Circuit Q and Field Energy

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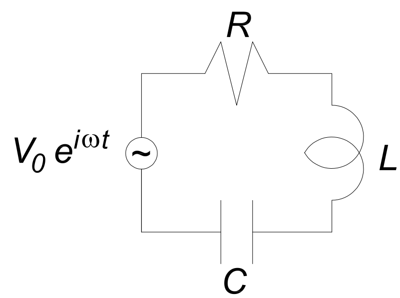
1 Problem

In a series R-L-C circuit, as sketched below, the maximum power is delivered to the load resistor R at the resonant (angular) frequency $\omega_0 = 1/\sqrt{LC}$, where L is the inductance and C is the capacitance of the circuit.[1] The frequency bandwidth 2δ is defined by the relation that the power delivered into the resistor at frequencies $\omega_0 \pm \delta$ is one half the maximum. The Q of this circuit is defined as the reciprocal of the relative bandwidth, $\omega_0/2\delta$.[2]



Deduce the Q of this circuit, and relate it to the electromagnetic field energy of the circuit.

Generalize this result to a circuit with frequency-dependent reactance $X(\omega)$ for which $X(\omega_0) = 0$.

2 Solution

2.1 Series R-L-C Circuit

The impedance Z of the circuit when operated at angular frequency ω is,[3]

$$Z = R + iX = R + i\left(\omega L - \frac{1}{\omega C}\right) = R + i\frac{\omega^2 LC - 1}{\omega C}, \tag{1}$$

where $X = \omega L - 1/\omega C$ is the reactance. The resonant frequency is,

$$\omega_0 = \frac{1}{\sqrt{LC}}\,, \tag{2}$$

[1]If the circuit represents an antenna (built from perfect conductors), the load power is radiated to "infinity" and R is the radiation resistance of the antenna.

[2]For the origin of this usage of the symbol Q, see [1].

[3]The term *impedance* was introduced by Heaviside [2]. See also, [4, 5].

at which the reactance vanishes, the impedance is simply $Z = R$ and the drive voltage $V_0\, e^{i\omega t}$ leads to current $I_0\, e^{i\omega t}$ given by,[4]

$$I_0 = \frac{V_0}{R}\,. \tag{3}$$

The time-average power delivered from the source to the load resistor is,

$$P_0 = \frac{Re(VI^\star)}{2} = \frac{V_0^2}{2R} = \frac{I_0^2 R}{2}\,. \tag{4}$$

The time-average (magnetic) energy stored in the inductor is,

$$\langle U_B(\omega_0)\rangle = \frac{LI_0^2}{4} = \frac{LV_0^2}{4R^2}\,, \tag{5}$$

while the time-average (electric) energy stored in the capacitor is,

$$\langle U_E(\omega_0)\rangle = \frac{Q_0^2}{4C} = \frac{\omega_0^2 I_0^2}{4C} = \frac{LI_0^2}{4C} = \langle U_L(\omega_0)\rangle = \langle U_0\rangle\,, \tag{6}$$

noting that $I = dQ/dt = i\omega Q$, so that I and Q are 90° out of phase, as are the magnetic and electric energies (5)-(6), and hence their equality at resonance implies that each of these are equal to the total (time-average) stored energy $\langle U_0\rangle$.

Away from resonance the current is given by,

$$I = \frac{V_0}{Z}\,, \tag{7}$$

and the power delivered to the load resistor is,

$$P = \frac{Re(VI^\star)}{2} = \frac{|I|^2 R}{2} = \frac{V_0^2}{2} Re\frac{1}{Z^\star} = \frac{V_0^2}{2} Re\frac{1}{R - iX} = \frac{V_0^2 R}{2(R^2 + X^2)} = \frac{P_0}{1 + X^2/R^2}\,. \tag{8}$$

The power P is one half of P_0 at frequencies where $X = \pm R$. That is, for,

$$\pm R = X = \frac{\omega^2 LC - 1}{\omega C} = \frac{\omega^2 - \omega_0^2}{\omega\omega_0^2 C}\,, \tag{9}$$

which leads to the quadratic equation,

$$\omega^2 \pm \omega_0^2 RC\omega - \omega_0^2 = 0. \tag{10}$$

The solutions are,

$$\omega = \frac{\pm\omega_0^2 RC \pm \sqrt{\omega_0^4 R^2 C^2 + 4\omega_0^2}}{2}\,. \tag{11}$$

[4]If the circuit is small compared to the wavelength $\lambda = 2\pi c/\omega_0$, where c is the speed of light, then the current is the same throughout the circuit. Otherwise, I_0 is strictly just the current at the terminals of the voltage source.

We now restrict the analysis to cases where the $\tau = RC$ time constant is small compared to the period $2\pi/\omega_0$, and we take the second $\pm$ sign in eq. (11) to be $+$, finding,

$$\omega \approx \omega_0 \pm \frac{\omega_0^2 \tau}{2} \equiv \omega_0 \pm \delta. \tag{12}$$

The quantity $2\delta \approx \omega_0^2 \tau$ is called the bandwidth of the circuit.

The reciprocal of the relative bandwidth is called the Q of the resonance,

$$Q = \frac{\omega_0}{2\delta} \approx \frac{1}{\omega_0 \tau} = \frac{\omega_0 P_0}{\omega_0^2 P_0 RC} = \frac{\omega_0 I_0^2 L}{2P_0} = \frac{2\omega_0 \langle U_0 \rangle}{P_0} = \frac{\omega_0 L}{R} = \frac{X_L(\omega_0)}{R} = \frac{X_C(\omega_0)}{R}. \tag{13}$$

This demonstrates the usual lore that the Q of a series R-L-C circuit can be expressed in terms of (time-average) quantities at resonance, or in terms of component impedances, as,

$$\begin{aligned} Q \equiv \frac{\omega_0}{\text{bandwidth}} &\approx \frac{2\omega_0 \times \text{stored energy at resonance}}{\text{power delivered to the load at resonance}} \\ &\approx \frac{\text{reactance of } L \text{ or of } C \text{ at resonance}}{R}. \end{aligned} \tag{14}$$

2.2 Circuit with Impedance $Z = R + iX(\omega)$ and $X(\omega_0) = 0$

For a more general circuit only some of the analysis of sec. 2.1 holds, namely eqs. (3)-(4) and (7)-(8).

In general, we cannot identify an effective inductance L and effective capacitance C such that the reactance has the form $X(\omega) = \omega L - 1/\omega C$ (see footnote 4 of [6]). If the reactance vanishes at some "resonant" frequency ω_0 then for nearby frequencies it is a good approximation to write,

$$X(\omega) \approx (\omega - \omega_0)\frac{dX(\omega_0)}{d\omega}. \tag{15}$$

Then, eq. (9) becomes,

$$\pm R = X(\omega \pm \delta) \approx \pm\delta \frac{dX(\omega_0)}{d\omega}, \tag{16}$$

and hence,

$$\delta \approx \frac{R}{|dX(\omega_0)/d\omega|}, \tag{17}$$

and,

$$Q = \frac{\omega_0}{2\delta} \approx \frac{\omega_0 \, |dX(\omega_0)/d\omega|}{2R} = \frac{\pi f_0 \, |dX(f_0)/d\omega|}{R}, \tag{18}$$

where $\omega_0 = 2\pi f_0$. The form (18) is often used to evaluate the Q of an antenna system via computation of the (terminal) reactance near resonance.[5]

Comparison of eq. (18) with the last form of eq. (14) permits us to say that close to the resonant frequency ω_0 the general circuit is equivalent to a series R-L-C circuit with,[6]

$$L_0 = \frac{1}{2}\frac{dX(\omega_0)}{d\omega}, \qquad \text{and} \qquad C_0 = \frac{1}{\omega_0^2 L_0} = \frac{2}{\omega_0^2\, dX(\omega_0)/d\omega}. \tag{21}$$

In addition, we can deduce from the complex version of Poynting's theorem (see, for example, [6]) that,

$$X(\omega) = \frac{4\omega[\langle U_B(\omega)\rangle - \langle U_E(\omega)\rangle]}{|I(\omega)|^2}, \tag{22}$$

where now $\langle U_B\rangle$ and $\langle U_E\rangle$ are the (time-average) energies associated with the magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$.[7] Using eq. (22) in (18), we have that,

$$Q = \frac{\omega_0}{P_0}\left|\frac{d(\omega[\langle U_B(\omega)\rangle - \langle U_E(\omega)\rangle])}{d\omega}\right|_{\omega_0}. \tag{24}$$

Alternatively, eq. (9) can be written as,

$$\begin{aligned} \pm R = X(\omega_0 \pm \delta) &= \frac{4(\omega_0 \pm \delta)[\langle U_B(\omega_0 \pm \delta)\rangle - \langle U_E(\omega_0 \pm \delta)\rangle]}{|I(\omega_0 \pm \delta)|^2} \\ &= \frac{4(\omega_0 \pm \delta)R[\langle U_B(\omega_0 \pm \delta)\rangle - \langle U_E(\omega_0 \pm \delta)\rangle]}{P_0}, \end{aligned} \tag{25}$$

[5]Expression (18) can be generalized to a "potential Q" of a circuit at angular frequency ω_0 even if $X(\omega_0) \neq 0$, in that if the circuit were "tuned" by adding (lossless) inductors or capacitors as needed to bring $X(\omega_0)$ to 0, the (potential) Q of the "tuned" circuit would be [7],

$$Q_{\text{tuned}} \approx \frac{\omega_0\,|dX(\omega_0)/d\omega| + |X(\omega_0)|}{2R}. \tag{19}$$

[6]It is proposed in [8] that the equivalent inductance and capacitance at any frequency be defined by,

$$L(\omega) = \frac{1}{2}\left(\frac{dX(\omega)}{d\omega} + \frac{X(\omega)}{\omega}\right), \qquad \text{and} \qquad C(\omega) = \frac{2}{\omega^2\,(dX(\omega)/d\omega - X(\omega)/\omega)}, \tag{20}$$

for which $L(\omega) = L$ and $C(\omega) = C$ when $X = \omega L - 1/\omega C$. The utility of such a definition is unclear.

[7]In [13] it is also proposed that we identify,

$$L = \frac{4(\langle U_B\rangle - \langle U_{B,\text{rad}}\rangle)}{|I|^2}, \qquad \text{and} \qquad C = \frac{|I|^2}{4\omega^2(\langle U_E\rangle - \langle U_{E,\text{rad}}\rangle)}, \tag{23}$$

but as discussed in footnote 4 of [6], this is unsatisfactory since in general $\langle U_{B,\text{rad}}\rangle \neq \langle U_{E,\text{rad}}\rangle$ (see the Appendix of [6]), so that we do not recover eq. (22) when writing $X = \omega L - 1/\omega C$. That is, the "reactive" field energy $\langle U_B\rangle - \langle U_E\rangle$ includes contributions from "radiation" field energy (contrary to the assumptions of many workers [8, 10, 9, 11, 12], who may have been misled by the special case of a single, small dipole radiator).

noting that since $P(\omega_0 \pm \delta) = P_0/2$, eq. (8) tell us that $|I(\omega_0 \pm \delta)|^2 = P_0/R$. The two cases of eq. (25) can now be written as,

$$\omega_0 + \delta = \frac{P_0}{4[\langle U_B(\omega_0 + \delta)\rangle - \langle U_E(\omega_0 + \delta)\rangle]}, \tag{26}$$

$$\omega_0 - \delta = \frac{P_0}{4[\langle U_B(\omega_0 - \delta)\rangle - \langle U_E(\omega_0 - \delta)\rangle]}. \tag{27}$$

For δ small compared to ω_0 it is a good approximation that,

$$\langle U_B(\omega_0 + \delta)\rangle - \langle U_E(\omega_0 + \delta)\rangle \approx - \langle U_B(\omega_0 - \delta)\rangle - \langle U_E(\omega_0 - \delta)\rangle, \tag{28}$$

such that,

$$\delta \approx \frac{P_0}{4[\langle U_B(\omega_0 + \delta)\rangle - \langle U_E(\omega_0 + \delta)\rangle]}, \tag{29}$$

and the Q of the circuit is given by,

$$Q = \frac{\omega_0}{2\delta} \approx \frac{2\omega_0[\langle U_B(\omega_0 + \delta)\rangle - \langle U_E(\omega_0 + \delta)\rangle]}{P_0}. \tag{30}$$

The forms (24) and (30) differ from that of eq. (14) in that the energy in the numerator is the so-called (time-average) reactive field energy $\langle U_B\rangle - \langle U_E\rangle$ (see, for example, sec. 3 of [6]),[8] rather than the total stored energy at resonance.[9,10]

References

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[2] O. Heaviside, *Electromagnetic Induction and Its Propagation*, part 29, Electrician **17**, 212 (1886), http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_17_212_86.pdf
Also p. 61 of [3].

[8] Evaluated on resonance in eq. (24) and off resonance by the half-bandwidth δ in eq. (30).

[9] The concept of circuit Q originated in circuit examples where radiation is ignored, and the "stored energy" does not include the (infinite) energy stored in the far ("radiation") field of a circuit that has been oscillating forever. When the reactance (22) is evaluated via computations of field energies (as might be done for antenna systems), the infinite, total stored energy is not useful for calculating the Q.

[10] A different definition of the Q of an antenna system, not intrinsically related to the relative fractional bandwidth, is often used, following [9, 10],

$$Q = \frac{2\omega_0 \max[\langle U_B(\omega_0)\rangle - \langle U_{B,\mathrm{rad}}(\omega_0)\rangle, \langle U_E(\omega_0)\rangle - \langle U_{E,\mathrm{rad}}(\omega_0)\rangle]}{P_0}. \tag{31}$$

Still other definitions for the Q of antenna systems exist (see, for example, [11, 12, 13]), based on the notion that the Q of a system can be defined independently of considerations of bandwidth (which seems misguided to this author).

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