from $\pi N \to 2\pi N$ (p 209). We suspect that the $\rho(750)$ resonance has $J=1$, but would like to confirm this.

If we keep the S and P wave terms in the partial wave expansion, then

$$\frac{d\sigma}{d\Omega} \sim | A_S P_0(\cos\theta) + A_P P_1(\cos\theta) |^2$$
$$\sim A_S^2 + 2 \mathrm{Re}\, A_S^* A_P \cos\theta + A_P^2 \cos^2\theta$$

The data are well fit by this form, and with $A_P \gg A_S$.

A $J=3$ assignment for the ρ would lead to $\cos^6\theta$ terms, which are 'clearly' absent.



$\omega \equiv M_{\pi\pi}$

The angular distribution of pion-pion scattering taken from Carmony and Van de Walle (1962)