Reactance of a Sinusoidally Driven Antenna

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1 Problem

Use Poynting's theorem for complex, time-harmonic fields to deduce general expressions for the reactance of an antenna that is operated at angular frequency ω. Discuss whether this reactance can be separated into capacitive and inductive reactances. Consider also possible meanings of the concept of "reactive field energy".

2 Solution

This problem is based on the overoptimistic claim in [1] that the reactance of an antenna can be decomposed into capacitive and inductive reactances. Deduction of antenna reactance via so-called complex Poynting theorem (sec. 13.14 of [2], sec. 2.20 of [3]) goes back at least to sec. 5 of [4].[1] *See also chap. 8 of [6]. We consider only media with unit relative permittivity and permeability.*

For any system in which the charges and currents have time dependence $e^{j\omega t}$ it is convenient to consider fields as complex vectors, of which only their real parts have physical meaning. For example, the real part of the complex Poynting vector,

$$\tilde{\mathbf{S}} = \frac{\mathbf{E} \times \mathbf{B}^\star}{2\mu_0}\,, \tag{1}$$

is the time-average flow of energy in the electromagnetic fields.

Poynting's theorem [7] can be expressed in terms of complex fields as follows,

$$\begin{aligned} \boldsymbol{\nabla} \cdot \tilde{\mathbf{S}} &= \boldsymbol{\nabla} \cdot \frac{\mathbf{E} \times \mathbf{B}^\star}{2\mu_0} = \frac{\mathbf{B}^\star}{2\mu_0} \cdot \boldsymbol{\nabla} \times \mathbf{E} - \mathbf{E} \cdot \boldsymbol{\nabla} \times \frac{\mathbf{B}^\star}{2\mu_0} \\ &= -j\omega \frac{|B|^2}{2\mu_0} + j\omega \frac{\epsilon_0\,|E|^2}{2} - \frac{\mathbf{E} \cdot \mathbf{J}^\star}{2} = -2j\omega(\langle u_B \rangle - \langle u_E \rangle) - \frac{\mathbf{E} \cdot \mathbf{J}^\star}{2}\,, \end{aligned} \tag{2}$$

where the time-average densities of energy in the electromagnetic fields are,

$$\langle u_B \rangle = \frac{|B|^2}{4\mu_0}\,, \qquad \text{and} \qquad \langle u_E \rangle = \frac{\epsilon_0\,|E|^2}{4}\,. \tag{3}$$

When we integrate eq. (2) over some volume we obtain,

$$\begin{aligned} \int \boldsymbol{\nabla} \cdot \tilde{\mathbf{S}}\, d\mathrm{Vol} = \oint \tilde{\mathbf{S}} \cdot d\mathbf{Area} &= -2j\omega \int (\langle u_B \rangle - \langle u_E \rangle)\, d\mathrm{Vol} - \int \frac{\mathbf{E} \cdot \mathbf{J}^\star}{2}\, d\mathrm{Vol} \\ &= -2j\omega(\langle U_B \rangle - \langle U_E \rangle) - \int \frac{\mathbf{E} \cdot \mathbf{J}^\star}{2}\, d\mathrm{Vol}, \end{aligned} \tag{4}$$

[1] Much of this note is discussed in chap. 5 of [5], which omits any assignment of physical significance to the imaginary part of the complex Poynting vector.

where $\langle U_B\rangle = \int \langle u_B\rangle\, d\mathrm{Vol}$ and $\langle U_E\rangle = \int \langle u_E\rangle\, d\mathrm{Vol}$ are the total, time-average energies of the magnetic and electric fields in that volume.

In the case of an antenna, we take the volume to be all the space outside the antenna and outside its power source, where we suppose that the power source for the antenna fits in the (small) space between its two terminals. Then, $\mathbf{J} = 0$ everywhere in this volume, so $\int \mathbf{E}\cdot\mathbf{J}^\star\, d\mathrm{Vol} = 0$.

The surface of this volume has three regions: a sphere at "infinity," the surface of the conductors of the antenna (excluding the small surfaces of the terminals that face the power supply), and the (small) surface of the power supply that does not face the terminals. Then,

$$\begin{aligned} \oint \tilde{\mathbf{S}}\cdot d\mathbf{Area} &= \oint_\infty \tilde{\mathbf{S}}\cdot d\mathbf{Area} + \oint_{\text{antenna}} \tilde{\mathbf{S}}\cdot d\mathbf{Area} + \oint_{\text{power supply}} \tilde{\mathbf{S}}\cdot d\mathbf{Area} \\ &= \langle P_{\rm rad}\rangle + \langle P_{\rm Ohmic}\rangle - \langle P_{\text{power supply}}\rangle\,. \end{aligned} \tag{5}$$

The integral of the Poynting vector over a sphere at "infinity" is the (to,e-average) power $\langle P_{\rm rad}\rangle$ that is "radiated to infinity". The integral of the Poynting vector into the surface of the antenna is $\langle P_{\rm Ohmic}\rangle$, taking note of the fact that the flow of energy across the surface of the antenna is zero in the limit of a perfect conductor (since then $\mathbf{E}_{\rm tangential} = 0$ and therefore $\mathbf{S}_\perp = 0$), and that energy must flow into a resistive conductor to replace the Ohmic losses. Similarly, the power supply does not deliver energy into the antenna, but rather energy flows directly from the power supply into the volume outside it (and the antenna).[2]

Defining I to be the (complex) current at the terminals of the antenna, we write the (time-average) power as $Z\,|I|^2/2$, where for the surface at "infinity" we define $Z = R_{\rm rad} = 2\,\langle P_{\rm rad}\rangle / |I|^2$ = radiation resistance, for the antenna conductors we define $Z = R_{\rm Ohmic} = 2\,\langle P_{\rm Ohmic}\rangle / |I|^2$ = terminal resistance,[3] and for the power supply we define $Z = Z_{\rm antenna}$ = total terminal impedance of the antenna as seen by the power supply.

Combining eqs. (4) and (5) we have,

$$Z_{\rm antenna} = \frac{2\,\langle P_{\text{power supply}}\rangle}{|I|^2} = R_{\rm rad} + R_{\rm Ohmic} + \frac{4j\omega(\langle U_B\rangle - \langle U_E\rangle)}{|I|^2} \equiv R_{\rm antenna} + jX_{\rm antenna}\,, \tag{6}$$

where,

$$R_{\rm antenna} = R_{\rm rad} + R_{\rm Ohmic}\,, \qquad \text{and} \qquad X_{\rm antenna} = \frac{4\omega(\langle U_B\rangle - \langle U_E\rangle)}{|I|^2} \equiv \omega L - \frac{1}{\omega C}\,, \tag{7}$$

introducing the antenna reactance $X_{\rm antenna}$, inductance L and capacitance C. The total energies $\langle U_B\rangle$ and $\langle U_E\rangle$ are infinite, so we cannot immediately identify $L = 4\omega\,\langle U_B\rangle / |I^2|$ and $C = |I|^2/4\omega^2\,\langle U_E\rangle$.

[2]The conductors of the antenna guide the energy from the power supply into the space around the antenna, but they do not generate this power. The antenna can be thought of as a waveguide, or an inside-out resonant cavity, which suggests that the concepts of capacitance and inductance may be relevant here.

[3]The terminal resistance depends on details of the current distribution in the antenna, and is not directly measurable by an "Ohm-meter".

We note that the electric and magnetic fields can be related to the charge and current densities ρ and $\mathbf{J}$ in the antenna and the power supply according to [8],

$$
\begin{aligned}
\mathbf{E}(\mathbf{x},t) &= \frac{1}{4\pi\epsilon_0}\int \frac{\rho\hat{\mathbf{R}}}{R^2}\, e^{j(\omega t-kR)}\, d^3\mathbf{x}' + \frac{\mu_0 c}{4\pi}\int \frac{(\mathbf{J}\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} + (\mathbf{J}\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R^2}\, e^{j(\omega t-kR)}\, d^3\mathbf{x}' \\
&\quad + \frac{j\mu_0\omega}{4\pi}\int \frac{(\mathbf{J}\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R}\, e^{j(\omega t-kR)}\, d^3\mathbf{x}', \qquad (8)
\end{aligned}
$$

$$
\mathbf{B}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\times\hat{\mathbf{R}}}{R^2}\, e^{j(\omega t-kR)}\, d^3\mathbf{x}' + \frac{j\mu_0\omega}{4\pi c}\int \frac{\mathbf{J}\times\hat{\mathbf{R}}}{R}\, e^{j(\omega t-kR)}\, d^3\mathbf{x}', \qquad (9)
$$

where $\mathbf{R} = \mathbf{x} - \mathbf{x}'$ and $k = \omega/c$, and c is the speed of light in vacuum. At large distances from the sources, the magnitudes of the electric and magnetic fields are related by $E = cB$. Hence, the difference $U_B - U_E$ in the field energies is finite, and the reactance X_{antenna} is meaningfully calculated according to eq. (7).

It is common to identify the "radiation fields" as the terms in eqs. (8)-(9) whose integrand varies as $1/R$,

$$
\mathbf{E}_{\mathrm{rad}} = \frac{j\mu_0\omega}{4\pi}\int \frac{(\mathbf{J}\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R}\, e^{j(\omega t-kR)}\, d^3\mathbf{x}', \qquad \mathbf{B}_{\mathrm{rad}} = \frac{j\mu_0\omega}{4\pi c}\int \frac{\mathbf{J}\times\hat{\mathbf{R}}}{R}\, e^{j(\omega t-kR)}\, d^3\mathbf{x}'. \quad (10)
$$

We note that while the total fields are asymptotically equal to the "radiation fields", no measurement can distinguish between them at any finite distance from the sources. This warns us that results based on use of the "radiation fields" at finite distances from the source may not be physically meaningful.

The energy densities $\langle u_{B,\mathrm{rad}}\rangle$ and $\langle u_{E,\mathrm{rad}}\rangle$ are asymptotically equal, so the difference $\langle U_{B,\mathrm{rad}}\rangle - \langle U_{E,\mathrm{rad}}\rangle$ between the total energies in the "radiation fields" is finite (but nonzero in general[4]). If we define the "nonradiative" field energies as,

$$
\langle U_{B_{\mathrm{nonrad}}}\rangle = \langle U_B\rangle - \langle U_{B_{\mathrm{rad}}}\rangle\,, \qquad \text{and} \qquad \langle U_{E_{\mathrm{nonrad}}}\rangle = \langle U_E\rangle - \langle U_{E_{\mathrm{rad}}}\rangle\,, \qquad (11)
$$

then,

$$
\langle U_{B_{\mathrm{nonrad}}}\rangle - \langle U_{E_{\mathrm{nonrad}}}\rangle = \langle U_B\rangle - \langle U_E\rangle - \langle U_{B_{\mathrm{rad}}}\rangle + \langle U_{E_{\mathrm{rad}}}\rangle\,. \qquad (12)
$$

In [1] it is proposed that the antenna inductance and capacitance be identified as,

$$
L = \frac{4\langle U_{B,\mathrm{nonrad}}\rangle}{|I^2|}\,, \qquad \text{and} \qquad C = \frac{|I|^2}{4\omega^2\langle U_{E,\mathrm{nonrad}}\rangle} \qquad \text{(as proposed in [1]).} \qquad (13)
$$

However, because $\langle U_{B_{\mathrm{rad}}}\rangle \neq \langle U_{E_{\mathrm{rad}}}\rangle$,

$$
\omega L - \frac{1}{\omega C} = \frac{4\omega}{|I|^2}(\langle U_{B_{\mathrm{nonrad}}}\rangle - \langle U_{E_{\mathrm{nonrad}}}\rangle) = \frac{4\omega}{|I|^2}(\langle U_B\rangle - \langle U_E\rangle - \langle U_{B_{\mathrm{rad}}}\rangle + \langle U_{E_{\mathrm{rad}}}\rangle) \neq X_{\mathrm{antenna}}\,, \quad (14)
$$

for the inductance and capacitance given in eq. (13).

[4]Only for an idealized point antenna, such as a Hertzian dipole, does $\langle U_{B,\mathrm{rad}}\rangle = \langle U_{E,\mathrm{rad}}\rangle$. See also the Appendices.

While the concept of antenna impedance is well defined, the relation of that impedance to an inductance and a capacitance is ambiguous.[5] The latter are well defined only for antennas that are small compared to a wavelength, such that $|E| \approx c|B|$ everywhere, and hence $\langle U_{B_{\rm rad}}\rangle - \langle U_{E_{\rm rad}}\rangle \approx 0$. In this case the capacitance and inductance of the antenna can be evaluated by quasistatic methods [11].

3 Can We Identify Reactive Field Energy?

The form of the antenna reactance found in eq. (7) suggests that we identify $\langle U_B\rangle - \langle U_E\rangle$ as the time average of the reactive field energy [4, 6], in which case $\langle u_B\rangle - \langle u_E\rangle$ is the (time-average) density of reactive field energy.[6] Can we also identify,

$$u_{\rm reactive} = u_B - u_E = \frac{(Re\mathbf{B})^2}{2\mu_0} - \frac{\epsilon_0 (Re\mathbf{E})^2}{2} \tag{15}$$

as the instantaneous density of reactive field energy?[7] Because the energy densities $u_{B,{\rm rad}}$ and $u_{E,{\rm rad}}$ of the radiation fields are asymptotically equal the density of reactive field energy falls off faster than $1/r^2$ from source charges and currents, and we infer that any flow of reactive field energy does not extend to "infinity".[8]

Writing the electric and magnetic fields as,

$$\mathbf{E}(\mathbf{x},t) = \tilde{\mathbf{E}}(\mathbf{x})\, e^{j\omega t} = Re\tilde{\mathbf{E}}\cos\omega t - Im\tilde{\mathbf{E}}\sin\omega t + j(Im\tilde{\mathbf{E}}\cos\omega t + Re\tilde{\mathbf{E}}\sin\omega t), \tag{16}$$

$$\mathbf{B}(\mathbf{x},t) = \tilde{\mathbf{B}}(\mathbf{x})\, e^{j\omega t} = Re\tilde{\mathbf{B}}\cos\omega t - Im\tilde{\mathbf{B}}\sin\omega t + j(Im\tilde{\mathbf{B}}\cos\omega t + Re\tilde{\mathbf{B}}\sin\omega t), \tag{17}$$

eq. (15) becomes,

$$\begin{aligned} u_{\rm reactive} = u_B - u_E \ &= \ \frac{(Re\tilde{\mathbf{B}})^2\cos^2\omega t + (Im\tilde{\mathbf{B}})^2\sin^2\omega t - Re\tilde{\mathbf{B}}\cdot Im\tilde{\mathbf{B}}\sin 2\omega t}{2\mu_0} \\ &\quad -\epsilon_0 \frac{(Re\tilde{\mathbf{E}})^2\cos^2\omega t + (Im\tilde{\mathbf{E}})^2\sin^2\omega t - Re\tilde{\mathbf{E}}\cdot Im\tilde{\mathbf{E}}\sin 2\omega t}{2}, \end{aligned} \tag{18}$$

[5]The relation $X_{\rm antenna} = \omega L - 1/\omega C$ could be satisfied by defining $L\,|I|^2/4 = \langle U_{B_{\rm nonrad}}\rangle + \alpha(\langle U_{B_{\rm rad}}\rangle - \langle U_{E_{\rm rad}}\rangle)$ and $|I|^2/4\omega^2 C = \langle U_{E_{\rm nonrad}}\rangle + (\alpha - 1)(\langle U_{B_{\rm rad}}\rangle - \langle U_{E_{\rm rad}}\rangle)$ for any α. One possible prescription is to take $\alpha = 1$ if $\langle U_{B_{\rm rad}}\rangle - \langle U_{E_{\rm rad}}\rangle \ge 0$, and $\alpha = 0$ if $\langle U_{B_{\rm rad}}\rangle - \langle U_{E_{\rm rad}}\rangle < 0$. An alternative [9] is to define $L = (dX/d\omega + X/\omega)/2$ and $C = 2/\omega^2(dX/d\omega - X/\omega)$. However, the utility of any such choice is limited in that the capacitance and inductance could only be calculated via integrals of the fields, which requires knowledge of the charges and currents in the system. If these are known (via a computer program such as NEC4 [10] given the terminal voltage V), the (complex) terminal current I is known and the terminal impedance can be directly calculated as $Z_{\rm antenna} = V/I$.

[6]See [12] for discussion of the relation between reactive field energy and the Q of a resonant circuit.

[7]A different definition of "reactive field energy" is advocated in [13], $u'_{\rm reactive} = \sqrt{u^2_{\rm field} - S^2/c^2} = \epsilon_0\sqrt{(E^2 - c^2B^2)^2/4 + (\mathbf{E}\cdot c\mathbf{B})^2} = \sqrt{u^2_{\rm reactive} + (\epsilon_0\mathbf{E}\cdot c\mathbf{B})^2}$, where $u'_{\rm reactive}/c^2$ is identified with the density of "inertia" in the electromagnetic field. However, this implies that any system "at rest" with a nonzero Poynting vector (such as a battery connected to a resistor) does not obey Einstein's relation $E = mc^2$ [14]. Furthermore, such systems would contain "nonreactive" field energy associated with the Poynting vector, which latter flows from one part of the system to another, which seems to be "reactive".

[8]The so-called radiation fields contribute to reactive field energy (15), as shown in Appendix B below.

whose time average is,

$$\langle u_{\rm reactive}\rangle = \frac{\left|\tilde{\mathbf{B}}\right|^2}{4\mu_0} - \frac{\epsilon_0 \left|\tilde{\mathbf{E}}\right|^2}{4} = \frac{|\mathbf{B}|^2}{4\mu_0} - \frac{\epsilon_0\, |\mathbf{E}|^2}{4}\,. \tag{19}$$

For definition (15) to be consistent we should also be able to identify a part, $\mathbf{S}_{\rm reactive}$, of the (real) Poynting vector $\mathbf{S} = Re\mathbf{E} \times Re\mathbf{B}/\mu_0$ that obeys Poynting's theorem in the form,

$$\boldsymbol{\nabla} \cdot \mathbf{S}_{\rm reactive} = -\frac{\partial u_{\rm reactive}}{\partial t} - Re\mathbf{E} \cdot Re\mathbf{J}, \tag{20}$$

where $\mathbf{J}$ is the current density. From (18) we find,

$$\begin{aligned} -\frac{\partial u_{\rm reactive}}{\partial t} = &= \omega \frac{[(Re\tilde{\mathbf{B}})^2 - (Im\tilde{\mathbf{B}})^2]\sin 2\omega t + 2Re\tilde{\mathbf{B}} \cdot Im\tilde{\mathbf{B}} \cos 2\omega t}{2\mu_0} \\ &\quad -\epsilon_0\omega \frac{[(Re\tilde{\mathbf{E}})^2 - (Im\tilde{\mathbf{E}})^2]\sin 2\omega t - 2Re\tilde{\mathbf{E}} \cdot Im\tilde{\mathbf{E}} \cos 2\omega t}{2}\,, \end{aligned} \tag{21}$$

which has terms of time dependence $\cos 2\omega t$ and $\sin 2\omega t$.[9]

If we apply eq. (4) to a volume that does not contain any currents or the power source we obtain,

$$\oint Im\tilde{\mathbf{S}} \cdot d\mathbf{Area} = -2\omega(\langle U_B\rangle - \langle U_E\rangle). \tag{22}$$

Equation (22) suggests that the imaginary part of the complex Poynting vector (1) is related to the flow of reactive field energy, in that we have come to associate the Poynting vector with flow of electromagnetic field energy. This relation is not immediately evident in that the reactive field energy $\langle U_B\rangle - \langle U_E\rangle$ is constant in time, and cannot be said to flow.

The (real) Poynting vector is, recalling eqs. (16)-(17),

$$\begin{aligned} \mathbf{S} &= \frac{Re\mathbf{E} \times Re\mathbf{B}}{\mu_0} = \frac{(Re\tilde{\mathbf{E}}\cos\omega t - Im\tilde{\mathbf{E}}\sin\omega t) \times (Re\tilde{\mathbf{B}}\cos\omega t - Im\tilde{\mathbf{B}}\sin\omega t)}{\mu_0} \\ &= \frac{Re\tilde{\mathbf{E}} \times Re\tilde{\mathbf{B}} + Im\tilde{\mathbf{E}} \times Im\tilde{\mathbf{B}}}{2\mu_0} + \frac{Re\tilde{\mathbf{E}} \times Re\tilde{\mathbf{B}} - Im\tilde{\mathbf{E}} \times Im\tilde{\mathbf{B}}}{2\mu_0}\cos 2\omega t \\ &\quad - \frac{Re\tilde{\mathbf{E}} \times Im\tilde{\mathbf{B}} + Im\tilde{\mathbf{E}} \times Re\tilde{\mathbf{B}}}{2\mu_0}\sin 2\omega t, \end{aligned} \tag{23}$$

and the complex Poynting vector is,

$$\begin{aligned} \tilde{\mathbf{S}} &= \frac{\mathbf{E} \times \mathbf{B}^\star}{2\mu_0} = [Re\tilde{\mathbf{E}}\cos\omega t - Im\tilde{\mathbf{E}}\sin\omega t + j(Im\tilde{\mathbf{E}}\cos\omega t + Re\tilde{\mathbf{E}}\sin\omega t)] \\ &\qquad \times \frac{Re\tilde{\mathbf{B}}\cos\omega t - Im\tilde{\mathbf{B}}\sin\omega t - j(Im\tilde{\mathbf{B}}\cos\omega t + Re\tilde{\mathbf{B}}\sin\omega t)}{2\mu_0} \\ &= \frac{Re\tilde{\mathbf{E}} \times Re\tilde{\mathbf{B}} + Im\tilde{\mathbf{E}} \times Im\tilde{\mathbf{B}}}{2\mu_0} - j\frac{Re\tilde{\mathbf{E}} \times Im\tilde{\mathbf{B}} + Im\tilde{\mathbf{E}} \times Re\tilde{\mathbf{B}}}{2\mu_0}\,. \end{aligned} \tag{24}$$

[9]The presence of terms of frequency 2ω in the Poynting vector $\mathbf{S}$ and in $\partial u/\partial t$ seems disconcerting to many people, some of whom chose to ignore the time dependence of these quantities, which are quadratic in the fields and so "naturally" include second-harmonic terms.

Hence, the imaginary part of the complex Poynting vector, $Im\tilde{\mathbf{S}}$, is the coefficient of the term in the real Poynting vector $\mathbf{S}$ that varies as $\sin 2\omega t$. However, we cannot consistently identify this term with the "reactive" part of the (real) Poynting vector, in that its divergence would also have time dependence $\sin 2\omega t$ while $\partial u_{\rm reactive}/\partial t$ of eq. (21) has a term of time dependence $\cos 2\omega t$.

This leaves us without a crisp physical interpretation of $Im\tilde{\mathbf{S}}$, and without a fully consistent identification of reactive field energy. Such difficulties are typical of attempts to partition quadratic quantities like field energy and Poynting flux. See also the discussion in [15] and in Appendix B below.

A Appendix: Hertzian (Electric) Dipole

We consider a single charge q that oscillates in vacuum about the origin with (complex) oscillating dipole moment $\mathbf{p} = -p\,\hat{\mathbf{z}}$ and amplitude small compared to the wavelength $\lambda = 2\pi c/\omega$. Then, in spherical coordinates (r, θ, ϕ), the electromagnetic fields are the real parts of the complex quantities,[10]

$$\begin{aligned}
\mathbf{E} &= -k^2 p(\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \times \hat{\mathbf{r}}\, \frac{e^{j(\omega t - kr)}}{4\pi\epsilon_0 r} - p[3(\hat{\mathbf{p}} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{p}}] \left(\frac{1}{r^3} + \frac{jk}{r^2}\right) \frac{e^{j(\omega t - kr)}}{4\pi\epsilon_0} \\
&= k^2 p \left(1 - \frac{j}{kr} - \frac{1}{k^2 r^2}\right) \frac{e^{j(\omega t - kr)}}{4\pi\epsilon_0 r} \sin\theta\, \hat{\boldsymbol{\theta}} - k^2 p \left(\frac{j}{kr} + \frac{1}{k^2 r^2}\right) \frac{e^{j(\omega t - kr)}}{2\pi\epsilon_0 r} \cos\theta\, \hat{\mathbf{r}}, \qquad (25) \\
\mathbf{B} &= \frac{\mu_0 c k^2 p}{4\pi} (\hat{\mathbf{r}} \times \hat{\mathbf{p}}) \left(\frac{1}{r} + \frac{1}{jkr^2}\right) e^{j(\omega t - kr)} \\
&= \frac{\mu_0 c k^2 p}{4\pi} \left(1 - \frac{j}{kr}\right) \frac{e^{j(\omega t - kr)}}{r} \sin\theta\, \hat{\boldsymbol{\phi}}. \qquad (26)
\end{aligned}$$

A.1 Poynting Vector

The complex Poynting vector of eq. (1) is,

$$\tilde{\mathbf{S}} = \frac{\mathbf{E} \times \mathbf{B}^\star}{2\mu_0} = \frac{ck^4 p^2 \sin^2\theta}{32\pi^2\epsilon_0 r^2} \left(1 + \frac{j}{k^3 r^3}\right) \hat{\mathbf{r}} + j\frac{ck^3 p^2 \sin 2\theta}{32\pi^2\epsilon_0 r^3} \left(1 + \frac{1}{k^2 r^2}\right) \hat{\boldsymbol{\theta}}. \qquad (27)$$

The real part of the complex Poynting vector is the time-averaged Poynting vector,

$$\langle \mathbf{S} \rangle = Re\tilde{\mathbf{S}} = \frac{ck^4 p^2 \sin^2\theta}{32\pi^2\epsilon_0 r^2}\, \hat{\mathbf{r}}, \qquad (28)$$

which is purely radial, and varies as $1/r^2$, corresponding to a conserved flow of energy out from the nominally point, oscillating dipole.

The imaginary part of the complex Poynting vector is,

$$Im\tilde{\mathbf{S}} = \frac{ck^4 p^2 \sin^2\theta}{32\pi^2\epsilon_0 r^2} \frac{1}{k^3 r^3}\, \hat{\mathbf{r}} + \frac{ck^4 p^2 \sin 2\theta}{32\pi^2\epsilon_0 r^2} \left(\frac{1}{kr} + \frac{1}{k^3 r^3}\right) \hat{\boldsymbol{\theta}}. \qquad (29)$$

[10]See, for example, sec. 9.2 of [16], with the convention that j of electrical engineering equals $-i$ of physics.

To assess the possible physical significance of eq. (28), we consider the actual Poynting vector, $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$, where the fields $\mathbf{E}$ and $\mathbf{B}$ are real, *i.e.*, the real parts of eqs. (25)-(26),

$$\begin{aligned}
\mathbf{E} &= k^2 p \left(1 - \frac{1}{k^2 r^2}\right) \frac{\cos(\omega t - kr)}{4\pi\epsilon_0 r} \sin\theta\, \hat{\boldsymbol{\theta}} + \frac{k^2 p}{kr} \frac{\sin(\omega t - kr)}{4\pi\epsilon_0 r} \sin\theta\, \hat{\boldsymbol{\theta}} \\
&\quad + \frac{k^2 p}{kr} \frac{\sin(\omega t - kr)}{2\pi\epsilon_0 r} \cos\theta\, \hat{\mathbf{r}} - \frac{k^2 p}{k^2 r^2} \frac{\cos(\omega t - kr)}{2\pi\epsilon_0 r} \cos\theta\, \hat{\mathbf{r}}, \qquad (30) \\
\mathbf{B} &= \frac{\mu_0 c k^2 p}{4\pi} \frac{\cos(\omega t - kr)}{r} \sin\theta\, \hat{\boldsymbol{\phi}} + \frac{\mu_0 c k^2 p}{4\pi kr} \frac{\sin(\omega t - kr)}{r} \sin\theta\, \hat{\boldsymbol{\phi}}. \qquad (31) \\
\mathbf{S} &= \frac{c k^4 p^2 \sin^2\theta}{16\pi^2 \epsilon_0 r^2} \left[\left(1 - \frac{1}{k^2 r^2}\right) \cos^2(\omega t - kr) + \frac{1}{k^2 r^2} \sin^2(\omega t - kr) \right. \\
&\quad \left. \left(\frac{2}{kr} - \frac{1}{k^3 r^3}\right) \cos(\omega t - kr) \sin(\omega t - kr) \right] \hat{\mathbf{r}} \\
&\quad + \frac{c k^4 p^2 \sin 2\theta}{16\pi^2 \epsilon_0 r^2} \left[-\frac{1}{k^2 r^2} \cos^2(\omega t - kr) + \frac{1}{k^2 r^2} \sin^2(\omega t - kr) \right. \\
&\quad \left. + \left(\frac{1}{kr} - \frac{1}{k^3 r^3}\right) \cos(\omega t - kr) \sin(\omega t - kr) \right] \hat{\boldsymbol{\theta}} \\
&= \frac{c k^4 p^2 \sin^2\theta}{16\pi^2 \epsilon_0 r^2} \left[\cos^2(\omega t - kr) - \frac{\cos 2(\omega t - kr)}{k^2 r^2} + \left(\frac{1}{kr} - \frac{1}{2k^3 r^3}\right) \sin 2(\omega t - kr) \right] \hat{\mathbf{r}} \\
&\quad + \frac{c k^4 p^2 \sin 2\theta}{16\pi^2 \epsilon_0 r^2} \left[-\frac{\cos 2(\omega t - kr)}{k^2 r^2} + \frac{1}{2} \left(\frac{1}{kr} - \frac{1}{k^3 r^3}\right) \sin 2(\omega t - kr) \right] \hat{\boldsymbol{\theta}}. \qquad (32)
\end{aligned}$$

The time average of eq. (32) is the same as eq. (28) as expected, but the full time-dependent flow of energy in the electromagnetic field, as described by eq. (32) is much more complicated than eq. (29). This reinforces that the imaginary part of the complex Poynting vector has no clear physical significance.

A.2 Field Energy

The density u_E of energy in the electric field follows from eq. (30) as,

$$\begin{aligned}
u_E(r, \theta, t) &= \frac{\epsilon_0 E^2}{2} = \frac{k^4 p^2}{32\pi^2 \epsilon_0 r^2} \left\{ \left[\left(1 - \frac{1}{k^2 r^2}\right)^2 \cos^2(\omega t - kr) \right. \right. \\
&\quad \left. + \left(1 - \frac{1}{k^2 r^2}\right) \frac{\sin 2(\omega t - kr)}{kr} + \frac{\sin^2(\omega t - kr)}{k^2 r^2} \right] \sin^2\theta \\
&\quad \left. + \left[\frac{4\sin^2(\omega t - kr)}{k^2 r^2} - \frac{4\sin 2(\omega t - kr)}{kr} + \frac{4\cos^2(\omega t - kr)}{k^4 r^4} \right] \cos^2\theta \right\}. \qquad (33)
\end{aligned}$$

It may be of interest to consider the energy density in a spherical shell of radius r,

$$
\begin{aligned}
u_E(r,t) &= 4\pi r^2 \int u_E\, d\Omega = \frac{k^4 p^2}{3\epsilon_0}\left[\left(1 - \frac{2}{k^2r^2} + \frac{3}{k^4r^4}\right)\cos^2(\omega t - kr)\right. \\
&\qquad \left. - \left(1 + \frac{1}{k^2r^2}\right)\frac{\sin 2(\omega t - kr)}{kr} + \frac{3\sin^2(\omega t - kr)}{k^2r^2}\right]. \qquad (34)
\end{aligned}
$$

The time average of this is,

$$
\langle u_E(r)\rangle = \frac{k^4p^2}{6\epsilon_0}\left(1 + \frac{1}{k^2r^2} + \frac{3}{k^4r^4}\right). \qquad (35)
$$

Similarly, the density u_B of energy in the magnetic field is,

$$
\begin{aligned}
u_B &= \frac{B^2}{2\mu_0} = \frac{k^4p^2}{32\pi^2\epsilon_0}\left[\frac{\cos^2(\omega t - kr)}{r^2} + \frac{\sin 2(\omega t - kr)}{k^3r^3} + \frac{\sin^2(\omega t - kr)}{k^4r^4}\right]\sin^2\theta, \qquad (36) \\
u_B(r) &= 4\pi r^2 \int u_B\, d\Omega = \frac{k^4p^2}{3\epsilon_0}\left[\cos^2(\omega t - kr) + \frac{\sin 2(\omega t - kr)}{kr} + \frac{\sin^2(\omega t - kr)}{k^2r^2}\right] \qquad (37) \\
\langle u_B(r)\rangle &= \frac{k^4p^2}{6\epsilon_0}\left(1 + \frac{1}{k^2r^2}\right). \qquad (38)
\end{aligned}
$$

If we suppose that the reactive energy density is given by eq. (15) (or by the negative of this), then we could have a simple result for its time average,

$$
\langle u_{\rm reactive}(r)\rangle = \langle u_E(r)\rangle - \langle u_B(r)\rangle = \frac{p^2}{3\epsilon_0 r^4}. \qquad (39)
$$

However, the instantaneous (radial) reactive energy density would be,

$$
\begin{aligned}
u_{\rm reactive}(r,t) &= u_E(r,t) - u_B(r,t) \\
&= \frac{k^4p^2}{3\epsilon_0}\left[\frac{3\cos^2(\omega t - kr)}{k^4r^4} - \left(2 + \frac{1}{k^2r^2}\right)\frac{\sin 2(\omega t - kr)}{kr} - \frac{2\cos 2(\omega t - kr)}{k^2r^2}\right], \qquad (40)
\end{aligned}
$$

which suffers from the issues discussed in sec. 3 above.[11]

B Appendix: Radiation Field Energy of a Pair of Oscillating Point Dipoles

We consider a system of only two charges, q_1 and q_2, at (average) positions $\mathbf{x}_1$ and $\mathbf{x}_2$, with (complex) oscillating dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$. Then,

$$
\mathbf{E}(\mathbf{x},t) = k^2 p_1(\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1)\times\hat{\mathbf{r}}_1\,\frac{e^{j(\omega t - kr_1)}}{4\pi\epsilon_0 r_1} + p_1[3(\hat{\mathbf{p}}_1\cdot\hat{\mathbf{r}}_1)\hat{\mathbf{r}}_1 - \hat{\mathbf{p}}_1]\left(\frac{1}{r_1^3} + \frac{jk}{r_1^2}\right)\frac{e^{j(\omega t - kr_1)}}{4\pi\epsilon_0} \qquad (41)
$$

[11]If one ignores the difficulties of interpretation of the time-dependent expression (40), the time-average result (39) seems appealing, as on p. 17 of [17].

$$
\begin{aligned}
& +k^2 p_1 (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \times \hat{\mathbf{r}}_2 \, \frac{e^{j(\omega t - kr_2)}}{4\pi\epsilon_0 r_2} + p_2 [3(\hat{\mathbf{p}}_2 \cdot \hat{\mathbf{r}}_2)\hat{\mathbf{r}}_2 - \hat{\mathbf{p}}_2] \left(\frac{1}{r_2^3} + \frac{jk}{r_2^2} \right) \frac{e^{j(\omega t - kr_2)}}{4\pi\epsilon_0} , \\
\mathbf{B}(\mathbf{x}, t) \;=\; & \frac{\mu_0 c k^2 p_1}{4\pi} (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \left(\frac{1}{r_1} + \frac{1}{jkr_1^2} \right) e^{j(\omega t - kr_1)} \\
& + \frac{\mu_0 c k^2 p_2}{4\pi} (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \left(\frac{1}{r_2} + \frac{1}{jkr_2^2} \right) e^{j(\omega t - kr_2)}, \qquad (42)
\end{aligned}
$$

where $\mathbf{r}_j = \mathbf{x} - \mathbf{x}_j$. The radiation fields are,

$$
\mathbf{E}_{\rm rad}(\mathbf{x}, t) = k^2 p_1 (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \times \hat{\mathbf{r}}_1 \, \frac{e^{j(\omega t - kr_1)}}{4\pi\epsilon_0 r_1} + k^2 p_1 (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \times \hat{\mathbf{r}}_2 \, \frac{e^{j(\omega t - kr_2)}}{4\pi\epsilon_0 r_2} , \qquad (43)
$$

$$
\mathbf{B}_{\rm rad}(\mathbf{x}, t) = \mu_0 c k^2 p_1 (\hat{\mathbf{r}}_1 \times \hat{\mathbf{p}}_1) \frac{e^{j(\omega t - kr_1)}}{4\pi r_1} + \mu_0 c k^2 p_2 (\hat{\mathbf{r}}_2 \times \hat{\mathbf{p}}_2) \frac{e^{j(\omega t - kr_2)}}{4\pi r_2} . \qquad (44)
$$

The corresponding time-average field-energy densities are,

$$
\begin{aligned}
\langle u_{E_{\rm rad}} \rangle \;=\; & \frac{\epsilon_0 \left| E_{\rm rad} \right|^2}{4} = \frac{k^4 p_1^2 \left(1 - |\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_1|^2 \right)}{64\pi^2 \epsilon_0 r_1^2} + \frac{k^4 p_2^2 \left(1 - |\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2|^2 \right)}{64\pi^2 \epsilon_0 r_2^2} \\
& + \frac{k^4}{32\pi^2 \epsilon_0 r_1 r_2} Re \left\{ p_1 p_2^\star \, e^{jk(r_2 - r_1)} \left[\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2^\star - (\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_2^\star) - (\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2^\star) \right. \right. \\
& \left. \left. \qquad + (\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_1)(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2^\star)(\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2) \right] \right\} , \qquad (45) \\
\langle u_{B_{\rm rad}} \rangle \;=\; & \frac{\left| B_{\rm rad} \right|^2}{4\mu_0} = \frac{k^4 p_1^2 \left(1 - |\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_1|^2 \right)}{64\pi^2 \epsilon_0 r_1^2} + \frac{k^4 p_2^2 \left(1 - |\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_2|^2 \right)}{64\pi^2 \epsilon_0 r_2^2} \\
& + \frac{k^4}{32\pi^2 \epsilon_0 r_1 r_2} Re \left\{ p_1 p_2^\star \, e^{jk(r_2 - r_1)} \left[(\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2)(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_2^\star) - (\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{p}}_2^\star)(\hat{\mathbf{r}}_2 \cdot \hat{\mathbf{p}}_1) \right] \right\} . \qquad (46)
\end{aligned}
$$

In general, $\langle u_{B_{\rm rad}} \rangle \neq \langle u_{E_{\rm rad}} \rangle$, but in the far zone they become equal since $\hat{\mathbf{r}}_1 = \hat{\mathbf{r}}_2$ there. Hence, radiation field energy contributes to reactive field energy according to the definition of eq. (15).

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