Resistance of a Disk

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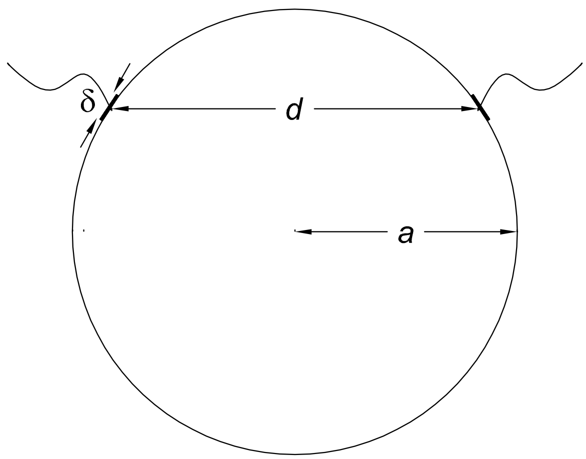
1 Problem

Calculate the resistance between two contacts on the rim of a disk of radius a, thickness $t \ll a$, and conductivity σ, when each (perfectly conducting) contact extends for a small distance δ around the circumference, and the distance along the chord between the contacts is $d \gg \delta$.



2 Solution

This problem is posed on p. 351 of *The Mathematical Theory of Electricity and Magnetism* (1908), by James Jeans,
http://kirkmcd.princeton.edu/examples/EM/jeans_electricity.pdf
It was the subject of Kirchhoff's first physics paper, Ann. d. Phys. **64**, 497 (1845),
http://kirkmcd.princeton.edu/examples/EM/kirchhoff_ap_64_497_45.pdf

We will evaluate the resistance R via Ohm's Law, $R = V/I$, by calculating the current I that flows when a potential difference V is established between the two contacts.

For a thin disk, the current flow is 2-dimensional. Since $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{J}$ is the current density and $\mathbf{E}$ is the electric field, the electric field is 2-dimensional also. And, since $\mathbf{E} = -\boldsymbol{\nabla}\phi$, where ϕ is the electric potential, the potential is 2-dimensional as well.

The form of the 2-dimensional potential is well approximated (for distances more than $\delta/2$ from the centers of the contacts) by considering a cylinder of radius a, rather than the disk, with a line charge density λ that passes through the center of one contact, and line charge $-\lambda$ that passes through the center of the other contact.

The electric field from the wire of charge density λ has magnitude (in Gaussian units),

$$E_1 = \frac{2\lambda}{r_1}, \tag{1}$$

according to Gauss' law, where r_1 is the distance from the wire to the observer. The corresponding electric potential is,

$$\phi_1 = 2\lambda \ln \frac{r_1}{r_0}, \tag{2}$$

where r_0 is a constant of integration. The potential due to the wire with charge density $-\lambda$ is similarly,

$$\phi_2 = -2\lambda \ln \frac{r_2}{r_0}, \tag{3}$$

where r_2 is the distance from the observer to wire 2. The potential at an arbitrary point is then given by,

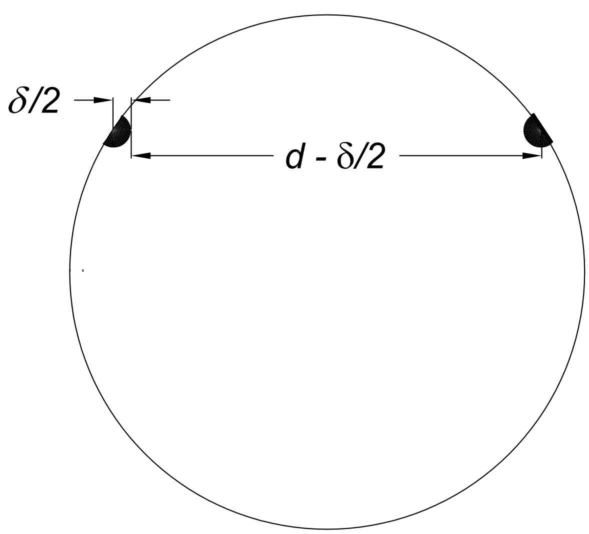
$$\phi = \phi_1 + \phi_2 = 2\lambda \ln \frac{r_1}{r_2}. \tag{4}$$

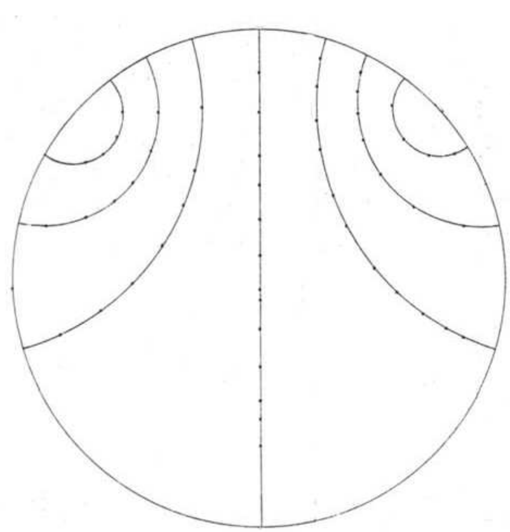
The total potential difference between the two line charges is formally divergent. To make physical sense, we can suppose that expression (4) holds only for r_1 and r_2 greater than $\delta/2$, the half width of the electrical contacts, and the potential is essentially constant for smaller values of r_1 and r_2. That is, we approximate the contacts of width δ by perfectly conducting wires of radii $\delta/2$, as shown in the figure below. Then, the potential of contact 2 is estimated from eq. (4) by setting $r_1 = d - \delta/2$ and $r_2 = \delta/2$,

$$\phi(\text{contact } 2) = 2\lambda \ln \frac{d - \delta/2}{\delta/2} \approx 2\lambda \ln \frac{2d}{\delta}. \tag{5}$$

The potential at the surface of contact 1 is just the negative of eq. (5), so the potential difference is,

$$V \approx 4\lambda \ln \frac{2d}{\delta}. \tag{6}$$





We note that the current and the electric field must be tangential to the edge of the disk. We recall that the equipotentials of eq. (4) are circles, as shown in the figure on the right above (from Kirchhoff), and that the corresponding electric field lines are also circles which, of course, pass through the line charges.[1] Hence, the boundary condition on the electric field at the edge of the disk is indeed satisfied.

To complete the solution, we must calculate the current I that is flowing. For this, we can integrate the current density $\mathbf{J}$ across any surface between the two contacts. For convenience, consider a cylindrical surface of radius r centered on one of the contacts, such that $\delta/2 < r \ll d$. Since $r \ll d$, this surface is essentially an equipotential, and the electric field is essentially that due to the nearby charge density λ. Namely, the electric field is normal to this surface, with magnitude,

$$E = \frac{2\lambda}{r}. \tag{7}$$

The current density across this surface is given by $J = \sigma E$. Restricting the problem to a disk of thickness t, the relevant area of the surface is $\pi r t$, so the total current is,

$$I = \pi r t \cdot \sigma \cdot \frac{2\lambda}{r} = 2\pi\sigma\lambda t, \tag{8}$$

which is independent of the choice of r.

Finally, the resistance is found by combining eqs. (6) and (8),

$$R = \frac{V}{I} \approx \frac{4\lambda \ln 2d/\delta}{2\pi\sigma\lambda t} = \frac{2}{\pi\sigma t} \ln \frac{2d}{\delta}, \tag{9}$$

independent of the radius a of the disk.

[1] See, for example, pp. 74a-b of http://kirkmcd.princeton.edu/examples/ph501/ph501lecture6.pdf