

Hydrogen-Atom Wavefunctions as Radiationless Modes

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(March 11, 2016)

1 Problem

Show that if the quantum probability current for hydrogen-atom wavefunctions is interpreted as an electric current, the implication is that this current does not emit radiation (*i.e.*, is a so-called radiationless mode).

2 Solution

After the discovery of the electron by J.J. Thomson [1], it was natural to consider that electrons are constituents of atoms, but if electrons in an atom are small electric charges moving in circular orbits, they would emit electromagnetic radiation and the radius of the orbit would quickly collapse to zero.¹ In 1903, Thomson pointed out [3, 4] that a uniform ring of electric charge, rotating about its symmetry axis at constant angular velocity, does not emit electromagnetic radiation although the moving charge is subject to a centripetal acceleration. This was the first discovery of a “classical radiationless mode”.² Thomson was inspired by this result to propose his “plum pudding” model of an atom [7] in which the positive electric charge was spread out in a continuous fashion, perhaps around rings of negative electrons.

Little survives of this model today, except that the quantum probability current,

$$\mathbf{J}_{\text{prob}} = \frac{i\hbar}{2M}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi), \quad (1)$$

of the wavefunction of an electron of mass M in, say, a hydrogen atom flows in rings, such that if $e\mathbf{J}_{\text{prob}}$ is interpreted as a classical electric current,³ this current emits no radiation.⁴ That, is the hydrogen-atom wavefunctions can be mapped onto classical radiationless states.

¹See, for example, [2].

²It was soon noted, most crisply by Ehrenfest [5], that a uniform spherical shell of electric charge can have arbitrary radial acceleration without emitting any radiation. See also, [6].

³The notion of the probability current (1) has its origins in Schrödinger’s interpretation of $e|\Psi|^2$ as electric charge density, and $e\mathbf{J}_{\text{prob}}$ as the electric current density. See sec. 7 of [8], which slightly predates Born’s interpretation [9] of $|\Psi|^2$ as a probability density, based on considerations of scattering rather than of the character of atomic states (where Schrödinger’s interpretation has considerable appeal).

⁴This conclusion appears on pp. 138-139 of [8], which has been translated as:

To given an example in the conservative one-electron problem ... the current density ... generally is a stationary current distribution. ... We may in a certain sense speak of a return to electrostatic and magnetostatic models. In this way the lack of radiation in the normal state would, indeed, find a startlingly simple explanation.

To see this, note that a hydrogen-atom wavefunction can be written in a spherical coordinate system (r, θ, ϕ) centered on the proton as,⁵

$$\Psi_{nlm} = R_{nl}(r)P_l^m(\cos\theta)e^{im\phi}, \quad (2)$$

where the associated Legendre polynomial P_l^m can be written as,

$$P_l^m(\cos\theta) = \sin^{l|m|}\theta \left(\frac{d}{d\cos\theta}\right)^{|m|} P_l(\cos\theta), \quad (3)$$

and P_l is a Legendre polynomial. Then, since the gradient operator in spherical coordinates is,

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \quad (4)$$

the probability current associated with the wavefunction Ψ_{nlm} is,

$$\mathbf{J}_{\text{prob},nlm} = \frac{m\hbar|\Psi_{nlm}|^2}{Mr\sin\theta}\hat{\boldsymbol{\phi}}. \quad (5)$$

This is well behaved for all θ , since $|\Psi_{nlm}|^2 \propto \sin^{2|m|}\theta$ according to eqs. (2)-(3). The probability current $\mathbf{J}_{\text{prob},nlm}$ is purely azimuthal, time-independent, and nonzero for nonzero (integer) index m (which labels the z -component of the orbital angular momentum of the atomic electron). So, if we interpret $e\mathbf{J}_{\text{prob},nlm}$ as the electric current density associated with the atomic electric, this is a superposition of steady ring currents, which do not emit electromagnetic radiation.

The spread-out character of the quantum electron in an atom implies that the wavefunctions are radiationless states for all values of indices n , m , and l (and not just for the ground state Ψ_{100}). A classical atom with current density $e\mathbf{J}_{\text{prob},nlm}$ would not radiate/decay, and there would be no classical explanation for the observed spectral lines of excited atomic states.

3 Comments

There is a long history of attempts to explain what are now considered to be “quantum” concepts by “classical” notions. The absence of radiation by the ground state of atoms is one of the phenomena that has attracted such attention. Prior to the initial development of a quantum view of atoms by Bohr [11], various attempts besides that of Thomson [3] were made to accommodate radiationless modes into the “classical” electron theory of small, charged spherical shells [12]-[19]. Even after the success of the “new” quantum theory of the 1920’s in explaining atomic states as stationary radiationless modes with a semiclassical interpretation via the probability current density (or Schrödinger’s electric current density), people have continued to seek purely classical models of electrons that can have radiationless orbital motion [20]-[25].⁶

⁵The essence of eq. (2) appears in Schrödinger’s first paper on wave mechanics [10].

⁶An “outsider science” effort, <http://brilliantlightpower.com/wp-content/uploads/theory/Volume1.djvu>, presents in overwhelming detail a theory in which it is claimed the wavefunctions (2) must have radial

References

- [1] J.J. Thomson, *Cathode Rays*, Phil. Mag. **44**, 293 (1897),
http://kirkmcd.princeton.edu/examples/EP/thomson_pm_44_293_97.pdf
- [2] J.D. Olsen and K.T. McDonald, *Classical Lifetime of a Bohr Atom* (Mar. 7, 2005),
<http://kirkmcd.princeton.edu/examples/orbitdecay.pdf>
- [3] J.J. Thomson, *The Magnetic Properties of Systems of Corpuscles describing Circular Orbits*, Phil. Mag. **6**, 673 (1903),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_6_673_03.pdf
- [4] K.T. McDonald, *Why Doesn't a Steady Current Loop Radiate?* (Dec. 1, 2001),
<http://kirkmcd.princeton.edu/examples/steadycurrent.pdf>
- [5] P. Ehrenfest, *Ungleichförmige Elektrizitätsbewegung ohne Magnet- und Strahlungsfeld*, Phys. Z. **11**, 708 (1910), http://kirkmcd.princeton.edu/examples/EM/ehrenfest_pz_11_708_10.pdf
- [6] K.T. McDonald, *Expanding Spherical Shell of Charge* (Aug. 22, 2014),
<http://kirkmcd.princeton.edu/examples/shell.pdf>
- [7] J.J. Thomson, *On the Structure of the Atom: an investigation of the Stability and periods of Oscillation of a number of Corpuscles arranged at equal intervals around the Circumference of a Circle; with Application of the results to the Theory of Atomic Structure*, Phil. Mag. **7**, 237 (1904),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_7_237_03.pdf
- [8] E. Schrödinger, *Quantisierung als Eigenwertproblem; Vierte Mitteilung*, Ann. d. Phys. **81**, 106 (1926), http://kirkmcd.princeton.edu/examples/QM/schroedinger_ap_81_109_26.pdf
http://kirkmcd.princeton.edu/examples/QM/schroedinger_ap_81_109_26_english.pdf
- [9] M. Born, *Quantenmechanik der Stoßvorgänge*, Z. Phys. **38**, 803 (1926),
http://kirkmcd.princeton.edu/examples/QM/born_zp_38_803_26.pdf
- [10] E. Schrödinger, *Quantisierung als Eigenwertproblem; Vierte Mitteilung*, Ann. d. Phys. **79**, 361 (1926), http://kirkmcd.princeton.edu/examples/QM/schroedinger_ap_79_361_26.pdf
http://kirkmcd.princeton.edu/examples/QM/schroedinger_ap_79_361_26_english.pdf
- [11] N. Bohr, *On the Constitution of Atoms and Molecules*, Phil. Mag. **26**, 1 (1913),
http://kirkmcd.princeton.edu/examples/QM/bohr_pm_26_1_13.pdf
- [12] G. Herglotz, *Zur Elektronentheorie*, Nachr. Gesell. Wiss. Göttingen, 357 (1903),
http://kirkmcd.princeton.edu/examples/EM/herglotz_ngwg_357_03.pdf
- [13] P. Hertz, *Untersuchungen über un stetige Bewegungen eines Elektrons*, Dissertation (Göttingen, 1904), http://kirkmcd.princeton.edu/examples/EM/hertz_dissertation_04.pdf

functions that are delta-functions, corresponding to superpositions of current rings of a single radius that somehow interpenetrate, to be radiationless. From this misunderstanding come claims of hydrogen-atom states with binding energy greater than a Rydberg.

- [14] A. Sommerfeld, *Zur Elektronentheorie. I. Allgemeine Untersuchung des Feldes eines beliebig bewegten Elektrons*, Nachr. Gesell. Wiss. Göttingen, 99 (1904),
http://kirkmcd.princeton.edu/examples/EM/sommerfeld_ngwg_99_04.pdf
- [15] A. Sommerfeld, *Zur Elektronentheorie. II. Grundlagen für eine allgemeine Dynamik des Elektrons*, Nachr. Gesell. Wiss. Göttingen, 363 (1904),
http://kirkmcd.princeton.edu/examples/EM/sommerfeld_ngwg_363_04.pdf
- [16] A. Sommerfeld, *Zur Elektronentheorie. III. Ueber Lichtgeschwindigkeits- und Ueberlichtgeschwindigkeits-Elektronen*, Nachr. Gesell. Wiss. Göttingen, 203 (1905),
http://kirkmcd.princeton.edu/examples/EM/sommerfeld_ngwg_203_05.pdf
- [17] A. Sommerfeld, *Simplified Deduction of the Field and the Forces of an Electron, moving in any given way*, Proc. Roy. Acad. Sci. Amsterdam **7**, 346 (1905),
http://kirkmcd.princeton.edu/examples/EM/sommerfeld_knaw_7_346_05.pdf
- [18] P. Hertz, *Die Bewegung eines Elektrons unter dem Einflusse einer stets gleich gerichteten Kraft*, Math. Ann. **65**, 1 (1907),
http://kirkmcd.princeton.edu/examples/EM/hertz_ma_65_1_07.pdf
- [19] G. Herglotz, *Über die Integralgleichungen der Elektronentheorie*, Math. Ann. **65**, 87 (1907), http://kirkmcd.princeton.edu/examples/EM/herglotz_ma_65_87_07.pdf
- [20] G.A. Schott, *The Electromagnetic Field of a moving uniformly and rigidly Electrified Sphere and its Radiationless Orbits*, Phil. Mag. **7**, 931 (1977),
http://kirkmcd.princeton.edu/examples/EM/schott_pm_15_752_33.pdf
- [21] D. Bohm and M. Weinstein, *The Self-Oscillations of a Charged Particle*, Phys. Rev. **74**, 1789 (1948), http://kirkmcd.princeton.edu/examples/EM/bohm_pr_74_1789_48.pdf
- [22] G.H. Goedecke, *Classically Radiationless Motions and Possible Implications for Quantum Theory*, Phys. Rev. **135**, B281 (1964),
http://kirkmcd.princeton.edu/examples/EM/goedecke_pr_135_b281_64.pdf
- [23] P. Pearle, *Absence of Radiationless Motions of Relativistically Rigid Classical Electron*, Found. Phys. **7**, 931 (1977),
http://kirkmcd.princeton.edu/examples/EM/pearle_fp_7_931_77.pdf
- [24] P. Pearle, *When Can a Classical Electron Accelerate Without Radiating?* Found. Phys. **8**, 879 (1978), http://kirkmcd.princeton.edu/examples/EM/pearle_fp_8_879_78.pdf
- [25] B.J. Hoenders and H.P. Baltes, *On the existence of non-radiating frequencies in the radiation from a stochastic current distributio*, J. Phys. A **13**, 995 (1980),
http://kirkmcd.princeton.edu/examples/EM/hoenders_jpa_13_995_80.pdf