An Antenna Reciprocity Theorem

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1 Problem

Consider any two antennas, labeled A and B, that can be operated as two-terminal devices. If antenna B is used as a receiver, the open-circuit (no load) voltage $V_B^{\rm oc}$ induced across its terminals by the radiation from antenna A with drive current I_A (between its terminals) is related to the open-circuit voltage $V'^{\rm oc}_A$ induced across the terminals of antenna A (when used as a receiver) by the radiation from antenna B (when used as a transmitter with drive current I'_B) according to the reciprocity relation,

$$V_B^{\rm oc} I'_B = V'^{\rm oc}_A I_A. \tag{1}$$

Show that the antenna reciprocity theorem (1) can be deduced both from Lorentz' reciprocity relation [1] and from Tellegen's network reciprocity relation [2].

2 Solution

A review of the history of reciprocity theorems in electrodynamics is given in the Appendix of [3]. The earliest statement of eq. (1) that I have found is in sec. 11.10 of [4], although something fairly similar appears in [5].

2.1 Lorentz Reciprocity

The generalization of previous reciprocity theorems to the vector fields of electromagnetism, including waves, is due to Lorentz (1896 [1]). He showed that if a (time-dependent) current distribution $\mathbf{J}_1$ leads to electric and magnetic fields $\mathbf{E}_1$ and $\mathbf{B}_1$ in a linear medium, and that current distribution $\mathbf{J}_2$ leads to electric and magnetic fields $\mathbf{E}_2$ and $\mathbf{B}_2$, then,

$$\int \mathbf{J}_1 \cdot \mathbf{E}_2 \, d{\rm Vol} = \int \mathbf{J}_2 \cdot \mathbf{E}_1 \, d{\rm Vol}. \tag{2}$$

A demonstration of eq. (2) invokes the vector calculus identity that,

$$\boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) - \mathbf{F} \cdot (\boldsymbol{\nabla} \times \mathbf{G}). \tag{3}$$

Thus, for fields with time dependence $e^{-i\omega t}$,

$$\begin{aligned} \boldsymbol{\nabla} \cdot (\mathbf{E}_1 \times \mathbf{B}_2) &= \mathbf{B}_2 \cdot (\boldsymbol{\nabla} \times \mathbf{E}_1) - \mathbf{E}_1 \cdot (\boldsymbol{\nabla} \times \mathbf{B}_2) \\ &= -\frac{1}{c}\mathbf{B}_2 \cdot \frac{\partial \mathbf{B}_1}{\partial t} - \mathbf{E}_1 \cdot \left(\frac{4\pi}{c}\mathbf{J}_2 + \frac{1}{c}\frac{\partial \mathbf{E}_2}{\partial t}\right) \\ &= -\frac{4\pi}{c}\mathbf{E}_1 \cdot \mathbf{J}_2 + \frac{i\omega}{c}(\mathbf{B}_2 \cdot \mathbf{B}_1 + \mathbf{E}_1 \cdot \mathbf{E}_2). \end{aligned} \tag{4}$$

Similarly,

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\mathbf{B}_1\times\mathbf{E}_2) &= \mathbf{E}_2\cdot(\boldsymbol{\nabla}\times\mathbf{B}_1)-\mathbf{B_1}\cdot(\boldsymbol{\nabla}\times\mathbf{E_2}) \\
&= \mathbf{E}_2\cdot\left(\frac{4\pi}{c}\mathbf{J}_1+\frac{1}{c}\frac{\partial\mathbf{E}_1}{\partial t}\right)-\frac{1}{c}\mathbf{B_1}\cdot\frac{\partial\mathbf{B}_2}{\partial t} \\
&= -\frac{4\pi}{c}\mathbf{E_2}\cdot\mathbf{J}_1+\frac{i\omega}{c}(\mathbf{E}_2\cdot\mathbf{E}_1+\mathbf{B_1}\cdot\mathbf{B}_2). \qquad (5)
\end{aligned}$$

Hence,

$$\boldsymbol{\nabla}\cdot(\mathbf{E}_1\times\mathbf{B}_2-\mathbf{B}_1\times\mathbf{E}_2)=\frac{4\pi}{c}(\mathbf{J_1}\cdot\mathbf{E}_2-\mathbf{J_2}\cdot\mathbf{E}_1), \qquad (6)$$

and so,

$$\int(\mathbf{J_1}\cdot\mathbf{E}_2-\mathbf{J_2}\cdot\mathbf{E}_1)\,d\mathrm{Vol}=\frac{c}{4\pi}\oint(\mathbf{E}_1\times\mathbf{B}_2-\mathbf{E}_1\times\mathbf{B}_2)\cdot d\mathbf{Area}. \qquad (7)$$

A small delicacy in the argument is that for sources $\mathbf{J}_1$ and $\mathbf{J}_2$ contained in a bounded region, the asymptotic radiation fields vary as $1/r$ and have the form $\mathbf{B}_{1(2)}=\hat{\mathbf{r}}\times\mathbf{E}_{1(2)}$, so the surface integral vanishes, and we obtain the reciprocity relation (2), independent of the angular frequency ω.

The assumption of linear media seems necessary only to insure that sources at frequency ω lead to fields only at this frequency. See [6] for a review of Lorentz reciprocity in the time domain.

Turning to reciprocity for antennas, we consider two antennas, A and B, which contain the only currents in our system. Then eq. (2) can be written as,

$$\int_A\mathbf{J}_{1A}\cdot\mathbf{E}_{2A}\,d\mathrm{Vol}+\int_B\mathbf{J}_{1B}\cdot\mathbf{E}_{2B}\,d\mathrm{Vol}=\int_A\mathbf{J}_{2A}\cdot\mathbf{E}_{1A}\,d\mathrm{Vol}+\int_B\mathbf{J}_{2B}\cdot\mathbf{E}_{1B}\,d\mathrm{Vol}. \qquad (8)$$

In situation 1, antenna A is the transmitter, and antenna B is the receiver, operated with no load (open circuit), while their roles are reversed in situation 2. Then the currents $\mathbf{J}_{1B}$ and $\mathbf{J}_{2A}$ exist only in the conductors of the receiving antennas, and not in the gap between the terminals of these antennas. Of course, these currents flow along the conductors. In the approximation of perfect conductors, the electric fields in/on these conductors have no component parallel to the conductors. Hence,

$$\mathbf{J}_{1B}\cdot\mathbf{E}_{2B}=0=\mathbf{J}_{2A}\cdot\mathbf{E}_{1A}, \qquad (9)$$

while the remaining integrals in eq. (8) have contributions only from the gap between the terminals (where the idealized power sources for the transmitting antennas are located),[1]

$$\int_{A,\mathrm{gap}}\mathbf{J}_{1A}\cdot\mathbf{E}_{2A}\,d\mathrm{Vol}=\int_{B,\mathrm{gap}}\mathbf{J}_{2B}\cdot\mathbf{E}_{1B}\,d\mathrm{Vol}. \qquad (10)$$

[1]The antenna reciprocity theorem (1) holds in cases where other conductors are present in the system beside those of the two antennas, provided those other conductors are "good" conductors. However, if there exist significant currents in poor conductors, such as lakes, oceans, or the ionosphere, then Lorentz' relation (2) still holds, but the form (1) does not.

We can write the currents in the gap as $\mathbf{J} = I\, d\mathbf{l}/d\mathrm{Vol}$, so that eq. (10) becomes,

$$I_{1A} \int_{A,\mathrm{gap}} \mathbf{E}_{2A} \cdot d\mathbf{l} = I_{1A} V_{2A}^{\mathrm{oc}} = I_{2B} V_{1B}^{\mathrm{oc}} = I_{2B} \int_{B,\mathrm{gap}} \mathbf{E}_{1B} \cdot d\mathbf{l}, \tag{11}$$

where V^{oc} is measured between the terminals of the receiving antennas. This is the reciprocity theorem (1), which holds for both "linear" and "loop" antennas.

2.2 Tellegen's Theorem

Tellegen [2] has given a network theorem that then leads to a kind of reciprocity theorem. See also [7].[2]

Consider a network with nodes, and links between some or all pairs of nodes. The network can consist of parts with no links between different parts.

"Currents" flow along links between pairs with nodes, with the same scalar value for the "current" at both nodes. A "current I_{jk} is defined to be positive if it flows from node j to node k. Then, $I_{ji} = -I_{ij}$. The only "physical" assumption underlying Tellegen's theorem is that,

$$\sum_{\text{nodes } k \text{ directly linked to node } j} I_{jk} = 0 \tag{12}$$

for all nodes j.

We also suppose that every node can be assigned a scalar "voltage" V_j. However, there is not necessarily any "physical" relation between "current" and "voltage".

It follows immediately from eq. (12 that,

$$\sum_j V_j \sum_{\text{nodes } k \text{ directly linked to node } j} I_{jk} = 0. \tag{13}$$

The "current" in a link appears exactly twice in the sum (13), in the form,

$$V_j I_{jk} = V_k I_{kj} = (V_j - V_k) I_{jk} \equiv V_{jk} I_{jk}. \tag{14}$$

Summing over all links, we obtain Tellegen's theorem,

$$\sum_{\text{links}} V_{jk} I_{jk} = 0. \tag{15}$$

Another consequence of the definition of the "voltage drop" $V_{jk} = V_j - V_k$ is that the directed sum of "voltage drops" around any closed loop of links is zero.

We can obtain a kind of reciprocity theorem from eq. (15) by considering a second set of "voltages" V'_j and the corresponding "voltage drops" V'_{jk}. Since eq. (15) does not require there to be any "physical" relation between "current" and "voltage", we also have that $\sum_{\text{links}} V'_{jk} I_{jk} = 0$. Likewise, we can consider another set of "currents" I'_{jk} that are not necessarily related to either the V_{jk} or the V'_{jk} (other than applying to the same network

[2](Jan. 30, 2020) A discussion of network reciprocity as a generalization of the so-called Shockley-Ramo theorem [8, 9], based on Lorentz reciprocity, is given in [10].

topology), for which we can write $\sum_{\rm links} V_{jk} I'_{jk} = 0$. Hence, we obtain Tellegen's reciprocity relation,

$$\sum_{\rm links} V'_{jk} I_{jk} = \sum_{\rm links} V_{jk} I'_{jk} = 0. \qquad (16)$$

However, we cannot deduce from this that $V'_{jk} I_{jk} = V_{mn} I'_{mn}$ for a pair of links jk and mn.

The "nonphysical" character of Tellegen's theorem clarifies how the reciprocity theorems are somewhat abstract "bookkeeping" constructs, rather than a manifestation of "cause and effect".

We can, however, obtain the antenna reciprocity theorem of sec. 2.1 from Tellegen's Theorem, but only for "linear" antennas. For this, we consider a network of two disconnected parts, antennas A and B, with three links, $A1$, $A2$, $A3$, *etc.* in each part in the case of a "linear" antenna. Link 2 is the gap between the physical conductors of the antenna (between its terminals).

In the unprimed situation 1, antenna A is the transmitter, and antenna B is the receiver, operated with no load (open circuit), while their roles are reversed in the primed situation. Then the "currents" in this example are the physical electrical currents, so that I_{B2} and I'_{A2} are zero, since the receiving antennas are operated "open circuit".[3] We define the "voltage difference" between the ends of a link to be $\int \mathbf{E} \cdot d\mathbf{l}$. Hence, $V_{A1} = V_{A3} = V_{B1} = V_{B1} = V'_{A1} = V'_{A3} = V'_{B1} = V'_{B1} = 0$, in the approximation that the conductors of the antenna are perfect. The voltages V_{jA} at the four nodes of antenna A are then $V_{1A} = V_{2A}$ and $V_{3A} = V_{4A}$.

Thus, of the 12 terms in eq. (16) for the pair of "linear" antennas only two are nonzero, and we have,

$$V'_{A2} I_{A2} = V_{B2} I'_{B2}, \qquad (17)$$

as in eq. (1), where again the "voltages" V'_{A2} and V_{B2} are "open circuit".

However, a similar argument for a "loop" antenna, representing it by two links connected to two nodes, fails. If link 2 again represent the gap between the terminals, then the "voltage" $V_{1A} = \int_{\rm link\ 1A} \mathbf{E} \cdot d\mathbf{l} = 0$, but $V_{2A} = \int_{\rm link\ 2A} \mathbf{E} \cdot d\mathbf{l}$ is nonzero in general, so a unique scalar "voltage" cannot be assigned to the two nodes of antenna A.

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[3] The currents along the conductors of the antenna are not constant but they do satisfy the node condition (12). The currents at the tips of a "linear" antenna are zero, where only one link is connected to the tip node. For the transmitting antennas the current I_{12} does not equal $-I_{21}$ and I_{34} does not equal $-I_{43}$. However, Tellegen's relation (16) still holds because $V_{12} = 0 = V_{34}$, as discussed below.

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