Newtonian Gravity with a Retarded Potential

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1 Problem

What is the gravitational force due to a pointlike mass in Newtonian gravity, supposing that gravity propagates at the speed c of light in vacuum, such that a retarded potential should be used?

Thanks to Jim Wertz for bringing this problem to the author's attention.

2 Solution

Newtonian gravity obeys a $1/r^2$ force law, like electrostatics, so a retarded potential Φ for Newtonian gravity has the same form as the retarded scalar potential in electromagnetism. The latter was first deduced by Liénard (1898), p. 6 of [1], and can be written for Newtonian gravity as,

$$\Phi(\mathbf{r}_m, t) = -\frac{GMm}{r - \mathbf{r}\cdot\mathbf{v}/c}\,, \tag{1}$$

where G is Newton's gravitational constant, M is the source mass at position $\mathbf{r}_M$, m is the test mass at position $\mathbf{r}_m$ at the present time t, $\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$ for $\mathbf{r}_M$ not at the present time t but at the retarded time $t' = t - r/c$, and $\mathbf{v}$ is the velocity of the source mass M at the retarded time t'. If the source mass M is at rest, this simplifies to the familiar Newtonian form $\Phi = -GMm/r$. with r as the present distance between masses M and m.

The force on mass m at time t is given by $\mathbf{F}(t) = -\boldsymbol{\nabla}\Phi(\mathbf{r}_m, t)$. The vector derivative $-\boldsymbol{\nabla}\Phi$ was not displayed by Liénard (nor by Wiechert [2]), but is given, for example, in eq. (6-3-7), p. 218 of [3], and eq. (10.62), p. 437 of [4]. For Newtonian gravity, we then have the retarded force on test mass m at present time t as,

$$\mathbf{F}(\mathbf{r}_m, t) = -\boldsymbol{\nabla}\Phi = -\frac{GMm}{(r - \mathbf{r}\cdot\mathbf{v}/c)^3}\left[\mathbf{r}\left(1 - \frac{v^2}{c^2} + \frac{\mathbf{a}\cdot\mathbf{r}}{c^2}\right) - \frac{\mathbf{v}}{c}\left(r - \frac{\mathbf{r}\cdot\mathbf{v}}{c}\right)\right], \tag{2}$$

where $\mathbf{v}$ and $\mathbf{a}$ are the velocity and acceleration of the source mass M at the retarded time t'. If $\mathbf{v}$ and $\mathbf{a}$ are both zero, *i.e.*, the source mass is at rest, the force (2) simplifies to the familiar form $\mathbf{F} = -GMm\,\mathbf{r}/r^3$, with $\mathbf{r}$ as the present distance between masses m and M.

As the ratio m/M goes to zero, the source mass M goes to rest in the center-of-mass frame, and there can be no effect of retardation in a Newtonian analysis (in this limit). This contrasts with Einstein's result from general relativity [5], in which a nonzero precession of orbits is predicted, independent of m/M for small m.

3 Comments

In the late 1800's many authors considered the effect on the precession of the perihelion of Mercury due to possible effects of a finite speed of gravity, which imply velocity-dependent corrections to the classic Newtonian analysis of orbits. A review is given in Chap. 6 of [6]; see also [7]. However, it seems that no effort of this type was made on the basis of the potentials of Liénard in the early 1900's.[1]

In 1975, Good [10] argued that adding retardation to Newtonian gravity implies unstable orbits, unlike in electromagnetism where v/c terms in the retarded force cancel the instability.[2] However, the retarded force (2) does contain terms of order v/c, so Good's argument is doubtful.

In recent years, three papers [13, 14, 15] have nominally considered the effect of a retarded potential, but none of these seemed aware of our eq. (2).[3,4]

One should work in the center-of-mass frame of the Sun and Mercury. Then, the Sun has a very small "orbit" about the center of mass of the system, which latter is taken to be at rest. The retarded force of the Sun (which moves in this frame) on Mercury then pulls Mercury slightly "forward" in its orbit, leading to a very slow advance of the perihelion.

3.1 Retarded Potential and Force to order $1/c^2$ (Apr. 5, 2023)

For small velocity and acceleration of masses M and m it may suffice to keep terms only to order $1/c^2$.

An approximation to the retarded potential (1) can be obtained as in §65 of [19], transcribing their eq. (65.5) as,

$$\Phi \approx -GMm\left(\frac{1}{r} + \frac{1}{2c^2}\frac{\partial^2 r}{\partial t^2}\right) \equiv -GMm\left(\frac{1}{r} + \frac{\ddot{r}}{2c^2}\right), \tag{3}$$

where $r = |\mathbf{r}_m - \mathbf{r}_M|$ is evaluated at the present time t. As discussed before eq. (65.3) of [19], the term of order $1/c$ in eq. (3) is zero because the source mass is constant in time. A consequence of this is that the approximate force (8) in terms of present quantities contains no term of order $1/c$, in contrast to the expression (2) in terms of retarded quantities.

Following an argument between eqs. (65.5) and 65.6) of [19], we can write, for present-time quantities, taking the observation point $\mathbf{r}_m$ as fixed when computing the retarded potential

[1] In 1918, Einstein [8] discussed gravitational waves in the context of his theory of general relativity, using a method of retarded potentials when considering the gravitational metric tensor as a kind of (tensor) potential. This was a correction to an earlier paper [9] in which he claimed that gravitational waves do not transmit energy.

[2] A variant of Good's argument is given in prob. 12.4 of [11]. See also [12].

[3] A method for numerical integration of differential equations with retardation is discussed in [16].

[4] Feb. 20, 2024. Onoochin has applied a version of retarded gravity to the problem of two equal masses in a circular orbit [17], finding violations of conservation of energy and angular momentum, and inferring that these can occur in gravitational interactions. More circumspect remarks on this problem are in sec. 3.6 of [18]. See also the end of sec. 3.1 below.

due to source mass M,[5]

$$\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M, \qquad \mathbf{v} = -\dot{\mathbf{r}} = -\dot{\mathbf{r}}_M, \tag{4}$$

$$r^2 = \mathbf{r}\cdot\mathbf{r}, \qquad r\dot{r} = \mathbf{r}\cdot\dot{\mathbf{r}} = -\mathbf{r}\cdot\mathbf{v}, \qquad \dot{r} = -\frac{\mathbf{r}\cdot\mathbf{v}}{r}, \tag{5}$$

$$r\ddot{r} + \dot{r}^2 = -\dot{\mathbf{r}}\cdot\mathbf{v} - \mathbf{r}\cdot\dot{\mathbf{v}}, \qquad \ddot{r} = -\frac{\dot{r}^2}{r} + \frac{v^2}{r} - \frac{\mathbf{r}\cdot\mathbf{a}}{r} = \frac{v^2}{r} - \frac{(\mathbf{r}\cdot\mathbf{v})^2}{r^3} - \frac{\mathbf{a}\cdot\mathbf{r}}{r}. \tag{6}$$

where $\mathbf{a} = \dot{\mathbf{v}} = -\ddot{\mathbf{r}}_M$ is the negative of the acceleration of the source mass M at present time t. Hence, the approximate retarded potential in terms of present-time quantities is,

$$\Phi \approx -\frac{GMm}{r}\left(1 + \frac{v^2}{2c^2} - \frac{(\mathbf{r}\cdot\mathbf{v})^2}{2c^2r^2} - \frac{\mathbf{a}\cdot\mathbf{r}}{2c^2}\right). \tag{7}$$

The retarded force on mass m, in terms of present quantities and to order $1/c^2$, is then,[6]

$$\mathbf{F}(\mathbf{r}_m, t) = -\boldsymbol{\nabla}\Phi \approx -\frac{GMm}{r^3}\left[\mathbf{r}\left(1 + \frac{v^2}{2c^2} + \frac{(\mathbf{r}\cdot\mathbf{v})^2}{c^2r^2} - \frac{\mathbf{a}\cdot\mathbf{r}}{2c^2}\right) - \frac{\mathbf{v}(\mathbf{r}\cdot\mathbf{v})}{c^2} - \frac{\mathbf{a}\,r^2}{2c^2}\right]. \tag{8}$$

Now that we have the force (8) on mass m, deduced from a retarded potential for a fixed point $\mathbf{r}_m$, we can use it in the equation of motion $\mathbf{F} = m\,\ddot{\mathbf{r}}_m$ in which the position $\mathbf{r}_m$ of mass m changes with time (assuming that the gravitational force on a mass is independent of its velocity, at least for low velocity [21]). We are working in the center-of-mass frame, where $m\,\mathbf{r}_m + M\,\mathbf{r}_M = 0$, so,

$$\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M = \frac{M+m}{M}\mathbf{r}_m = -\frac{M+m}{m}\mathbf{r}_M, \tag{9}$$

$$\mathbf{v} = -\dot{\mathbf{r}}_M = \frac{m}{M+m}\dot{\mathbf{r}}, \qquad \mathbf{a} = -\ddot{\mathbf{r}}_M = \frac{m}{M+m}\ddot{\mathbf{r}}. \tag{10}$$

The Newtonian equation of motion is then,

$$\frac{Mm}{M+m}\ddot{\mathbf{r}} \approx$$
$$-\frac{GMm}{r^3}\left[\mathbf{r} + \left(\frac{m}{M+m}\right)^2\left(1 + \mathbf{r}\frac{\dot{\mathbf{r}}^2}{2c^2} + \mathbf{r}\frac{(\mathbf{r}\cdot\dot{\mathbf{r}})^2}{c^2r^2} - \dot{\mathbf{r}}\frac{(\mathbf{r}\cdot\dot{\mathbf{r}})}{c^2}\right) - \frac{m}{M+m}\left(\mathbf{r}\frac{\ddot{\mathbf{r}}\cdot\mathbf{r}}{2c^2} - \frac{\ddot{\mathbf{r}}\,r^2}{2c^2}\right)\right]. \tag{11}$$

If $m \ll M$, and we neglect the terms of order m^2/M^2, we obtain,

$$\ddot{\mathbf{r}} \approx -\frac{GM}{r^3}\left[\mathbf{r} - \frac{m}{M}\left(\mathbf{r}\frac{\ddot{\mathbf{r}}\cdot\mathbf{r}}{2c^2} - \frac{\ddot{\mathbf{r}}\,r^2}{2c^2}\right)\right]. \tag{12}$$

[5] In the analysis of Liénard-Wiechert potentials, which are part of a field theory, the observation point is regarded as at rest, even if it happens that a moving charge/mass is there at time t.

This contrasts with Weber's, instantaneous, action-at-a-distance, electromagnetic potential, which is not a field-theory concept, $U = (ee'/r)(1 - \dot{r}^2/2c^2)$, where r is the present distance between (moving) electric charges e and e', and $\dot{r} = (\dot{\mathbf{r}}_e - \dot{\mathbf{r}}_{e'})\cdot(\mathbf{r}_e - \mathbf{r}_{e'})/r$. See, for example, p. 229 of [20].

[6] This is not what would be obtained from eq. (2) by approximating the retarded quantities there as present quantities, and keeping only terms to order $1/c^2$.

As previously noted at the end of sec. 2 above, in the limit of zero mass m there is no deviation of the orbit from a Newtonian ellipse (in contrast to observation, and to the prediction of general relativity [5]). This argues against the validity of Newtonian gravity with a retarded potential.

Feb. 20, 2024. For the case of equal masses, $m = M$, and a nominally circular orbit of radius $\mathbf{r}_m = \mathbf{r}/2$ about their center of mass, eq. (11) for $\mathbf{F}_m = m\ddot{\mathbf{r}}_m = m\ddot{\mathbf{r}}/2$ includes the term $Gm^2(\mathbf{r}_m \cdot \dot{\mathbf{r}}_m)\,\dot{\mathbf{r}}_m/4c^3 r_m^3$ is in the direction of motion of mass m, and which would imply nonconservation of energy and angular momentum. Now, the factor $\mathbf{r}_m \cdot \dot{\mathbf{r}}_m$ is negligible for near-circular orbits, so this defect of the model of the retarded potential can be ignored in this special case (where the effect of the retarded potential analysis is only to imply a correction of order $1/c^2$ to the centripetal acceleration, which makes a very tiny correction to the Newtonian value of the separation r between the two orbiting masses). However, for elliptical orbits, the violation of conservation of energy and angular momentum would be significant, which further suggests that the assumption that gravity can be described by a retarded scalar potential is not valid.

According to general relativity (in which the gravitational potential is essentially the metric tensor), a binary-mass system emits gravitational radiation as well as angular momentum, so the system spirals in towards its center of mass very slowly, while the mechanical angular momentum of the system decreases. This has famously been well studied for the binary pulsar system PSR 1913+16 [22], which is in excellent agreement with general relativity (but not the model of gravity with a retarded scalar potential).

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