Off the Rim

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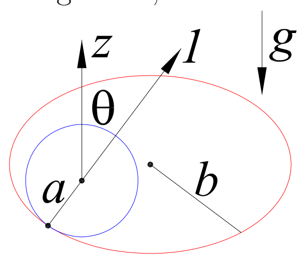
1 Problem

A frequent occurrence in basketball or golf is that the ball rolls around in the rim of the hoop/cup for a while, then sometimes goes in, sometimes not...



Consider a sphere of radius a that rolls without slipping on a horizontal hoop of radius $b > a$. An equilibrium of steady rolling exist with zero "spin" component, $\omega_0 = \boldsymbol{\omega} \cdot \hat{\mathbf{1}} = 0$, where $\boldsymbol{\omega}$ is the total angular velocity of the sphere and $\hat{\mathbf{1}}$ is directed from the point of contact with the hoop to the center of the sphere. Show that in the case the angular velocity of the point of contact about the vertical is,

$$\Omega = \sqrt{\frac{3g \tan\theta_0}{5(b - a\sin\theta_0)}}, \tag{1}$$

for a spherical shell, where θ is the angle of $\hat{\mathbf{1}}$ to the vertical.

For a basketball of radius 12 cm and a hoop of radius 24 cm, $\Omega_0 \approx 0.6$ revolution per second at $\theta_0 = 45°$.

Show that this equilibrium is unstable (for $b/a = 2$). That is, for Ω greater/less than ω_0, the sphere rises/falls, and only in the latter case does it pass through the hoop as desired.

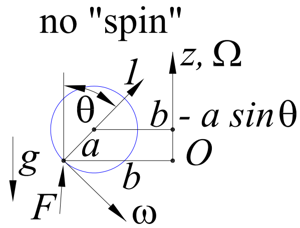
2 Solution

We consider a sphere of radius a that rolls without slipping on a horizontal hoop of radius $b > a$.

2.1 Steady Motion with No "Spin"

Before treating the general motion, we consider the special case of steady motion with no "spin" about the line, $\hat{\mathbf{1}}$, between the point of contact of the sphere with the hoop and the center of the sphere.

In this case, the angle θ_0 between the vertical, $\hat{\mathbf{z}}$, and $\hat{\mathbf{1}}$ is constant, and the center of the sphere moves in a horizontal circle of radius $b - a\sin\theta_0$ with constant angular velocity Ω.



The rolling constraint is,

$$\mathbf{v}_{\rm contact} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{a} = 0, \tag{2}$$

where $\mathbf{v}$ is the velocity of the center of the sphere, $\boldsymbol{\omega}$ is its total angular velocity, and $\mathbf{a} = -a\,\mathbf{1}$ points from the center of the sphere to the point of contact.

In the case of no "spin" about $\hat{\mathbf{1}}$, the angular velocity $\boldsymbol{\omega}$ is perpendicular to $\hat{\mathbf{1}}$ and in the vertical plane that contains the center of the hoop and the point of contact, as shown in the figure above. Then, the rolling constraint (2) implies,

$$\Omega(b - a\sin\theta_0) = \omega a. \tag{3}$$

The torque equation of (steady) motion about the point of contact is,

$$\boldsymbol{\tau}_{\rm contact} = -\mathbf{a} \times m\mathbf{g} = -mag\,\hat{\mathbf{1}} \times \hat{\mathbf{z}} = mag\sin\theta_0\,\hat{\mathbf{2}}$$

$$= \frac{d\mathbf{L}_{\rm contact}}{dt} = I_{\rm contact}\frac{d\boldsymbol{\omega}}{dt} = \left(I + ma^2\right)\Omega\,\hat{\mathbf{z}} \times \boldsymbol{\omega} = \left(I + ma^2\right)\Omega\,\omega\,\cos\theta_0\,\hat{\mathbf{2}}, \tag{4}$$

$$\Omega = \frac{mag\sin\theta_0}{(I + ma^2)\,\omega\,\cos\theta_0}\,, \qquad \Omega^2 = \frac{ma^2 g\tan\theta_0}{(I + ma^2)\,(b - a\sin\theta_0)}\,, \tag{5}$$

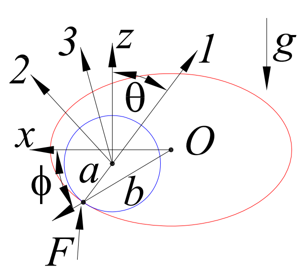
using eq. (3), and defining $\hat{\mathbf{2}} = \hat{\mathbf{z}} \times \hat{\mathbf{1}}/\sin\theta_0$, which is into the page in the figure above.

For a spherical shell of radius $a = 12$ cm, $I = 2ma^2/3$, with $b = 2a$ and $\theta_0 = 45^\circ$, the frequency of revolution of the sphere about the center of the hoop is $2\pi/\Omega = 2\pi\sqrt{5a(4 - \sqrt{2})/6g} \approx 0.6$ Hz.

Also, there is a formal equilibrium with $\theta_0 = 0 = \Omega$, at which the sphere is perched on a point on the rim. For "spin" $\omega_1 = 0$ this equilibrium is unstable, but we need to consider whether it might be stable for large enough ω_1.

2.2 General Equations of Motion

Turning to the general case when θ and $\dot\phi = \Omega$ vary with time, we introduce the principal axes (not body axes) $\hat{\mathbf{1}}$,$\hat{\mathbf{2}}$,$\hat{\mathbf{3}}$, with origin at the center of the sphere. $\hat{\mathbf{1}}$ points from the point of contact with the hoop to the center of the sphere, $\hat{\mathbf{2}} = \hat{\mathbf{z}} \times \hat{\mathbf{1}}/\sin\theta$ is horizontal, and $\hat{\mathbf{3}} = \hat{\mathbf{1}} \times \hat{\mathbf{2}}$ is in the vertical plane containing the centers of the hoop and the sphere. Also, $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{1}} + \sin\theta\,\hat{\mathbf{3}}$.



The velocity of the center of the sphere is,

$$\mathbf{v} = -\dot{\phi}(b - a\sin\theta)\,\hat{\mathbf{2}} - a\,\dot{\theta}\,\hat{\mathbf{3}}. \tag{6}$$

where $\Omega = \dot{\phi}$ is the angular velocity of the sphere about the center of the hoop.

From the rolling constraint (2) we have, recalling that $\mathbf{a} = -a\,\hat{\mathbf{1}}$,

$$\mathbf{1} \times (\boldsymbol{\omega} \times \mathbf{a}) = -a\,\boldsymbol{\omega} - \omega_1\,\mathbf{a} = -\hat{\mathbf{1}} \times \mathbf{v} = -a\,\dot{\theta}\,\hat{\mathbf{2}} + \dot{\phi}(b - a\sin\theta)\,\hat{\mathbf{3}}, \tag{7}$$

$$\boldsymbol{\omega} = \omega_1\,\hat{\mathbf{1}} + \dot{\theta}\,\hat{\mathbf{2}} - \dot{\phi}\frac{b - a\sin\theta}{a}\,\hat{\mathbf{3}}. \tag{8}$$

The angular velocity of the triad $\hat{\mathbf{1}}$,$\hat{\mathbf{2}}$,$\hat{\mathbf{3}}$ is,

$$\boldsymbol{\omega}_{123} = \dot{\theta}\,\hat{\mathbf{2}} + \dot{\phi}\,\hat{\mathbf{z}} = \dot{\phi}\cos\theta\,\hat{\mathbf{1}} + \dot{\theta}\,\hat{\mathbf{2}} + \dot{\phi}\sin\theta\,\hat{\mathbf{3}}. \tag{9}$$

The time rate of change of the principal axes is related by,

$$\frac{d\hat{\mathbf{i}}}{dt} = \boldsymbol{\omega}_{123} \times \hat{\mathbf{i}}, \tag{10}$$

$$\frac{d\hat{\mathbf{1}}}{dt} = (\dot{\phi}\cos\theta\,\hat{\mathbf{1}} + \dot{\theta}\,\hat{\mathbf{2}} + \dot{\phi}\sin\theta\,\hat{\mathbf{3}}) \times \hat{\mathbf{1}} = \dot{\phi}\sin\theta\,\hat{\mathbf{2}} - \dot{\theta}\,\hat{\mathbf{3}}, \tag{11}$$

$$\frac{d\hat{\mathbf{2}}}{dt} = (\dot{\phi}\cos\theta\,\hat{\mathbf{1}} + \dot{\theta}\,\hat{\mathbf{2}} + \dot{\phi}\sin\theta\,\hat{\mathbf{3}}) \times \hat{\mathbf{2}} = \dot{\phi}\sin\theta\,\hat{\mathbf{1}} + \dot{\phi}\cos\theta\,\hat{\mathbf{3}}, \tag{12}$$

$$\frac{d\hat{\mathbf{3}}}{dt} = (\dot{\phi}\cos\theta\,\hat{\mathbf{1}} + \dot{\theta}\,\hat{\mathbf{2}} + \dot{\phi}\sin\theta\,\hat{\mathbf{3}}) \times \hat{\mathbf{3}} = \dot{\theta}\,\hat{\mathbf{1}} - \dot{\phi}\cos\theta\,\hat{\mathbf{2}}. \tag{13}$$

The force and torque equations of motion of (the center of) the sphere of radius a are,

$$\begin{aligned}
\mathbf{F} - mg\,\hat{\mathbf{z}} = m\frac{d\mathbf{v}}{dt} &= -m\,\ddot{\phi}(b - a\sin\theta)\,\hat{\mathbf{2}} + ma\,\dot{\phi}\,\dot{\theta}\cos\theta\,\hat{\mathbf{2}} - ma\,\ddot{\theta}\,\hat{\mathbf{3}} \\
&\quad -m\dot{\phi}(b - a\sin\theta)(\dot{\phi}\sin\theta\,\hat{\mathbf{1}} + \dot{\phi}\cos\theta\,\hat{\mathbf{3}}) - ma\dot{\theta}(\dot{\theta}\,\hat{\mathbf{1}} - \dot{\phi}\cos\theta\,\hat{\mathbf{2}}) \\
&= -m(\dot{\phi}^2(b - a\sin\theta)\sin\theta + a\,\dot{\theta}^2)\,\hat{\mathbf{1}} + m(2a\,\dot{\phi}\,\dot{\theta}\cos\theta - \ddot{\phi}(b - a\sin\theta))\,\hat{\mathbf{2}} \\
&\quad -m(a\,\ddot{\theta} + \dot{\phi}^2(b - a\sin\theta)\cos\theta)\,\hat{\mathbf{3}},
\end{aligned} \tag{14}$$

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} = I\frac{d\boldsymbol{\omega}}{dt} &= I\dot{\omega}_1\,\hat{\mathbf{1}} + I\ddot{\theta}\,\hat{\mathbf{2}} + I\left(\dot{\phi}\,\dot{\theta}\cos\theta - \ddot{\phi}\frac{b - a\sin\theta}{a}\right)\hat{\mathbf{3}} + I\omega_1(\dot{\phi}\sin\theta\,\hat{\mathbf{2}} - \dot{\theta}\,\hat{\mathbf{3}}) \\
&\quad +I\dot{\theta}(\dot{\phi}\sin\theta\,\hat{\mathbf{1}} + \dot{\phi}\cos\theta\,\hat{\mathbf{3}}) - I\dot{\phi}\frac{b - a\sin\theta}{a}(\dot{\theta}\,\hat{\mathbf{1}} - \dot{\phi}\cos\theta\,\hat{\mathbf{2}}) \\
&= I\left(\dot{\omega}_1 - \dot{\phi}\,\dot{\theta}\frac{b - 2a\sin\theta}{a}\right)\hat{\mathbf{1}} + I\left(\ddot{\theta} + \omega_1\,\dot{\phi}\cos\theta + \dot{\phi}^2\frac{b - a\sin\theta}{a}\cos\theta\right)\hat{\mathbf{2}} \\
&\quad +I\left(2\dot{\phi}\,\dot{\theta}\cos\theta - \omega_1\,\dot{\theta} - \ddot{\phi}\frac{b - a\sin\theta}{a}\right)\hat{\mathbf{3}} \\
&= \boldsymbol{\tau} = \mathbf{a} \times \mathbf{F}
\end{aligned} \tag{15}$$

$$= -ma(2a\,\dot{\phi}\,\dot{\theta}\cos\theta - \ddot{\phi}(b - a\sin\theta))\,\hat{\mathbf{3}} - ma(a\,\ddot{\theta} + \dot{\phi}^2(b - a\sin\theta)\cos\theta)\,\hat{\mathbf{2}} + mag\sin\theta\,\hat{\mathbf{2}},$$

where I is the moment of inertia of the sphere about its center. The components of the equation of motion (15) are:

$$\hat{\mathbf{1}}: \qquad \dot{\omega}_1 = \dot{\phi}\,\dot{\theta}\frac{b - 2a\sin\theta}{a}, \tag{16}$$

$$\hat{\mathbf{2}}: \qquad \left(I + ma^2\right)\left(\ddot{\theta} + \dot{\phi}^2\frac{b - a\sin\theta}{a}\cos\theta\right) + I\,\omega_1\,\dot{\phi}\cos\theta = mag\sin\theta, \tag{17}$$

$$\hat{\mathbf{3}}: \qquad \left(I + ma^2\right)\left(2\dot{\phi}\,\dot{\theta}\cos\theta - \ddot{\phi}\frac{b - a\sin\theta}{a}\right) = I\,\omega_1\,\dot{\theta}. \tag{18}$$

For steady motion with $\theta = \theta_0 =$ constant and $\dot{\phi} = \Omega =$ constant, we have that $\omega_1 =$ constant from eq. (16), eq. (18) is trivial, and eq. (17) leads to,

$$\Omega^2 + \frac{I\Omega\,\omega_1}{I + ma^2} = \frac{ma^2 g\tan\theta_0}{(I + ma^2)\,(b - a\sin\theta_0)} \tag{19}$$

which reduces to eq. (5) for the special case of no "spin", *i.e.*, $\omega_1 = 0$.

The coupled equations of motion (16)-(18) are intricate, and we limit further discussion to two special cases: either the "spin" ω_1 is negligible, or the equilibrium is with $\theta_0 = 0 = \Omega$.

2.2.1 Stability of Steady Motion with No "Spin"

For ω_1 negligible, we consider possible nutations of the form,

$$\theta = \theta_0 + \epsilon\sin\alpha t, \qquad \dot{\phi} = \Omega + \delta\sin\alpha t, \tag{20}$$

$$\sin\theta \approx \sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t, \qquad \cos\theta \approx \cos\theta_0 - \epsilon\sin\theta_0\sin\alpha t, \tag{21}$$

for small constants ϵ and δ. Then, to first order in ϵ and δ, eq. (18) becomes,

$$2\Omega\alpha\,\epsilon\cos\alpha t\cos\theta_0 \approx \alpha\,\delta\cos\alpha t\frac{b - a\sin\theta_0}{a}, \qquad \delta \approx 2\,\epsilon\,\Omega\frac{a}{b - a\sin\theta_0}\cos\theta_0. \tag{22}$$

and eq. (17) becomes,

$$\begin{aligned} mag(\sin\theta_0 + \epsilon\cos\theta_0\sin\alpha t) &\approx -\alpha^2\,\epsilon\left(I + ma^2\right)\sin\alpha t \\ + \left(I + ma^2\right)(\Omega^2 + 2\Omega\,\delta\sin\alpha t)&\left(\frac{b}{a} - \sin\theta_0 - \epsilon\cos\theta_0\sin\alpha t\right)(\cos\theta_0 - \epsilon\sin\theta_0\sin\alpha t), \end{aligned} \tag{23}$$

$$\begin{aligned} \epsilon\,m\,a\,g\cos\theta_0 &\approx -\alpha^2\,\epsilon\left(I + ma^2\right) - \epsilon\left(I + ma^2\right)\Omega^2\left[\frac{b - a\sin\theta_0}{a}\sin\theta_0 + \cos^2\theta_0\right] \\ &\quad + 2\Omega\,\delta\left(I + ma^2\right)\frac{b - a\sin\theta_0}{a}\cos\theta_0, \end{aligned} \tag{24}$$

$$\alpha^2 \approx -\frac{m\,a\,g\cos\theta_0}{I + ma^2} - \Omega^2\left[\frac{b - a\sin\theta_0}{a}\sin\theta_0 + \cos^2\theta_0\right] + 4\Omega^2\cos^2\theta_0. \tag{25}$$

For this case we also have Ω^2 given by eq. (5), so,

$$\begin{aligned} \alpha^2 &\approx -\frac{m\,a\,g\cos\theta_0}{I + ma^2} - \frac{mag\sin^2\theta_0}{(I + ma^2)\cos\theta_0} + 3\frac{mag\sin\theta_0\cos\theta_0}{(I + ma^2)\,(b/a - \sin\theta_0)} \\ &= \frac{mag}{(I + ma^2)\,(b/a - \sin\theta_0)\cos\theta_0}[3\sin\theta_0\cos^2\theta_0 - \sin^2\theta_0 - \cos^2\theta_0(b/a - \sin\theta_0)]. \end{aligned} \tag{26}$$

A numerical calculation (https://kirkmcd.princeton.edu/examples/rim.xlsx) indicates that $\alpha^2 < 0$ for any angle θ_0 when $b/a > 1.88$.[1] In regulation basketball, b/a is very close to 2, so $\alpha^2 < 0$ for any θ_0, and the equilibrium with $\omega_1 = 0$ is unstable. That is, if a basketball starts to roll around the hoop, it quickly falls in or out.

2.2.2 Sphere Directly above a Point on the Hoop

We now turn to the equilibrium of a sphere whose center is at rest directly above some point on the hoop, with the sphere spinning about the vertical.

We consider possible, small nutations about this equilibrium as in eqs. (20)-(21), but here, $\theta_0 = 0 = \Omega$. Then, the right side of eq. (16) is of second order, so in the first approximation ω_1 is constant. Equation (18) now implies that,

$$-\,\delta\left(I + ma^2\right)\frac{b}{a}\cos\alpha t = \epsilon I\,\omega_1\,\cos\alpha t, \qquad \delta = -\epsilon\frac{I}{I+ma^2}\frac{a\omega_1}{b}\,, \tag{27}$$

and eq. (17) leads to,

$$\epsilon\, mag\sin\alpha t \approx -\alpha^2\,\epsilon\left(I + ma^2\right)\sin\alpha t + \delta\, I\,\omega_1\sin\alpha t,$$

$$mag \approx -\alpha^2\,\left(I + ma^2\right) - \frac{I}{I+ma^2}\frac{a\omega_1}{b}I\,\omega_1, \tag{28}$$

$$\alpha^2 \approx \left(\frac{I}{I+ma^2}\right)^2\omega_1^2\frac{a}{b} - \frac{m\,a\,g}{I+ma^2}\,. \tag{29}$$

This equilibrium is stable for,

$$\omega_1 > \frac{I+ma^2}{I}\sqrt{\frac{gb}{a^2}\frac{ma^2}{I+ma^2}}\,. \tag{30}$$

For a basketball of radius $a = 12$ cm and a hoop with $b/a = 2$, the minimum "spin" ω_1 for stability of this equilibrium is only 2 Hz.[2]

Note, however, that no stability is possible in the limit $b \to \infty$, which corresponds to a spinning sphere on a straight wire.[3]

A Appendix: The Limit of $b \to \infty$

In the limit of $b \to \infty$, the hoop becomes a long, straight wire for small ϕ, say along the x direction. Then, the horizontal vector $\hat{\mathbf{2}}$ is $\hat{\mathbf{x}}$, and the quantity $\phi(b - a\sin\theta \approx \phi\, b$ takes on the significance of the position x of the center of the sphere along the wire (for small ϕ).

[1] When stability is possible, it is most stable angle for $\theta_0 \approx 42^\circ$.

[2] Gyroscopic stability of a basketball on a curved hoop occurs for smaller ω_1 than when balancing it on your finger. For the latter, see http://kirkmcd.princeton.edu/examples/basketball.pdf.

[3] This case was discussed on pp. 212-214 of http://kirkmcd.princeton.edu/examples/Ph205/ph205l20.pdf. See also §424, p. 360 of Milne's *Vectorial Mechanics*, especially eq. (8), http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf.

Furthermore, $\dot{\phi}(b - a\sin\theta) \to \dot{x}$ and $\ddot{\phi}(b - a\sin\theta) \to \ddot{x}$. Then, the component equations of motion (16)-(18) become,

$$\hat{\mathbf{1}}: \qquad a\,\dot{\omega}_1 = \dot{\theta}\,\dot{x}, \tag{31}$$

$$\hat{\mathbf{2}}: \qquad \left(I + ma^2\right)\ddot{\theta} = mag\sin\theta, \tag{32}$$

$$\hat{\mathbf{3}}: \qquad -\left(I + ma^2\right)\ddot{x} = aI\,\omega_1\,\dot{\theta}. \tag{33}$$

These are the equations of motion found on p. 214 of
http://kirkmcd.princeton.edu/examples/Ph205/ph205l20.pdf, but with $x \to -x$. See also §424, p. 360 of Milne's *Vectorial Mechanics*, especially eqs. (7)-(9),
http://kirkmcd.princeton.edu/examples/mechanics/milne_mechanics.pdf.

In particular, eq. (32) indicates that once θ is nonzero, further motion only increases θ (until the sphere loses contact with the wire).

A finite radius of curvature b of the (horizontal) wire leads to more intricate 2- and 3-componets of the equations of motion, *i.e.*, eqs. (17)-(18), which permit gyroscopic stabilization of the sphere for large enough "spin" ω_1 (at some values of θ_0).