The Pin and the Pendulum

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(October 30, 2012)

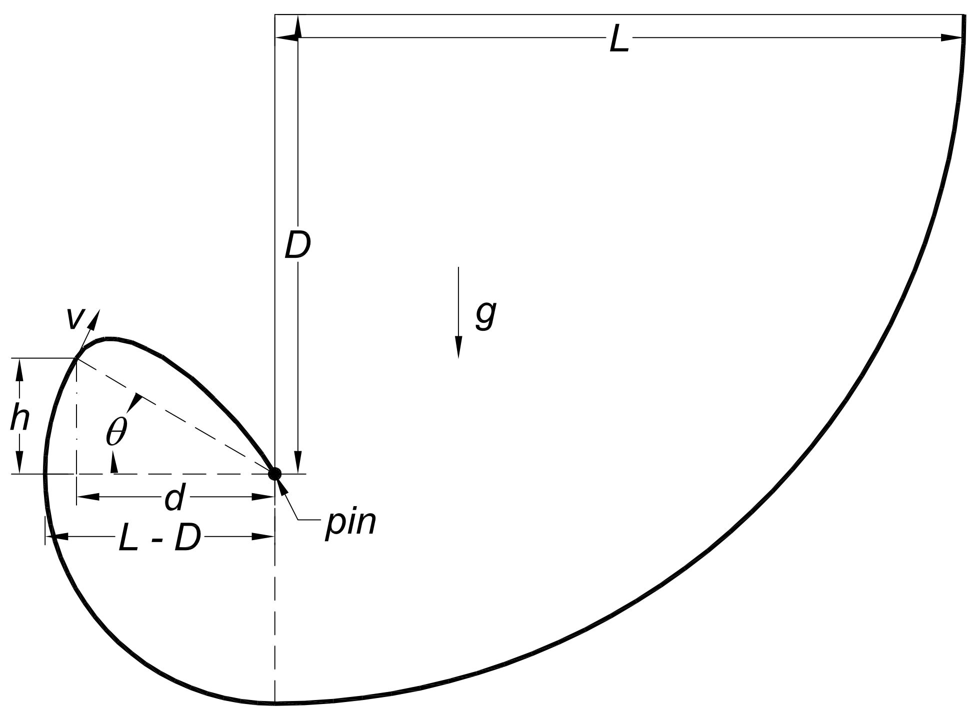
1 Problem

A particle attached to the end of a light string of length L forms a simple pendulum. A horizontal pin is located an unknown distance D directly underneath the pivot point. The particle is held so that the string is taut and horizontal and then released. What is the minimum distance D that would allow the string to wrap around the pin at least once?

This problem was posed by John Mallinckrodt.[1]

2 Solution

The bob of the pendulum initially moves in a circle of radius L, but after the string encounters the pin the bob moves in a circle of radius $L - D$. For D not too large the bob ceases to move in a circle when at angle θ with respect to a horizontal axis through the pin, and follows a parabolic trajectory thereafter, under the influence of gravity, as sketched below.



We seek the special case that the bob just passes over the pin, such that the string ends up wrapped 3/4 of the way around the pin when the bob finally comes to rest.

[1]As quoted in *Physics Challenges for Teachers and Students*, B. Korsunksy, ed., Phys. Teacher **50**, 506 (2012).

At angle θ the bob is still in circular motion about the pin, with radius $L - D$ and instantaneous velocity v. The string goes slack at this angle, so that gravity alone provides the centripetal acceleration,

$$\frac{v^2}{L - D} = g \sin\theta. \tag{1}$$

Assuming that energy is conserved, the velocity v is also related by,

$$v^2 = 2g(D - h) = 2g[D(1 + \sin\theta) - L\sin\theta], \tag{2}$$

noting that the height h of the bob above the pin is,

$$h = (L - D)\sin\theta. \tag{3}$$

Combining eqs. (1) and (2), we find that,

$$D = \frac{3L}{3 + 2/\sin\theta}. \tag{4}$$

The parabolic trajectory of the pin, taking the origin to be at the start of the parabola, is given by,

$$x = v\sin\theta\, t, \qquad y = v\cos\theta\, t - \frac{gt^2}{2}, \tag{5}$$

where g is the acceleration due to gravity. The desired parabola intersects the pin, which is at $(d, -h) = (L - D)(\cos\theta, -\sin\theta)$. The time t when the bob passes by the pin is given by setting $y = -h = -(L - D)\sin\theta$,

$$\frac{gt^2}{2} - v\cos\theta\, t - h = 0, \tag{6}$$

$$t = \frac{v\cos\theta + \sqrt{v^2\cos^2\theta + 2gh}}{g}. \tag{7}$$

Then, the horizontal distance d is related by,

$$d = (L - D)\cos\theta = v\sin\theta\, t = \frac{v\sin\theta}{g}\left(v\cos\theta + \sqrt{v^2\cos^2\theta + 2g(L - D)\sin\theta}\right). \tag{8}$$

Using eq. (1) for v^2, eq. (8) simplifies to,

$$\begin{aligned} g(L - D)\cos^3\theta &= \sin\theta\sqrt{g(L - D)\sin\theta}\sqrt{g(L - D)\sin\theta(2 + \cos^2\theta)} \\ &= g(L - D)\sin^2\theta\sqrt{2 + \cos^2\theta} = g(L - D)(1 - \cos^2\theta)\sqrt{2 + \cos^2\theta}. \end{aligned} \tag{9}$$

On squaring this we find that $0 = 2 - 3\cos^2\theta$,

$$\cos^2\theta = \frac{2}{3}, \qquad \sin^2\theta = \frac{1}{3}, \qquad \sin\theta = \frac{1}{\sqrt{3}}, \qquad \theta \approx 35°. \tag{10}$$

Finally, eq. (4) tells us that,

$$\frac{D}{L} = \frac{3}{3 + 2/\sin\theta} = \frac{3}{3 + 2\sqrt{3}} \approx 0.46. \tag{11}$$