Vector Potential of a Long Solenoid in the Poincaré Gauge

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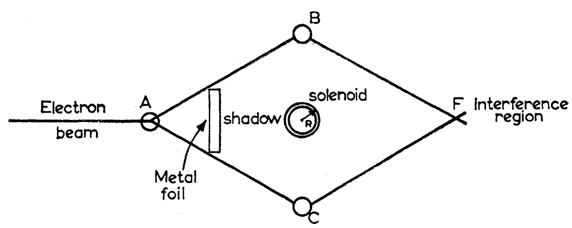
1 Problem

In a well-known paper [1], Aharonov and Bohm predicted an interesting quantum interference effect for an electron that passes around (but not through) a long magnetic solenoid, which effect has been verified by experiment [2, 3].[1]

In particular, the amplitudes for an electron of charge e to travel from A to F in the figure below (from [1]) on the paths ABF or ACF differ by a phase $\Delta\varphi$ that can be calculated as,

$$\Delta\varphi = \frac{e}{\hbar}\oint_{\rm ACFBA} \mathbf{A}\cdot d\mathbf{l} = \frac{e}{\hbar}\int_{\rm ACFBA} d\mathbf{Area}\cdot\boldsymbol{\nabla}\times\mathbf{A} = \frac{e}{\hbar}\int_{\rm ACFBA} \mathbf{B}\cdot d\mathbf{Area} = \frac{e\Phi_M}{\hbar}, \qquad (1)$$

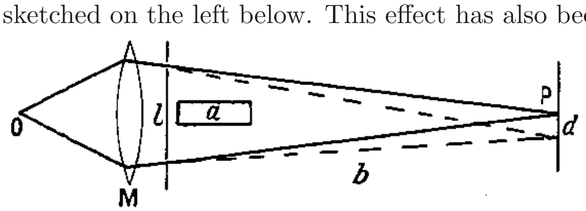
where $\mathbf{A}$ is the electromagnetic vector potential, $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ is the (essentially uniform) static magnetic field inside the solenoid (which field is essentially zero outside it), and Φ_M is the total magnetic flux inside the solenoid.

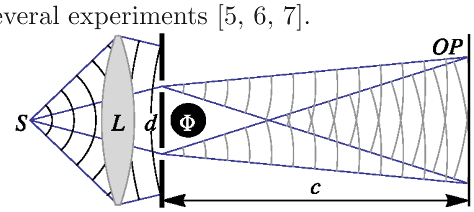


The phase difference $\Delta\varphi$ is "measurable" by observing the probability that the electron is detected at F. This phase difference depends on the classical quantity Φ_M, and so it gauge invariant, although it can be computed via the vector potential $\mathbf{A}$ outside the solenoid where $\mathbf{B} \approx 0$.

While Aharonov and Bohm did not explicitly argue that their effect shows that the vector potential $\mathbf{A}$ is "measurable", others have been inspired by them to say it is. See, for example, [8]. Aharonov and Bohm did say (right column, p. 490 of [1]) "the potentials must, in certain cases, be considered as physically effective", which apparently has been interpreted as "measurable" or "observable" by some. Perhaps the more relevant comment by Aharonov

[1]The Aharonov-Bohm effect was anticipated much earlier by Ehrenberg and Siday [4], who predicted an interference effect in an experiment with a biprism/double slit that also contained a solenoid magnet, as sketched on the left below. This effect has also been observed in several experiments [5, 6, 7].





and Bohm (left column, p. 490 of [1]) was that "according to current relativistic notions, all fields must interact locally".[2] Indeed, even prior to the development of the theory of relativity, Faraday and Maxwell devised field theory to avoid the notion of (nonlocal) action at a distance. The Aharonov-Bohm effect is an example of action at a distance if one considers only the electric and magnetic fields, while consideration of the vector potential permits analysis in which the effect derives from the "local" interaction of the electron with a field = the vector potential.

Deduce the form of the vector potential $\mathbf{A}$ of a long/infinite solenoid in the Poincaré gauge to show that the region in which an electron can interact locally with the vector potential is dependent on the choice of origin of the coordinate system. Hence, the location of the local e-$\mathbf{A}$ interaction is not well defined, such that the use of the vector potential in discussion the Aharonov-Bohm effect does not lead to a crisp "local" understanding thereof.

2 Solution

2.1 Electromagnetic Potentials in the Poincaré Gauge

We recall that in cases where the fields $\mathbf{E}$ and $\mathbf{B}$ are known, we can compute the electromagnetic potentials V and $\mathbf{A}$ in the Poincaré gauge (see sec. 9A of [11] and [12, 13, 14, 15]),[3] for which the gauge condition is $\mathbf{A}\cdot\mathbf{r} = 0$,

$$V^{(\mathrm{P})}(\mathbf{r},t) = -\mathbf{r}\cdot\int_0^1 du\,\mathbf{E}(u\mathbf{r},t), \qquad \mathbf{A}^{(\mathrm{P})}(\mathbf{r},t) = -\mathbf{r}\times\int_0^1 u\,du\,\mathbf{B}(u\mathbf{r},t) \qquad \text{(Poincaré)}. \quad (2)$$

These forms are remarkable in that they depend on the instantaneous value of the fields only along a line between the origin and the point of observation.[4]

[2] Aharonov and Bohm may have had in mind Einstein's objections to quantum theory, particularly as expressed in [9], in that quantum correlations of entangled objects persist even when those objects are spacelike separated, and can no longer have "local" interactions. For recent discussion of the continuing disquiet caused by such phenomena, see [10].

[3] The Poincaré gauge is also called the Hamilton or temporal gauge [11], and the multipolar gauge [16, 17].

[4] We transcribe Appendices C and D of [17] to verify that $\mathbf{E}$ and $\mathbf{B}$ indeed follow from the Poincaré potentials (2).

$$\begin{aligned}
-\boldsymbol{\nabla}V^{(\mathrm{P})} - \frac{1}{c}\frac{\partial\mathbf{A}^{(\mathrm{P})}}{\partial t} &= \int_0^1 du\left\{\boldsymbol{\nabla}[\mathbf{r}\cdot\mathbf{E}(u\mathbf{r},t)] + \mathbf{r}\times\frac{u}{c}\frac{\partial\mathbf{B}(u\mathbf{r},t)}{\partial t}\right\} = \int_0^1 du\,\{\boldsymbol{\nabla}[\mathbf{r}\cdot\mathbf{E}(u\mathbf{r},t)] - \mathbf{r}\times[\boldsymbol{\nabla}\times\mathbf{E}(u\mathbf{r},t)]\} \\
&= \int_0^1 du\,\{(\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{E}(u\mathbf{r},t) + [\mathbf{E}(u\mathbf{r},t)\cdot\boldsymbol{\nabla}]\mathbf{r} + \mathbf{E}(u\mathbf{r},t)\times(\boldsymbol{\nabla}\times\mathbf{r})\} = \int_0^1 du\left\{u\frac{d(ux_i)}{du}\frac{\partial\mathbf{E}(u\mathbf{r},t)}{\partial(ux_i)} + \mathbf{E}(u\mathbf{r},t)\right\} \\
&= \int_0^1 du\,\frac{d}{du}u\mathbf{E}(u\mathbf{r},t) = \mathbf{E}(\mathbf{r},t). \qquad (3)
\end{aligned}$$

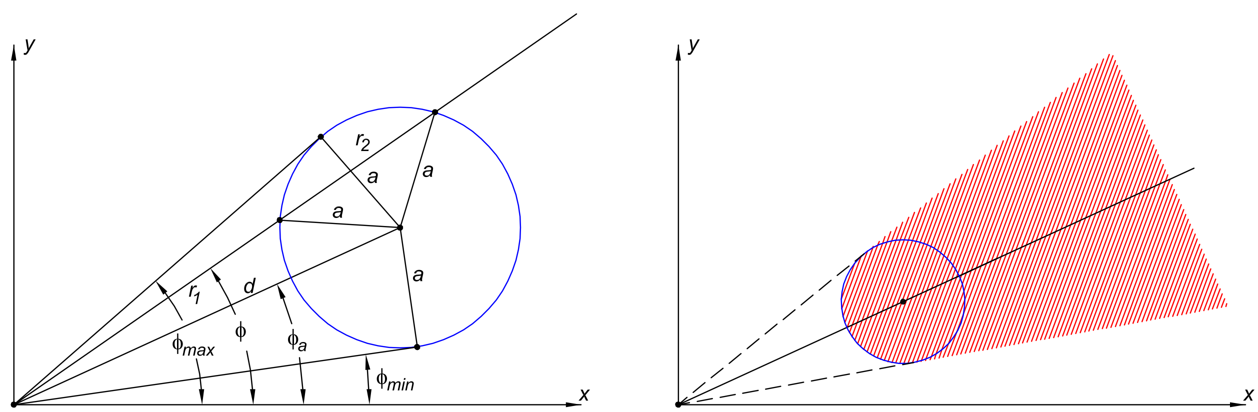
$$\begin{aligned}
\boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{P})} &= -\int_0^1 u\,du\,\boldsymbol{\nabla}\times[\mathbf{r}\times\mathbf{B}(u\mathbf{r},t)] \\
&= -\int_0^1 u\,du\,\{\mathbf{r}[\boldsymbol{\nabla}\cdot\mathbf{B}(u\mathbf{r},t)] - \mathbf{B}(u\mathbf{r},t)[\boldsymbol{\nabla}\cdot\mathbf{r}] + [\mathbf{B}(u\mathbf{r},t)\cdot\boldsymbol{\nabla}]\mathbf{r} - (\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{B}(u\mathbf{r},t)\} \\
&= \int_0^1 u\,du\left\{2\mathbf{B}(u\mathbf{r},t) + ux_i\frac{\partial\mathbf{B}(u\mathbf{r},t)}{\partial(ux_i)}\right\} = \int_0^1 u\,du\left\{\frac{1}{u}\frac{d}{du}u^2\mathbf{B}(u\mathbf{r},t)\right\} = \mathbf{B}(\mathbf{r},t). \qquad (4)
\end{aligned}$$

The potentials in the Poincaré gauge depend on the choice of origin. If the origin is inside the region of electromagnetic fields, then the Poincaré potentials are nonzero throughout all space. If the origin is to one side of the region of electromagnetic fields, then the Poincaré potentials are nonzero only inside that region, and in the region on the "other side" from the origin.

2.2 Poincaré Potentials for an Infinite Solenoid

We consider an electrically neutral "infinite" solenoid of radius a, with axis parallel to the z-axis and magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}$ inside, and $\mathbf{B} \approx 0$ outside the solenoid. The electric field associated with the solenoid is everywhere zero, so the Poincaré scalar potential $V^{(\mathrm{P})}$ is identically zero.

As we are particularly interested in cases where the vector potential is nonzero only in some part of space, we suppose the origin of the coordinate system is outside the solenoid, as distance $d > a$ from the axis of the solenoid.[5] The figure below illustrates the case when the axis of the solenoid is at angle ϕ_a to the x-axis in a cylindrical coordinate system (r, ϕ, z). The Poincaré vector potential is nonzero only in the shaded region ($r > r_1$, $\phi_{\min} < \phi < \phi_{\max}$).



Any ray from the origin that passes through the solenoid has $\phi_{\min} < \phi < \phi_{\max}$ where $\phi_{\max,\min} = \phi_a \pm \sin^{-1}(a/b)$. Such a ray intersects that solenoid at radii r_1 and r_2 related by,

$$a^2 = d^2 + r_i^2 - 2dr_i\cos(\phi - \phi_a), \qquad r_i = d\left(\cos(\phi - \phi_a) \pm \sqrt{\frac{a^2}{d^2} - \sin^2(\phi - \phi_a)}\right). \tag{5}$$

The vector potential of eq. (2) at (r, ϕ, z) is nonzero along such a ray for all $r > r_1$, *i.e.*, only within the "shadow", with,

$$\mathbf{A}^{(\mathrm{P})}(r_1 < r < r_2) = -\mathbf{r} \times B\,\hat{\mathbf{z}} \int_{r_1/r}^{1} u\,du = B\frac{r^2 - r_1^2}{2r}\,\hat{\boldsymbol{\phi}} \qquad \text{(Poincaré)}, \tag{6}$$

$$\mathbf{A}^{(\mathrm{P})}(r > r_2) = -\mathbf{r} \times B\,\hat{\mathbf{z}} \int_{r_1/r}^{r_2/r} u\,du = B\frac{r_2^2 - r_1^2}{2r}\,\hat{\boldsymbol{\phi}} \qquad \text{(Poincaré)}, \tag{7}$$

[5]If the origin is on the axis of the solenoid, then the Poincaré vector potential is $rB/2$ for $r < a$ and $a^2B/2r$ for $r > a$. This is the same potential as would be computed in the Coulomb, Lorenz and the Gibbs/Hamiltonian gauges [11, 15].

noting that $\mathbf{r} = r\,\hat{\mathbf{r}} + z\,\hat{\mathbf{z}}$. The forms (6)-(7) obey $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ with nonzero $\mathbf{B} = B\,\hat{\mathbf{z}}$ only inside the solenoid, recalling that $B_z = (1/r)\partial(rA_\phi)/\partial r$.

2.3 Comments

For any choice of the origin outside the solenoid, the path ABFCA in the figure on p. 1 passes through the wedge-shaped region in which the Poincaré vector potential is nonzero, so we can say that the electron interacts "locally" with the vector potential (either on path ABF or path ACF, or both). However, for different choices of the origin, this "local" interaction occurs at different places along the path. Thus, the "local" interaction is not "localized", such that the use of the vector potential in the analysis does not fully satisfy one's "classical" desire for explanations based on "local" interactions at well defined places.

It seems to this author that the argument of Aharonov and Bohm about the significance of the vector potential in quantum mechanics is off the mark. But, this is not to say that vector potentials are unimportant. Rather, a different line of thought, prior to the paper of Aharonov and Bohm, had already indicated a much greater significance to the potentials than possibly avoiding action at a distance in certain special cases.

In 1926 Fock noted [18, 19, 20] that Schrödinger's equation for an electric charge e of mass m in electromagnetic fields described by potentials $A_\mu = (\phi, \mathbf{A})$ can be written,

$$\frac{(-i\mathbf{D})^2}{2m}\psi = iD_0\psi, \qquad \text{using the "altered" (covariant) derivative} \qquad D_\mu = \partial_\mu + ieA_\mu, \quad (8)$$

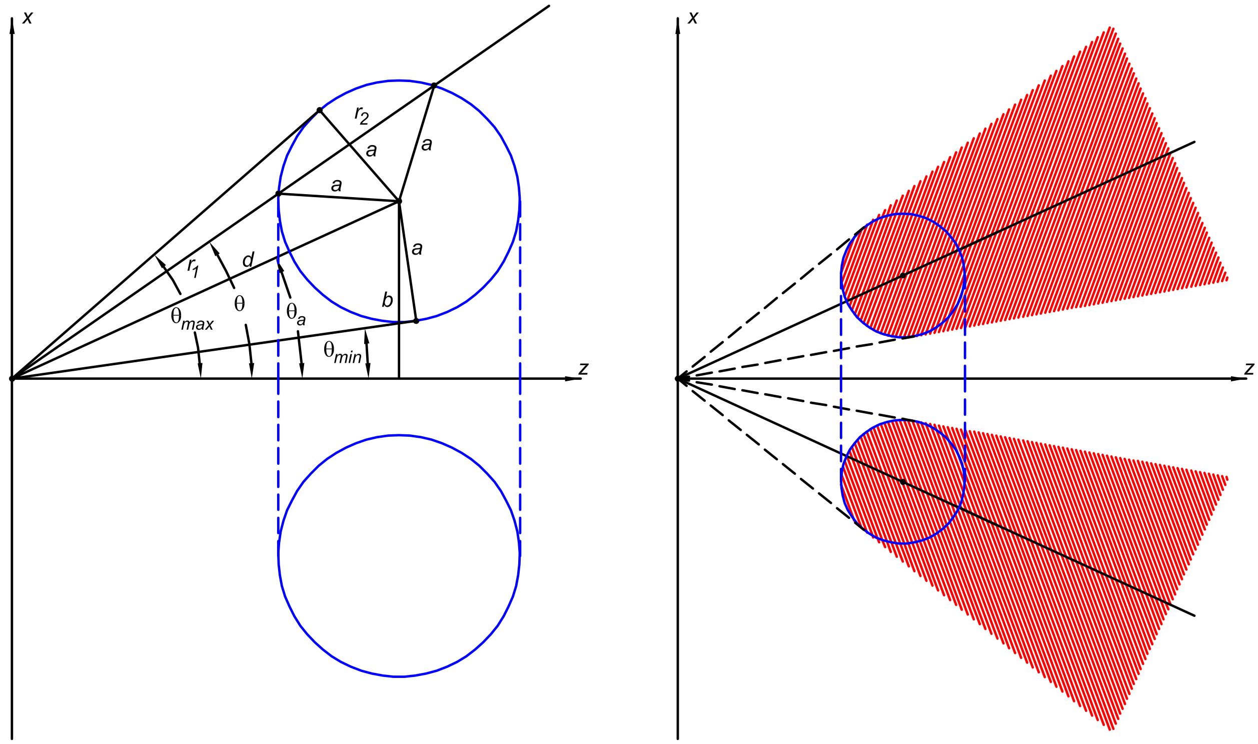
which is gauge invariant only if the gauge transformation of the potentials, $A_\mu(x_\nu) \to A_\mu + \partial_\mu\Omega(x_\nu)$, is accompanied by a phase change of the wavefunction, $\psi(x_\nu) \to e^{-ie\Omega(x_\nu)}\psi$. Yang and Mills (1954) [21, 22] may have been the first to point out that Fock's argument can be inverted such that a requirement of local phase invariance of the form $\psi(x_\nu) \to e^{-ie\Omega(x_\nu)}\psi$ implies the existence of an interaction described by a potential A_μ (and charge e) which satisfies gauge invariance and modifies Schrödinger's equation via the altered derivative D_μ.

This led to a greater appreciation of the significance of potentials in the quantum realm, that the nonobservability of potentials associated with their gauge invariance, together with a requirement of local phase invariance of the wave function, restricts the possible forms of interactions. The great successes of Weinberg and Salam [23, 24] in formulating the now-Standard electroweak theory, and of Gross, Wilczek and Politzer [25, 26] in deducing the theory of quantum chromodynamics, were the result of this more insightful view of the role of the potentials.

A Appendix: Poincaré Potentials of a Toroid

The Aharonov-Bohm effect (1959) also exists in case the magnetic field is due to a toroidal current, as well as that due to a long solenoid. The interaction of a moving electron with a toroidal magnet has also been considered "paradoxical" by Cullwick [27, 28, 29] (1952).

The Poincaré vector potential of such a toroid, with azimuthal magnetic field $B_\phi = kr\sin\theta$ in its interior, and zero exterior magnetic field, can be computed as in eqs. (6)-(7), referring to the figure on the next page, if we interpret the z-axis as the axis of the toroid and θ as the polar angle with respect to the z-axis in a spherical coordinate system $(r, \theta\, \phi)$. As for the long solenoid, the vector potential depends on the choice of origin relative to the toroid, and is zero except within the "shadow" of the toroid with respect to the origin.



The vector potential of eq. (2) at (r, θ, ϕ) is nonzero along a ray from the origin for all $r > r_1$ of eq. (4), *i.e.*, only within the "shadow", with,

$$\mathbf{A}^{(\mathrm{P})}(r_1 < r < r_2) = -\mathbf{r} \times \int_{r_1/r}^{1} u(kur \sin\theta\, \hat{\boldsymbol{\phi}})\, du = k \sin\theta \frac{r^3 - r_1^3}{3r}\, \hat{\boldsymbol{\theta}} \qquad \text{(Poincaré)}, \qquad (9)$$

$$\mathbf{A}^{(\mathrm{P})}(r > r_2) = -\mathbf{r} \times \int_{r_1/r}^{r_2/r} u(kur \sin\theta\, \hat{\boldsymbol{\phi}})\, du = k \sin\theta \frac{r_2^3 - r_1^3}{3r}\, \hat{\boldsymbol{\theta}} \qquad \text{(Poincaré)}. \qquad (10)$$

The forms (9)-(10) obey $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ with nonzero $\mathbf{B} = kr \sin\theta\, \hat{\boldsymbol{\phi}}$ only inside the solenoid, recalling that $B_\phi = (1/r)\partial(rA_\theta)/\partial r$.

If an electron moves parallel to the axis of the toroid and passes inside the latter, the Poincaré vector potential is zero along its entire path. In contrast, if the electron's path is at distance r from the x-axis larger than the inner radius of the toroid, the electron traverses a region of nonzero $\mathbf{A}$, but the location of this region depends on the choice of origin.

As in the usual Aharonov-Bohm effect, there is a quantum phase difference (1) associated with motion along these two paths, independent of the choice of origin, but the region in space where this phase difference is accumulated according to eq. (1) depends on the choice of origin. One cannot say that measurement of this phase difference has "measured" the vector potential in a manner independent of the choice of origin.

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