Gauging Away Polarization States of Waves

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1 Problem

Discuss how the concept of gauge invariance can lead to an understanding of how/why electromagnetic waves can have only two independent polarization states.

Comment also on gravitational waves.

2 Solution

2.1 Electromagnetic Waves

A familiar argument considers a unit, plane electromagnetic wave in free space with electric field given by the real part of,

$$\mathbf{E} = \hat{\mathbf{E}}_0 \, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{1}$$

where $\hat{\mathbf{E}}_0$ is a unit vector (possibly complex), $\mathbf{k}$ is the wave vector and $\omega = kc$ is the angular frequency, with c being the speed of light in vacuum. This field obeys the first Maxwell equation, $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ (in empty space), which implies that,

$$\hat{\mathbf{E}}_0 \cdot \mathbf{k} = 0, \tag{2}$$

and hence there are only two independent possibilities for the unit vector $\hat{\mathbf{E}}_0$, which are correspond to two independent "polarization" states of the wave, both of which have electric field transverse to the wave vector. Whereas, for waves inside matter, in general $\boldsymbol{\nabla}\cdot\mathbf{E} \neq 0$, so there can be waves with longitudinal, as well as transverse, polarization.

Here, we consider a longer argument based on the scalar and vector potentials ϕ and $\mathbf{A}$, which has the possible merit of being extendable to the case of gravitational waves.

The potentials can be considered as components of a 4-vector potential,

$$\phi_\mu = (\phi, \mathbf{A}). \tag{3}$$

Plane waves of the potentials have the form,

$$\phi_\mu = \epsilon_\mu \, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{4}$$

where ϵ_μ is a constant 4-vector. In principle, there are 4 independent types of polarization for waves of the 4-potential.

Because the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be deduced from derivatives of the potentials, the latter have some degree of arbitrariness, which fact has come to be discussed under the theme of gauge invariance.[1] One consequence of gauge invariance is that one can

[1] For a historical review, see [1].

choose to enforce one relation among the derivatives of the potentials, now called a gauge condition, or choice of gauge. When electromagnetic waves are concerned, a particularly useful choice is the Lorenz condition [2],[2]

$$\partial_\mu \phi^\mu = 0 = \frac{1}{c}\frac{\partial \phi}{\partial t} - \boldsymbol{\nabla} \cdot \mathbf{A} \qquad \text{(Lorenz)}, \tag{5}$$

in Gaussian units. Applying the Lorenz-gauge condition (5) to the 4-potential wave (4), we have that,

$$k_\mu \phi^\mu = 0, \tag{6}$$

where,

$$k_\mu = (\omega, \mathbf{k}c), \tag{7}$$

is the wave 4-vector.

We can say that the Lorenz condition (5) has eliminated one of the four possible polarization states, leaving three. In the rest of this note, we suppose that the wave vector $\mathbf{k}$ is in the z-direction. Then we can write one basis for the three remaining polarization states ϵ_μ of the 4-potential as,

$$\epsilon_\mu^{(1)} = (0, 1, 0, 0), \tag{8}$$
$$\epsilon_\mu^{(2)} = (0, 0, 1, 0), \tag{9}$$
$$\epsilon_\mu^{(3)} = (kc/\omega, 0, 0, 1). \tag{10}$$

We now show that for waves with $\omega = kc$, as holds in vacuum, the longitudinal polarization state $\epsilon_\mu^{(3)}$ can be eliminated by a gauge transformation (while staying within the Lorenz gauge).[3]

The (gauge) transformation,

$$\phi_\mu \to \phi_\mu + \partial_\mu \Omega, \qquad \phi \to \phi + \frac{1}{c}\frac{\partial \Omega}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} - \boldsymbol{\nabla}\Omega, \tag{11}$$

does not change the electromagnetic fields,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{12}$$

and the revised potentials (11) still satisfy the Lorenz condition (5) provided that,

$$\partial_\mu \partial^\mu \Omega = 0 = \frac{1}{c^2}\frac{\partial^2 \Omega}{\partial t^2} - \nabla^2 \Omega. \tag{13}$$

For example, we can choose,

$$\Omega = \frac{1}{ik}\, e^{ik(z-ct)}, \qquad \partial_\mu \Omega = -(1, 0, 0, 1)\, e^{ik(z-ct)}. \tag{14}$$

[2]For a survey of several gauge conditions, see [3, 4].

[3]The choice of a gauge condition is not sufficient for the potentials to be unique. For an example of two rather different sets of potentials in the Lorenz gauge for waves inside a rectangular metallic cavity, see sec. 2.2.3 of [5].

Then, for waves with $\omega = kc$, the revised polarization state 3 vanishes,

$$\epsilon_\mu^{(3)} = (1, 0, 0, 1) \to \epsilon_\mu^{(3)} + \partial_\mu \Omega = (0, 0, 0, 0). \tag{15}$$

This confirms that familiar result that for electromagnetic waves which obey the free-space relation that $\omega = kc$ there is no longitudinal polarization state. Loosely speaking, we can gauge away the longitudinal polarization of waves in free space, but not for those in matter. *The transverse polarizations states (9)-(10) are altered by the gauge transformation (11), but it is usual to redefine the transverse polarization states after the gauge transformation to have their original forms again.*

2.2 Gravitational Waves

In Einstein's theory of gravitation one considers waves as weak perturbations of the metric tensor,

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_{\mu\nu}, \tag{16}$$

where $\eta_{\mu\nu}$ is the (Euclidean) metric for empty space,

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{17}$$

as was tacitly assumed in sec. 2.1. For isotropic spacetime, we infer that the "potential" tensor $\phi_{\mu\nu}$ is symmetric, with only 10 independent components. We can enforce a Lorenz-like gauge condition on the derivatives of $\phi_{\mu\nu}$ that,

$$\partial_\mu \eta^{\mu\lambda} \phi_{\lambda\nu} = \partial^\mu \phi_{\mu\nu} 0. \tag{18}$$

The four conditions (18) reduce the number of independent components of $\phi_{\mu\nu}$ to six.

We now wish to argue that further use of gauge transformations, within the Lorenz-like gauge, reduce the number of independent components of $\phi_{\mu\nu}$ to two.

However, arguments based on consideration of waves $\phi_{\mu\nu}$ in otherwise empty space miss a noteworthy issue: that weak gravitational waves inside low-density matter can have five independent polarization states. This is clearer in a quantum view in which gravitational waves are associated with spin-2 quanta, which have five independent spin components, in general. Hence, we infer that there exists one more condition on the $\phi_{\mu\nu}$ which holds even for waves inside low-density matter. Here, we simply state this condition to be that $\phi_{\mu\nu}$ is traceless.

We now consider gravitational waves in free space that propagate in the z-direction,

$$\phi_{\mu\nu} = \epsilon_{\mu\nu}\, e^{ik(z-ct)}, \tag{19}$$

where $\epsilon_{\mu\nu}$ is the (constant) polarization tensor. The Lorenz-like condition (18) tells us that,

$$k^{\mu}\epsilon_{\mu\nu} = 0, \qquad k_{\mu} = kc(1,0,0,1), \qquad \Rightarrow \epsilon_{0\nu} = \epsilon_{3\nu}. \tag{20}$$

The requirement of gauge invariance of the potentials $\phi_{\mu\nu}$ tells us that the transformation,

$$\phi_{\mu\nu} \to \phi'_{\mu\nu} = \phi_{\mu\nu} + \partial_{\mu}\Omega_{\nu} + \partial_{\nu}\Omega_{\mu} \tag{21}$$

does not change the physics provided the 4-vector Ω_{μ} satisfies the free-space wave equation,

$$\partial_{\nu}\partial^{\nu}\Omega_{\mu} = 0 = \frac{1}{c^2}\frac{\partial^2\Omega_{\mu}}{\partial t^2} - \nabla^2\Omega_{\mu}. \tag{22}$$

Hence, we can consider,

$$\Omega_{\mu} = \chi_{\mu}\, e^{ik(z-ct)}, \tag{23}$$

for any constant 4-vector χ_{μ}. Applying the gauge transformation (21) to the wave potentials (20), the transformed polarization states are,

$$\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} + k_{\mu}\chi_{\nu} + k_{\nu}\chi_{\mu}. \tag{24}$$

We choose the four constants χ_{μ} to eliminate the $\epsilon'_{0\nu}$:

$$\epsilon'_{00} = \epsilon_{00} + 2k_0\chi_0 = \epsilon_{00} + 2kc\chi_0, \qquad \Rightarrow \chi_0 = -\epsilon_{00}/2kc, \tag{25}$$

$$\epsilon'_{01} = \epsilon_{01} + k_0\chi_1 + k_1\chi_0 = \epsilon_{01} + kc\chi_1, \qquad \Rightarrow \chi_1 = -\epsilon_{01}/kc, \tag{26}$$

$$\epsilon'_{02} = \epsilon_{02} + k_0\chi_2 + k_2\chi_0 = \epsilon_{02} + kc\chi_2, \qquad \Rightarrow \chi_2 = -\epsilon_{02}/kc, \tag{27}$$

$$\epsilon'_{03} = \epsilon_{03} + k_0\chi_3 + k_3\chi_0 = \epsilon_{03} + kc\chi_3 - \epsilon_{00}/2, \qquad \Rightarrow \chi_3 = (\epsilon_{00}/2 - \epsilon_{03})/kc. \tag{28}$$

So far,

$$\epsilon'_{00} = \epsilon'_{01} = \epsilon'_{02} = \epsilon'_{03} = 0. \tag{29}$$

The Lorenz-like condition (20), applied to $\epsilon'_{\mu\nu}$, tells us that,

$$\epsilon'_{30} = \epsilon'_{31} = \epsilon'_{32} = \epsilon'_{33} = 0. \tag{30}$$

Since $\epsilon'_{\mu\nu}$ is symmetric, we also have that,

$$\epsilon'_{10} = \epsilon'_{20} = \epsilon'_{13} = \epsilon'_{23} = 0. \tag{31}$$

The remaining nonzero components are ϵ'_{11}, ϵ'_{22} and $\epsilon'_{12} = \epsilon'_{21}$. Since $\epsilon'_{\mu\nu}$ is traceless, $\epsilon'_{22} = -\epsilon'_{11}$, and the polarization tensor $\epsilon'_{\mu\nu}$ has only two independent (transverse) components, ϵ'_{11} and ϵ'_{12},

$$\epsilon'_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon'_{11} & \epsilon'_{12} & 0 \\ 0 & \epsilon'_{12} & -\epsilon'_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

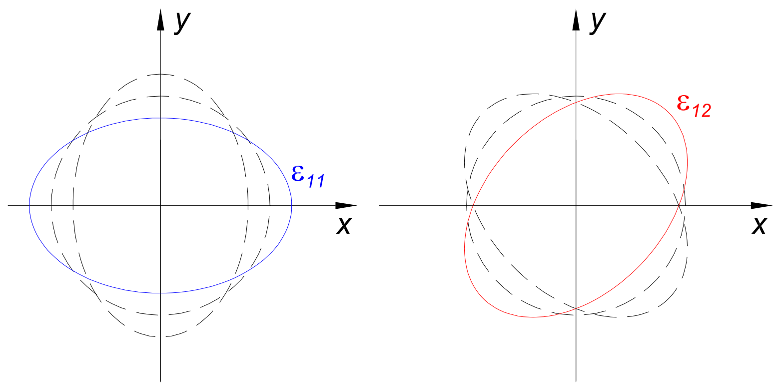
The fact that a massless particle with spin S can have only two spin (polarization) states $S_z = \pm S$ when propagating in the z direction is discussed by Bargmann and Wigner [6] from the perspective of relativistic invariance.

2.2.1 Physical Significance of the Two Polarizations

The gravitational waves parameterized by ϵ_{11} and ϵ_{12} (dropping the $'$s in eq. (32)) perturb the weak-field metric tensor according to eq. (16), and so affect the invariant interval between two events,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{33}$$

If a gravitational plane wave of polarization ϵ_{11} is incident on a massive sphere, the x-separation between pairs of points increases, while the y-separation decreases. As a result, the sphere is (slightly) deformed into an ellipsoid, as sketched on the left below. *Of course, half a wave period later, the x-separation has decreased and the y-separation has increased.*



In contrast, a wave of polarization ϵ_{12} increases the separation between points for which $dx = dy$, and decreases the separation when $dx = -dy$, as shown on the right in the sketch above.

These oscillatory deformations have a quadrupole character, with the two polarization states rotated by 45° with respect to one another (compared to the 90° rotation between x and y linear polarizations of electromagnetic waves). *Likewise, the lowest multipole of gravitational waves emitted by an oscillating mass is the quadrupole.*

References

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[3] J.D. Jackson, *From Lorenz to Coulomb and other explicit gauge transformations*, Am. J. Phys. **70**, 917 (2002), http://kirkmcd.princeton.edu/examples/EM/jackson_ajp_70_917_02.pdf

[4] K.-H. Yang and K.T. McDonald, *Formal Expressions for the Electromagnetic Potentials in Any Gauge* (Feb. 25, 2015), http://kirkmcd.princeton.edu/examples/gauge.pdf

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[6] V. Bargmann and E.P. Wigner, *Group Theoretical Discussion of Relativistic Wave Equations*, Proc. Nat. Acad. Sci. **34**, 211 (1948).
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